ELECTRICAL ENGINEERING TEXTS

# APPLICATIONS OF THE METHOD OF SYMMETRICAL COMPONENTS

BY

WALDO V. LYON

*Professor of Electrical Machinery of the Massachusetts Institute of Technology, Fellow of the American Institute of Electrical Engineers*

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# PREFACE

The subject matter presented in this book has been developed during the past fifteen years for the use of graduate students at the Massachusetts Institute of Technology. It is designed to acquaint these students with some of the problems arising when polyphase circuits are unsymmetrical and to show how these problems can be solved by means of the Method of Symmetrical Components. The author has tried to develop the subject from fundamental principles in an orderly and logical manner, and, so far as possible, he has arranged the material in the order of its difficulty. About forty-five numerical illustrations are given indicating how the method may be applied in typical concrete cases.

The reader is expected to have a thorough grounding in the theory of single-phase and symmetrical polyphase circuits, as well as of transformers and the symmetrical operation of three-phase induction and synchronous machines.

The author wishes to acknowledge the helpful suggestions that members of the teaching staff have offered. In particular he is grateful to C. Kingsley, Jr., who computed the effects of simultaneous faults on the power network at the end of Chapter X, and to E. W. Kimbark, who measured these effects by means of the network analyzer. The chart for determining the symmetrical components was designed by R. E. Peters.

In completing the book the author regretfully brings to a close a series of many pleasant and exceedingly helpful conferences with the editor in chief, Prof. H. E. Clifford, Dean of the Harvard Engineering School.

W. V. Lyon.

Massachusetts Institute of Technology,
*January*, 1937.

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# APPLICATIONS OF THE METHOD OF SYMMETRICAL COMPONENTS

## INTRODUCTION

Substantially all the progress during the past 50 years in determining the electrical characteristics of alternating-current power networks and their connected machines has resulted from applying to these problems analytical methods of greater and greater power.[1] The necessary fundamental electric and magnetic principles had already been discovered in the preceding years of the nineteenth century. Furthermore, nearly all the mathematical processes that were to prove so useful had already been developed, either as pure mathematics or for the purpose of solving other problems.

Although, with few exceptions, the original development of electrical apparatus has preceded the mathematical determination of its characteristics, it is generally true that apparatus does not reach its highest development until the electrical reactions can be quantitatively analyzed and computed. It has

[1] In the following list there are set down some of the more important of these analytical methods and ideas. They have proved so useful that they are now considered the indispensable tools of a well-equipped electrical engineer.

1. Vector representation of alternating currents, including the symbolic notation of complex numbers.
2. Bessel's functions.
3. Conception of leakage reactance.
4. Development of equivalent circuits.
5. Blondel double-reaction method.
6. Heaviside operational method.
7. Hyperbolic functions of complex variable.
8. Graphical methods for mapping electric and magnetic fields.
9. Application of harmonic analysis.
10. Use of per unit values for potential, current, and impedance.
11. Method of symmetrical components.
12. Operational methods for determining transient conditions.

happened often that those who sought to analyze the behavior of a machine did not know the method that was best suited to their problem, and either were deterred by the problem's seeming complexity in terms of the mathematics with which they were familiar, or were forced to make such simplifying assumptions as would bring the problem within the scope of their own mathematical ability. In such instances, the complete solution of the problem has been deferred until the discovery of the mathematical method by which the desired analysis could be made. An outstanding illustration of such a discovery is the Method of Symmetrical Components.

The Method of Symmetrical Components had its origin in the mathematical analysis of induction machines operating under unbalanced conditions. The following brief historical account of the genesis of the induction motor is abstracted from S. P. Thompson, "Polyphase Electric Currents," 1895 (1st ed.). It records the early development of the application of a scientific principle (concerning induced currents) discovered more than half a century before. It would have been difficult to realize at the time (1879) of the first induction motor that the scientific toy would grow and, in less than a generation, become a dominating factor in the distribution of electric energy.[1]

"Considering how much attention has been devoted to magnetic fields and to the combinations of the fields of two or more magnets, it is surprising how little thought has been given to the properties of rotating magnetic fields. The essence of the modern polyphase motor is the production of rotating magnetic fields within which masses of metal are set into forcible rotation by reason of the currents induced in them. These rotating

[1] LAMME, B. G. The Story of the Induction Motor, *Jour. A.I.E.E.*, March, 1921. The opening paragraph is as follows: "Looking back over the technical history of the induction motor, we see an apparatus which is of extreme theoretical interest, and, at the same time, is of inestimable value. It is impossible to define just what effect it has had upon the general development of the electrical industry. It is probably safe to say that without this motor the whole trend of the art would have been greatly modified; for unquestionably the induction motor has had a very great part in placing the polyphase alternating current system in its dominating position of today. Practically 95 per cent of the generated electric power of today is by alternating current. It is beyond belief that the a-c system could have reached anything like this dominating position if there had been no induction motor for the utilization of alternating current for power purposes."

magnetic fields are generated by the artifice of combining together two or more oscillating magnetic fields by the use of alternate currents in different phases, as already shown. But the principal properties of rotatory fields can be investigated and demonstrated without any such artifices, by very simple means. All that is required is an apparatus for spinning a magnet, the field of which is then investigated while it revolves at some known speed

"The subject appears to have first attracted notice about the year 1825, in the discussion of the phenomenon of Arago's rotations, of which, accordingly, some account is here given.

"**Arago's Rotations.**—As has so often occurred in other branches of science, the discovery of the magnetic rotations was made nearly simultaneously by several persons, for all of whom priority has been claimed. About 1824, Gambey,[1] the celebrated instrument-maker of Paris, had made the casual observation that a compass-needle, if disturbed, and set oscillating around its pivot, comes to rest sooner if the bottom of the compass-box is of copper than if it is of wood or other material. Barlow and Marsh,[2] at Woolwich, had at the same time been observing the effect on a magnetic needle of rotating in its neighbourhood a sphere of iron. Arago,[3] the astronomer, who is said to have learned of the phenomenon from Gambey, but who is also said[4] to have independently discovered it in 1822, when working with Humboldt at magnetic determinations, was beyond question the first to publish an account of the observation, which he did verbally before the Académie des Sciences of Paris, on November 22, 1824. He hung a compass-needle within rings of different materials, pushed the needle aside to about 45°, and counted the number of oscillations made by the needle before the angle of swing decreased to 10°. In a wooden ring the number was 145, in a thin copper ring 66, and in a stout copper ring it was only 33.

"The effect of the presence of the mass of copper is to damp the vibrations of the needle. Each swing takes the same time as

[1] See JAMIN, *Cours de Physique*, iii, 296, 1869, and VERDET, *Conferences de Physique*, i, 415, 1872.

[2] *Edinburgh Philos. Journal*, xiii, 122, 1825

[3] *Annales de Chimie et de Physique*, xxvii, 363, xxviii, 325, xxxii, 213.

[4] ARAGO, "Oeuvres complètes," iv, 424.

before, but the amplitudes are lessened; the motion dying down, as though there were some invisible friction at work. Arago remarked that it gave evidence of the presence of a force which only existed whilst there was relative motion between the magnet-needle and the mass of copper. He gave the phenomenon the name of magnetism of rotation. In 1825 he published a further experiment, in which, arguing from the principle of action and reaction, he produced a reaction on a stationary needle by motion of a copper disk. Suspending a compass-needle in a glass jar closed at the bottom by a sheet of paper or of glass, he held it over a rotating disk of copper. If the latter turns slowly the needle is simply deviated out of the magnetic meridian, tending to turn in the sense of rotation of the disk, as though invisibly dragged by it. With quicker rotation the deviation is greater. If the rotation is so swift that the needle is dragged over 90° a continuous rotation ensues."

This phenomenon was studied by several experimenters in England and on the continent, but "the publication of Faraday's brilliant research on magneto-electric induction, in 1831, forestalled the complete explanation of which [they were] in search.

"Faraday, in fact, showed that relative motion between magnet and copper disk inevitably set up currents in the metal of the disk, which, in turn, reacted on the magnet pole with mutual forces tending to diminish the relative motion—that is, tending to drag the stationary part (whether magnet or disk) in the direction of the moving part, and tending always to oppose the motion of the moving part. In fact, the currents go eddying round in the moving disk, unless led off by sliding contacts. This, indeed, Faraday effected, when he inserted his copper disk edgeways between the poles of a powerful magnet, and spun it, while against edge and axle were pressed spring contacts to take off the currents. The electromotive-force, acting at right angles to the motion, and to the lines of the magnetic field, produces currents which flow along the radius of the disk. If no external path is provided, the currents must find for themselves internal return paths in the metal of the disk.

"With the explanation given by Faraday of the Arago rotations, as being merely due to induced eddy-currents, the peculiar interest which they excited whilst their cause was unknown, seems almost to have died out. True, a few years later some

interest was revived when Foucault showed that they were capable of heating the metal disk, if in spite of the drag the rotation were forcibly continued in the magnetic field. Why this observation should have caused the eddy-currents discovered by Faraday as the explanation of Arago's phenomenon to be dubbed Foucault's currents is not clear. If any one is entitled to the honour of having the eddy-currents named after him, it is obviously Faraday or Arago, not Foucault.

"**The First Induction Motor.**—Amidst the crowd of modern inventions little note has been taken of the modest beginnings of the polyphase motor, the birth of which dates from 1879. Figure 89[1] illustrates the elementary motor which Mr. Walter Baily exhibited to the Physical Society of London on June 28, 1879, on the occasion of his reading a paper entitled, 'A Mode of Producing Arago's Rotations.'

"Down to that date the only mode of producing the Arago-rotations of a copper disk had been by rotating beneath it a steel magnet. Baily, instead of rotating any material magnet below the disk used a fixed electromagnet, but caused its magnetism to shift progressively between four successive poles, thus producing in the copper disk pivoted above them eddy-currents, which by their reaction gave the disk a mechanical motion in the direction of the progression of the poles.

"The disk in this primitive model is about $2\frac{3}{8}$ inches in diameter; the four magnet cores are about 4 inches high, joined to a common yoke; and each is wound with about 150 turns of insulated copper wire 2.5 mm. in diameter. The coils are connected two and two in series, like two independent horseshoe magnets set diagonally across one another. The two circuits are brought down separately to an ingenious revolving commutator built up of a simple arrangement of springs and contact strips mounted on a bit of wood, with a wire handle by which it is turned. On rotating it, the currents from two batteries are caused to be reversed alternately in the two circuits, giving rise to the following successions of polarity in the four poles:

| | | | | |
|---|---|---|---|---|
| N O | N N | O N | S N | S O |
| ↘ | ↓ | ↙ | ← | ↖ |
| O S | S S | S O | S N | O N |

[1] Not reproduced.

and so forth. Mr. Baily had very clear views as to how far this really represented a rotatory magnetic field. His own words are as follows:

" 'The rotation of the disk is due to that of the magnetic field in which it is suspended, and we should expect that if a similar motion of the field could be produced by any other means the result would be a similar motion of the disk.

" 'Possibly the rotation of the magnet may be the only practicable way of producing a uniform rotation of the field; but it will be shown in this paper that the disk can be made to rotate by an intermittent rotation of the field effected by means of electromagnets.' The author then goes on to discuss the result of the increase in strength of a pole while a neighbouring pole of the same sign decreases in strength, and suggests that if a whole circle of poles were arranged under the disk, and successively excited in opposite pairs, the series of impulses all tend to make the disk move in one direction around the axis; adding: 'In one extreme case, *viz.* when the number of electromagnets is infinite, we have the case of a *uniform* rotation of the magnetic field, such as we obtain by rotating permanent magnets.'

"In 1883[1] Marcel Deprez communicated to the Académie des Sciences an important theorem on the production of a true rotatory magnetic field, by the combination of two alternating magnetic fields having as their difference of phase a quarter period.

"If an alternating current is led round a coil so as to produce an alternating or oscillating magnetic field along the (axis) $OX$, and a second alternating current is led round a second coil so as to produce a second alternating magnetic field along the (axis) $OY$, then the result will be a *rotatory* magnetic field, provided these two magnetic fields are of equal period and amplitude, and differ exactly a quarter in phase. If they are of equal period, but not of exactly equal amplitude, the result will be equivalent to an *elliptically* rotating magnetic field; that is to say, one in which the strength and direction of the field is represented by the successive values of the radius vector drawn to an ellipse from its centre, and sweeping over equal areas in equal times. An elliptically rotatory field will also be produced

[1] *Comptes rendus*, II, 1193, 1883.

if the two component magnetic fields, though equal in period and amplitude, do not differ by exactly a quarter period. For a perfect rotatory field, corresponding to uniform circular motion, the two components must vary precisely as the sine and the cosine[1] of an angle respectively.

"Deprez's theorem bore no fruit: it remained a geometrical abstraction.

"Professor Galileo Ferraris,[1] of Turin, had already in 1885, arrived at the same fundamental ideas as those of Baily and of Deprez. But the result was more fruitful, inasmuch as he, without knowing of the work of either, united both sets of ideas. Like Baily he proposed to produce rotation of a copper conductor by means of eddy-currents induced in it by a progressively shifted magnetic field; and this progressively shifted magnetic field he proposed to generate as a true rotatory field by combining at right angles to one another two alternate currents which differed by a quarter-period from one another.

"Ferraris's first publication was in March, 1888, entitled *Electrodynamic rotations produced by means of alternate currents.* After expounding the geometric theory of the rotatory magnetic field, he suggested that a simple way of procuring the desired phase-currents would be to branch the circuit of an alternate current into two parts, into one of which should be inserted a resistance without self-induction, into the other a coil of much self-induction but of small resistance. The two windings of the motor should be respectively introduced into these two branches. The difference of phase thus produced would be sufficiently near to 90° to be effective. He expressed the opinion that in this way one might obtain all the effects that can be obtained by the rotation of a magnet.[2]

"**Researches of Nikola Tesla.**—The work done by Nikola Tesla between the years 1887 and 1891, is of itself sufficient, had no other workers been occupied in the same field of research, to have established the rotatory-field motor upon a solid basis. He arrived in 1886 at the firm conviction that some method must exist of driving an armature by currents induced within it, instead of driving it by currents brought into it (as in the ordinary

[1] See also Ferráris, "Rotazioni elettrodynamiche," Turin Acad., March, 1888.

[2] This marks the birth of the split-phase induction motor.

electric motors), through the agency of metallic contacts, commutators and brushes. By October, 1887, Tesla's work was sufficiently advanced for him to apply to the United States Patent Office for patents covering numerous points of a more or less fundamental character.

The Westinghouse Company acquired the Tesla patents. "The development of the induction motor,[1] being in reality an analytical problem, did not make much headway in the 'cut and try' days of 1888 and 1889, when the Westinghouse Company was undertaking to put it into commercial form. Our only frequencies in those days were 133 and 125 cycles and the only alternating-current supply circuits were single phase. None of these was suitable for the new motor, and, therefore, it was badly handicapped at the start.

**"Polyphase Work at the Frankfort Exhibition, 1891.**[2]—No record of the development of polyphase currents would be complete without a reference to the Electrotechnical Exhibition at Frankfort on the Main in the summer of 1891. Though nominally an International Exhibition the exhibits were mainly by German firms; and the features of greatest interest were the polyphase apparatus contributed by many firms." Several firms exhibited 2-phase and 3-phase synchronous generators and motors. A number of different types of 3-phase induction motors were also shown.

"A smaller 3-phase motor[3] of about 3 horsepower, used to drive a small continuous-current dynamo with a load of lamps, had a construction the inverse of that now usual in induction motors. The currents were led by three slip-rings into a revolving armature, whilst a stationary external part, consisting of iron core-rings furnished with closed circuit winding, constituted an induced field-magnet. A still smaller motor with induction rotor without contacts served to drive a small fan. Other motors exhibited at the same place by the Oerlikon Company, constructed from Brown's designs, had the now usual construction of a fixed external armature built of core-rings pierced to receive the windings; whilst the rotor was also built of core-rings pierced with a simple copper circuit of bars short-circuited with two

[1] LAMME, B. G., "Story of the Induction Motor."

[2] THOMPSON, S P., "Polyphase Electric Currents," 1895.

[3] Now in the laboratory of the Technical College, Finsbury.

end-rings like a squirrel-cage. One of these, of 20 horsepower, at 1200 revolutions per minute, weighed only 420 kilogrammes.

"The Allgemeine Elektricitäts-Gesellschaft, of Berlin, had, in conjunction with the Oerlikon Machine Company, of Zurich, undertaken the striking demonstration of long distance transmission of power, at high voltage, from Lauffen to Frankfort.

"It had been suggested, in the autumn of 1890, to seize the opportunity afforded by the Frankfort Exhibition to show what could be done in the way of transmitting power to a long distance at high voltage, and at the same time to demonstrate the advantages of the Drehstrom or polyphase system. Lauffen is 110 miles from Frankfort. To transmit, as was proposed, 100 horsepower through three copper wires, each only 4 millimetres thick, and with an efficiency of at least 75 per cent, necessitated the employment of a pressure of no less than 8000 volts. This *tour de force* was nevertheless accomplished."

Two turbine-driven generators were installed at Lauffen. "Each was capable of furnishing three currents, each of 1400 amperes at about 55 volts, the frequency being 40 periods per second. At each end of the line 3-phase transformers were used: at Lauffen to raise the pressure to 8500 volts, at Frankfort to reduce it back to about 65 volts. These transformers (some built in Berlin, others in Oerlikon) were immersed, for better insulation, in oil. The connections of both low-pressure and high-pressure windings were star-wise, the common junctions being earthed in every case. The lines were carried on about 3000 poles at a height of about 25 feet, each pole supporting three porcelain insulators with internal rims for holding oil.

"In the exhibition there was [a load consisting of] a 100-horsepower 3-phase motor and other smaller motors to which allusion has already been made. In addition to these motors, there were about 1000 glow-lamps operated by the current so transmitted.

"Great scepticism prevailed at first as to the probable result of the transmission under the novel conditions of using such very high voltages over so long a line, and with polyphase currents. It was anticipated by some that the efficiency would be greatly reduced by possible disturbances due to the capacity of the lines acting as condensers, or to leakage over the 10,000 insulators on which the line was supported. In private, some who were very

closely connected with the enterprise expressed their fears lest the efficiency should fall below 50 per cent, and at one time there was some apprehension lest the jury would not be allowed to make full tests. However, as experience was gained these fears were found to be empty. The elaborate tests carried out by the commission in the autumn months, mostly at about 8000 volts pressure, showed that the energy given out electrically at Frankfort was as much (on the average) as 74 per cent of the energy given by the turbines at Lauffen to the generator." In the report on these tests there is a summary which concludes as follows:

"The electrical running with alternate currents of 7500 to 8500 volts in conductors of more than 100 miles in length, insulated by means of oil, porcelain and air, proceeds just as regularly, safely, and as free from disturbances as does running with alternate currents of a few hundred volts' pressure over conducting wires of a few metres' length.

"The Lauffen-Frankfort transmission was much more than a mere experiment. It was a daring and successful demonstration not only of the utility of high voltages in the transmission of power, but of the success of polyphase currents. As such it marked an epoch in the commercial development of electricity. It evoked an extraordinary interest throughout the continent of Europe, and in Germany in particular. One evidence of this is to be found in the circumstance that early in the history of the project the German Emperor himself made a contribution of 10,000 marks toward the cost of the scheme."

At the Chicago World's Fair in 1893 the Westinghouse Company exhibited a 300-hp. induction motor having a rotating primary and a stationary secondary. "During starting[1] the secondary circuits were closed through a series of long heavy carbon rods which were placed in the basement beneath the exhibit. Not much was known about starting resistances in those days, and, as this carbon-starting outfit would sometimes get red hot while the motor was being brought up to speed, it was not considered desirable to let the public see it." The Westinghouse brought out their first line of commercial motors in 1894.

The following quotation is from S. P. Thompson, "Polyphase Electric Currents," 1895:

[1] LAMME, B. G., "Story of the Induction Motor."

"**The 'Monocyclic' System.**—Mr. C. P. Steinmetz has proposed[1] a system of distributing electricity for lighting and power which, although its essence is that more than one cycle or phase is used, is called the 'monocyclic' system. By this system it is sought to obtain the advantages of a polyphase system for motor purposes, combined with the ease of regulation of a single-phase system. Under normal conditions the whole of the power is supplied by two mains, between which is maintained a constant alternating pressure. To those parts of the district where motive power is required, a third wire is carried, from which a current can be drawn, differing in phase from the main current, and by which motors can be started. The windings of the motor are arranged so that when full speed is attained the back E.M.F. is as great as the pressure from the third wire, so that no more current is drawn from it, the power being supplied by the principal mains. One of the ways of maintaining the difference of phase in the third wire is to wind on the alternator coils displaced in relation to the main coils, so as to generate an E.M.F. differing by 90° from principal E.M.F. One of the terminals of these displaced coils is joined to a middle point of the principal winding of the alternator, and the other to the third wire in question. The number of turns of the displaced coils is so arranged that the E.M.F. resulting from them and the half of the principal coils in series with them shall have the required difference in phase. Where a number of motors are installed, the back E.M.F. of any motor that happens to be running will be sufficient to start another motor, so that in cases where some of the motors are always running it will not be necessary to carry the third wire to the alternator."

It is of particular interest to note that these motors, in supplying the necessary out-of-phase starting emf to other motors, act as phase converters. It was his study of phase converters that led Fortescue to develop the Method of Symmetrical Components.

We can better appreciate the position that the Method of Symmetrical Components occupies if we first review the gradual development of the concepts upon which the method is founded. In 1825 Fresnel showed that a plane polarized ray of light is equivalent to two opposite circularly polarized rays. Mathe-

[1] See *Electrical World*, **25**, 182, February, 1895; also E. Boistel, "Distribution Monocyclique," *L'Eclairage Electrique*, **3**, 152, April, 1895.

matically this principle is similar to the one that Ferraris was to use 68 years later in analyzing the single-phase induction motor. (It should be understood that a single-phase induction motor has a single stator winding but a symmetrical polyphase rotor winding which is frequently of the cage type.) With both Fresnel and Ferraris the analysis depends upon the well-known trigonometrical relation that

$$B \sin \omega t \sin x = 0.5B \cos(x - \omega t) - 0.5B \cos(x + \omega t).$$

That is, it can be said that a *stationary* sinusoidally distributed field [in space ($x$)], the magnitude of which varies harmonically [in time ($t$)], is equivalent to two equal and constant sinusoidal fields that move at the same angular speed, $\omega$, but in opposite directions. Thus, Ferraris considered that the *stationary*, harmonically varying magnetic field produced in the air gap of the induction motor by the alternating current in the single-phase stator winding is equivalent to two *equal* and *constant* magnetic fields which *rotate* in *opposite* directions at the same electrical angular velocity $\omega$. This angular velocity equals the angular velocity of the alternating current in the stator winding. (The electrical angular velocity of an alternating current is understood to mean $2\pi$ times the frequency of the current. That is, it is the angular velocity of the rotating vector which is used to represent the alternating current.) The magnitude of each of these component fields is one-half the *maximum* value of the actual field. The *space* variation of the flux density in the air gap around the periphery is assumed to be sinusoidal. This treatment of the single-phase motor has been commonly referred to as the Method of Ferraris. If carried to its logical conclusion this method correctly analyzes the electromagnetic reactions within the motor and is similar to the Method of Symmetrical Components, which also resolves the *pulsating* magnetic field due to the stator current into two oppositely *rotating* fields. Thus it is true that Ferraris applied in a simple and restricted manner a principle which at a later time Fortescue was to develop so brilliantly and generalize. Unfortunately, however, Ferraris expressed the results of his analysis in such a form that they have been frequently misinterpreted and erroneous conclusions have been drawn from them. For this reason it is of considerable interest to review his analysis briefly and note its true signifi-

cance. First, it should be observed that there are two methods by which the current induced in a short-circuited winding can be computed. If there is a current $I_b$ induced in a short-circuited winding $b$ by the action of a current $I_a$ in a neighboring winding $a$, we can write

$$j\omega M_{ab}I_a + (r_b + j\omega L_b)I_b = 0 \tag{I}$$

where $M_{ab}$ is the mutual inductance between the two windings, $r_b$ and $L_b$ are the resistance and self- (not leakage) inductance of the $b$ winding, and $\omega$ is the angular velocity of the inducing current. That is

$$I_b = -\frac{j\omega M_{ab}I_a}{r_b + j\omega L_b} \tag{A}$$

By the simple device of adding and subtracting $j\omega M_{ab}I_b$ to the preceding equation (I), we have

$$j\omega M_{ab}(I_a + I_b) + [r_b + j\omega(L_b - M)]I_b = 0 \tag{II}$$

Whereas $j\omega M_{ab}I_a$ is the emf induced in the $b$ winding by the magnetic field that is produced solely by the current in the $a$ winding, $j\omega M_{ab}(I_a + I_b)$ is the emf induced in the $b$ winding by the resultant magnetic field that is produced by the joint action of the currents in the $a$ and $b$ windings. In equation (II) $(L_b - M)$ is interpreted as the leakage inductance of the $b$ winding, provided the $a$ and $b$ windings have the same number of effective turns. The value of the short-circuit current is now

$$I_b = -\frac{j\omega M_{ab}(I_a + I_b)}{r_b + j\omega(L_b - M_{ab})} \tag{B}$$

Although expressed in quite different form, these two values of the short-circuit current are identical. Ferraris used the first method ($A$) to determine the current in the short-circuited rotor winding, for at that time the conception of *leakage* inductance had not been clearly established.

In his analysis of the single-phase motor Ferraris assumed that there was impressed upon the short-circuited polyphase rotor winding a pulsating magnetic field which, by the principle already referred to, is equivalent to two equal and oppositely rotating fields. If the slip of the rotor with respect to one of

these fields is $s$, the slip with respect to the other field is $2 - s$. In this case there are two components in the short-circuit rotor current, one due to each rotating field. Thus

$$I_{b1} = -\frac{js\omega M_{ab}I_a}{r_b + js\omega L_b}$$

$$I_{b2} = \frac{-j(2 - s)\omega M_{ab}I_a}{r_b + j(2 - s)\omega L_b}$$

In these equations $M_{ab}$ is the mutual inductance between the stator winding and one phase of the rotor winding and thus $I_{b1}$ and $I_{b2}$ are the component currents in each phase of the rotor. Since the two *rotating* fields are each *one-half* the maximum value of the actual *stator* field, $I_a$ is one-half the actual stator current. In each case the rotor flux linkage with the inducing magnetic field is $M_{ab}I_a$ but the relative angular velocity of the rotor with respect to the field moving in the same direction is $s\omega$, while the relative angular velocity with respect to the oppositely rotating field is $(2 - s)\omega$. It will be noted that no distinction is made between the rotor resistances $r_b$ in these two expressions although the frequencies of the component rotor currents are very different when the slip $s$ is small. Since the average torque is proportional to the product of the current, the flux, and the sine of their relative phase displacement, the expressions for the average torque produced by these oppositely rotating fields are

$$T_1 = k\frac{s\omega r_b M_{ab}^2 I_a^2}{r_b^2 + s^2\omega^2 L_b^2}$$

$$T_2 = k\frac{(2 - s)\omega r_b M_{ab}^2 I_a^2}{r_b^2 + (2 - s)^2\omega^2 L_b^2}$$

where $k$ is a constant depending upon the number of rotor phases, the number of poles, the frequency, and the unit in which the torque is expressed. The net average torque is $T_1 - T_2$.

Except for the different nomenclature, these are the expressions Ferraris gave for the torque developed by a single-phase motor.[1] Note that the ratio of the component average torques is independent of the stator current and that the torques are equal at standstill, or when $s = 1$.

[1] See *Electrician*, **33**, 110, 129, 152, 184, 1894.

$$\frac{T_2}{T_1} = \frac{(2 - s)(r_b^2 + s^2\omega^2 L_b^2)}{s[r_b^2 + (2 - s)^2\omega^2 L_b^2]}$$

It is important to note that these expressions give the average torque. For any given value of the slip, the torque developed in a symmetrical *polyphase* motor when operating with balanced terminal potentials is constant throughout the cycle of the alternating current. This is not true, however of a single-phase motor. The torque expressions that have been given result from the reaction between the rotor current that is induced by one of the magnetic fields with *that field*. There is also a torque reaction between the rotor current induced by one field with the *oppositely rotating field*. Each of these latter component torques varies sinusoidally at double the applied frequency. Their average value is zero but they tend to make the motor vibrate and thus may be objectionable.[1]

It has been suggested that the single-phase motor is the equivalent of two polyphase motors, the stator windings of which are connected in series in such a way as to produce oppositely rotating magnetic fields and the rotors of which are mechanically coupled. The average torque developed by this combination corresponds to that developed by the single-phase motor, but the torque of the combination is constant throughout the cycle since there can be no reaction between the current in one rotor winding with the magnetic field in the other motor. In representing the net average torque graphically, Ferraris apparently assumed that the maximum value of the impressed pulsating field did not change. This is equivalent to assuming that the stator current $I_a$ is constant. Thus the analysis was incomplete since it is the *potential* applied to the stator that is nominally constant and not the stator current.

It is interesting to write the expressions for the two component average torques when the rotor currents are determined by the second method (*B*). They are

$$T_1 = k\frac{s\omega r_b M_{ab}^2(I_a + I_{b1})^2}{r_b^2 + s^2\omega^2(L_b - M_{ab})^2}$$

$$T_2 = k\frac{(2 - s)\omega r_b M_{ab}^2(I_a + I_{b2})^2}{r_b^2 + (2 - s)^2\omega^2(L_b - M_{ab})^2}$$

[1] See pp. 292 and 305.

In these expressions the terms $M_{ab}(I_a + I_{b1})$ and $M_{ab}(I_a + I_{b2})$ are the rotor flux linkages with the *resultant* magnetic fields due to both the stator and rotor currents. These fields rotate in opposite directions around the air gap of the motor. Expressions of this type are now more commonly used in computing the torque than are those of the type used by Ferraris. The two rotating-field components of the resultant air-gap flux are equal only when the rotor is not turning. When the motor is running at its normal speed near synchronism, the reaction produced by the polyphase currents induced in the rotor windings reduces the relative magnitude of the component field which rotates *oppositely* to the rotor. Thus this field becomes very small as compared with the field that rotates in the *same* direction as does the rotor. This reduction is so great that the negative-sequence field, as we shall call it is but approximately 3 or 4 per cent of the greater or positive-sequence field when the motor is developing no electromagnetic torque ($s = 0$). The relative value of the negative-sequence field increases to approximately 10 per cent when the motor develops its rated torque. That is, under normal operating conditions the ratio $\dfrac{I_a + I_{b2}}{I_a + I_{b1}}$ is not more than 0.10, approximately

In 1897 Behrend[1] applied Fresnel's theorem to the single-phase induction motor and carried the analysis to a successful conclusion. Behrend recognized the reactions of the rotor currents in causing the inequality in the oppositely rotating components of the air-gap flux, and showed how to determine the current and torque for a given applied stator potential. It is interesting to note that Behrend's result was essentially the same as would be obtained now by using the Method of Symmetrical Components. Many authors have employed this method of oppositely rotating fields when analyzing the performance of the single-phase motor. Some have made the error of assuming that the air-gap flux consisted of two equal, constant, but oppositely rotating fields, while others have noted this error but have failed to point out how it could be corrected.

[1] BEHREND, B. A, Asynchronous Alternating-current Motors, *ETZ*, Mar 25, 1897

The next important development in the theory of unbalanced circuits was due to Stokvis.[1] He showed that, when a 3-phase alternating-current generator supplies an unbalanced load, the armature magnetomotive force consists of two components, one of which rotates synchronously with the field poles as if the load were balanced and the other rotates at the *same* speed but in the *opposite* direction. The latter induces a double-frequency current in the field circuit. If the pulsating magnetic field which is produced by this double-frequency current is analyzed by the Ferraris Method, it is seen to consist of two component fields which rotate in opposite directions and at twice synchronous speed with respect to the field poles. Thus one of these component fields rotates at synchronous speed in the negative direction, *with respect to the armature*, and the other at three times synchronous speed in the positive direction. The latter generates triple frequency emfs in the armature windings. These emfs are displaced 120 degrees in time phase. They correspond to the emfs that would be generated in the armature windings on open circuit if the rotor speed were three times its normal value. These emfs should not be confused with the *third-harmonic* emfs which are sometimes present in the armature windings and are due to a third-harmonic component in the space distribution of the air-gap flux density. The third-harmonic emfs may be present under *balanced* conditions and are then in time-phase in the different windings of a 3-phase generator.

Stokvis also showed how the unbalanced armature currents can be resolved into two components so that one component will produce the magnetomotive force that rotates synchronously with the field poles, and the other will produce the magnetomotive force that rotates at synchronous speed but in the opposite or negative direction. It had been known from the beginning of the development of the polyphase induction motor that balanced currents would produce a rotating magnetic field of constant value and that the direction of rotation could be changed by

[1] Production of Triple-frequency Emfs in 3-phase Alternators by Unbalanced Loads, *Compt rend.*, **159**, 46–49, 1914.

*Rev. gen. elec.*, **22**, 261–262, 1914.

Analysis of Unbalanced 3-phase Systems, *Elec. World*, **65**, 1111–1115, 1915.

changing the phase order of the currents in the windings. Therefore Stokvis proposed to resolve the unbalanced armature currents into two sets of balanced currents, the phase orders of which were opposite. He first assumed that the neutral of the generator was free so that the sum of the armature currents was zero. Thus if the armature currents are

$$
\begin{aligned}
i_a &= I_a \sin(\omega t + \alpha) \\
i_b &= I_b \sin(\omega t + \beta) \\
i_c &= I_c \sin(\omega t + \gamma)
\end{aligned}
$$

we can write

$$
\begin{aligned}
i_a &= I_1 \sin(\omega t + \alpha_1) + I_2 \sin(\omega t + \alpha_2) \\
i_b &= I_1 \sin(\omega t + \alpha_1 - 120°) + I_2 \sin(\omega t + \alpha_2 + 120°) \\
i_c &= I_1 \sin(\omega t + \alpha_1 - 240°) + I_2 \sin(\omega t + \alpha_2 + 240°)
\end{aligned}
$$

Since the sum of the instantaneous currents in a balanced system of either phase order is zero, the condition that $i_a + i_b + i_c = 0$ is satisfied. Each of the first set of balanced currents has a maximum value $I_1$ and the current in the $b$ phase *lags* the current in the $a$ phase by 120 degrees. These are now known as the positive-sequence currents. Each of the second set of balanced currents, which are known as the negative-sequence currents, has a maximum value $I_2$ and the current in the $b$ phase *leads* the current in the $a$ phase by 120 degrees. That is, the phase orders of these two sets of balanced currents are opposite.

The following procedure can be used to determine the positive- and negative-sequence component systems of currents[1] when the expressions for the actual phase-currents are known. It is evident that the equation for $i_a$ is still true if a time angle $(\omega t + 120°)$ is substituted for $\omega t$ in *each* term. Thus

$$
I_a \sin(\omega t + \alpha + 120°) = I_1 \sin(\omega t + \alpha_1 + 120°) + I_2 \sin(\omega t + \alpha_2 + 120°)
$$

By combining this equation with that for $i_b$, the negative-sequence component can be eliminated. That is

$$
I_a \sin(\omega t + \alpha + 120°) - I_b \sin(\omega t + \beta) = I_1[\sin(\omega t + \alpha_1 + 120°) - \sin(\omega t + \alpha_1 - 120°)]
$$

from which

$$
\sqrt{3} I_1 \sin(\omega t + \alpha_1 + 90°) = I_a \sin(\omega t + \alpha + 120°) - I_b \sin(\omega t + \beta)
$$

[1] Stokvis used a different procedure.

On dividing each term by $\sqrt{3}$ and retarding each term by 90 degrees there results the positive-sequence component of $i_a$, or

$$I_1 \sin(\omega t + \alpha_1) = \frac{1}{\sqrt{3}}[I_a \sin(\omega t + \alpha + 30°) - I_b \sin(\omega t + \beta - 90°)]$$

By a similar process it can be shown that the negative-sequence component of $i_a$ is

$$I_2 \sin(\omega t + \alpha_2) = \frac{1}{\sqrt{3}}[I_a \sin(\omega t + \alpha - 30°) - I_b \sin(\omega t + \beta + 90°)]$$

These are the expressions that Stokvis gave for the positive- and negative-sequence components. There is another method of determining the sequence components, due to Fortescue, in which the resulting expressions appear more symmetrical in form. It can be used even when the sum of the currents is not zero. If in the expression for $i_b$ each term is advanced 120 degrees and if in the expression for $i_c$ each term is advanced 240 degrees and their sum is added to the expression for $i_a$, it will be observed that the sum of the negative-sequence components is zero and that the sum of the positive-sequence components is $3I_1 \sin(\omega t + \alpha_1)$. Thus it follows that the positive-sequence component of $i_a$ is

$$I_1 \sin(\omega t + \alpha_1) = \tfrac{1}{3}[I_a \sin(\omega t + \alpha) + I_b \sin(\omega t + \beta + 120°) + I_c \sin(\omega t + \gamma + 240°)]$$

If $i_b$ and $i_c$ are *retarded* 120 and 240 degrees, respectively, a similar expression can be derived for the negative-sequence component of $i_a$, in which there are negative signs before the phase angles of 120 and 240 degrees.

Thus Stokvis showed how any unbalanced 3-phase system of armature currents, the sum of which is zero, could be resolved into two balanced systems of currents of opposite phase order. Of these two systems the positive-sequence system of currents produces an armature magnetomotive force (armature reaction) which rotates synchronously with the field poles and must therefore be counteracted by a component in the direct current applied to the field winding if the terminal potential of the alternator is to be maintained constant. On the other hand, the negative-sequence system of currents produces an armature magnetomotive force which rotates at synchronous speed but in

the *opposite* direction to the rotation of the field poles and thus produces a current of double frequency in the field winding. If, as Stokvis assumed, the field winding is of the usual (single-phase) type, this double-frequency current produces two magnetic fields, one of which moves forward and the other backward with respect to the field poles at double synchronous speed. The magnetic field which moves forward induces a triple-frequency current in the armature winding while the backward rotating field induces a current of fundamental frequency in the armature winding. Since it has been assumed that the load circuits connected to the alternator are unbalanced, both of these induced currents will also be unbalanced and will thus be the cause of additional armature magnetomotive forces which rotate in *opposite* directions about the air gap. The negatively rotating magnetomotive force, due to the triple-frequency armature current, will induce a quadruple-frequency current in the field winding. It is now evident that there will be continued reflections between the field and armature windings with the result that there will be currents having frequencies of 0, $2f$, $4f$, etc., in the field winding and currents having frequencies of $f$, $3f$, $5f$, etc., in the armature winding. The amplitudes of these currents will ordinarily decrease as the frequency increases. Thus it is apparent that the exact solution for the problem, as it was stated by Stokvis, is practically impossible by employing the method of resolving the armature currents into positive- and negative-sequence components following his suggestion. In order to obtain a solution by this method it becomes necessary to make assumptions that limit the number of different frequencies in the field or the armature winding to not more than two in each. If this is done a solution can be obtained.

When the sum of the armature currents is not zero, as may be the case if the neutral of the generator is grounded, Stokvis suggested that it be assumed that the neutral current is supplied by the third phase and that the remainder of the current in this phase be combined with the currents in the first and second phases in the manner already described to resolve them into positive- and negative-sequence components. The pulsating magnetomotive force due to the neutral component of the current in the third phase could then be resolved into oppositely rotating magnetomotive forces according to the method of

Ferraris. This was in substance what Stokvis proposed. It is a workable method and will yield correct results wherever the Fortescue Method of Symmetrical Components is applicable. The Stokvis Method is not so convenient to use and lacks the convenient symmetry that the Method of Symmetrical Components possesses. Although the method proposed by Stokvis was a distinct advance in the theory of the unbalanced operation of a polyphase rotating machine, there was no indication that it had a wider field of application beyond the particular problem of the unsymmetrically loaded generator.

It remained for Fortescue to show how the principles upon which the method was founded could be extended so as to broaden the scope of the method and endow it with a vivifying generality.

In the opening paragraphs of his now classic paper on symmetrical components,[1] Fortescue, says:

"In the latter part of 1913 the writer had occasion to investigate mathematically the operation of induction motors under unbalanced conditions. The work was first carried out, having particularly in mind the determination of the operating characteristics of phase converters which may be considered as a particular case of unbalanced motor operation, but the scope of the subject broadened out very quickly and the writer undertook this paper in the belief that the subject would be of interest to many.

"The most striking thing about the results obtained was their symmetry; the solution always reduced to the sum of two or more symmetrical solutions. The writer was then led to inquire if there were no general principles by which the solution of unbalanced polyphase systems could be reduced to the solution of two or more balanced cases. The present paper is an endeavor to present a general method of solving polyphase networks which has peculiar advantages when applied to the type of polyphase networks which include rotating machines."

Thus the entire history of the development of the Method of Symmetrical Components has been connected with polyphase rotating machines and the need for a method of determining their characteristics when operating under unbalanced conditions. Furthermore it was particularly fortunate, as we shall presently

[1] Fortescue, C. L., Method of Symmetrical Co-ordinate Applied to the Solution of Polyphase Networks, *Am. Inst. Elec. Eng.*, **37**, 1027, 1918.

see, that the machines, the characteristics of which Fortescue was analyzing, were equipped with symmetrical polyphase windings. A symmetrical 2-phase winding consists of two like groups of coils spaced 90 degrees apart electrically. A symmetrical 3-phase winding consists of three like groups of coils displaced 120 degrees electrically one from the next. A symmetrical cage winding may have as many phases as there are rotor bars. We shall limit the following discussion to the consideration of the 3-phase case only and, following Fortescue, shall disregard all space and time harmonics in the quantities that are involved.

When balanced 3-phase currents flow in a symmetrical 3-phase winding, there is produced in a uniform air gap a magnetic field of constant magnitude which rotates at constant speed. The direction of rotation is determined by the phase order or sequence of the currents. It can be assumed that positive-sequence currents, for which the current in the *a* phase *leads* the current in the next or *b* phase by 120 degrees, produce a magnetic field which rotates in the positive direction. Negative-sequence currents have the opposite phase order, that is, the current in the *a* phase *lags* the current in the *b* phase by 120 degrees, and thus negative-sequence currents produce a magnetic field which rotates in the opposite or negative direction. The sum of the three positive-sequence currents is zero, as is also the sum of the three negative-sequence currents. Fortescue proposed to divide the three armature currents, in case their sum was zero (neutral free) into positive- and negative-sequence systems of currents. This was the same procedure that Stokvis had used.

When their sum was not zero, however, Fortescue proposed to divide the sum of the currents, that is to say, the neutral current, *equally* among the phases, instead of assuming, as Stokvis had done, that it was confined to one phase alone. That is, this so-called zero-sequence component of the armature currents is the same in magnitude and in time-phase in each of the phases of the armature and thus equals one-third of the neutral current. It takes its name from the fact that the time-phase angle between the components in the different phases is zero. If, as before, the phase currents are $I_a \sin \omega t$, $I_b \sin(\omega t + \beta)$, and $I_c \sin(\omega t + \gamma)$, the zero-sequence component of these currents is

$$\tfrac{1}{3}[I_a \sin \omega t + I_b \sin(\omega t + \beta) + I_c \sin(\omega t + \gamma)].$$

The resultant field in the air gap due to these zero-sequence currents is the sum of three component fields, one for each phase, which are equal in magnitude, are in *time-phase*, and are equally displaced in *space-phase*, owing to the arrangement of the armature windings. Therefore the resultant air-gap field due to the zero-sequence currents is zero and for this reason alone they would be aptly named.

Since the zero-sequence currents produce no (fundamental) field in the air gap, they induce no current in the rotor windings[1] and are thus the cause of no reaction between the armature (stator) and the field (rotor). If the zero-sequence component is subtracted from each armature current, it is evident that the sum of the three remainders is zero. Thus the remaining currents can be divided into positive- and negative-sequence components. That is, Fortescue proposed to divide each of the three armature currents into three components. The first set of components formed the zero-sequence system of currents. They produce zero air-gap flux. The second set of components formed the positive-sequence system of currents which produce a positively rotating field in the air gap. The third set of components formed the negative-sequence system of currents which produce a negatively rotating field in the air gap. The tremendous advantage possessed by this method in resolving the reactions of the unbalanced armature currents must now be evident since one component system of currents does not react upon the rotor and the two other component systems do react in the well-known manner of balanced currents. The reactions due to the positive- and negative-sequence systems of currents are in general not alike since the magnitudes of the two systems of currents are different and the magnetic fields they produce have different speeds with respect to the rotor windings. It is important to note that the space-phase relation of symmetrical armature windings is the same as the time-phase relation of balanced armature currents of either positive- or negative-sequence phase order.

Let us now consider the emfs that are produced by the magnetic fields that are set up by these three different systems of currents. Although the zero-sequence system of currents produces no air-gap flux, these currents do produce leakage fluxes

[1] For a more complete analysis, see Arts. 113, 127, 135, 139, 148, 149.

that link the armature (stator) windings alone. Since these leakage-flux linkages with the different armature phases are equal and in time-phase, the emfs that they generate in the windings are also *equal* and in *time-phase.* Thus these emfs can be considered to form a zero-sequence system in the same sense that the equal currents to which they are due form a zero-sequence system. The positive-sequence system of currents produces in the air gap a magnetic field of fixed magnitude which rotates in the positive direction at synchronous speed. This rotating field generates emfs in each phase which have equal magnitudes. Owing to the arrangement of the armature windings, these emfs differ in time-phase by 120 degrees. The phase order or sequence of these emfs is the same as that of the positive-sequence currents which produce them. That is, these emfs form a positive-sequence system. Similarly, a negative-sequence system of currents produces a magnetic field in the air gap which rotates in the negative direction and thus generates in the armature windings emfs which form a negative-sequence system.

Therefore Fortescue proposed to resolve *both* the currents and the emfs in a 3-phase system into zero-, positive-, and negative-sequence components. The method of resolution was exactly the same for emfs and for currents. He proposed to generalize still further the method of resolution and apply it to the circuit constants as well. The same formula was used to resolve the current vectors, the potential vectors, and the impedance vectors into their symmetrical components. Fortescue also showed how it is possible to resolve each one of any number of related quantities, for example, the impedances of an unbalanced polyphase load, into components such that the corresponding components in the several quantities form a symmetrical polyphase group. Indeed the problem of unbalanced circuits can be attacked in this purely mathematical manner without regard to its physical aspects.

The Method of Symmetrical Components has gathered power with use until it has become an indispensable tool for the analysis of unsymmetrical circuits. It is useful in the analysis of transient conditions that are due to the sudden alteration of the electric circuit of a rotating machine. It has been used successfully to resolve the unequal ratios of transformation of an unsymmetrical 3-phase group of transformers into zero-, posi-

tive-, and negative-sequence components. It has been extended in a modified form and applied to the analysis of an induction motor having *unsymmetrical* stator windings.

Although the Method of Symmetrical Components was devised for the specific purpose of analyzing rotating machines, it is probably more widely used in determining the conditions set up by unsymmetrical faults on power networks. An extensive technique has been developed for solving problems relating to circuit-breaker capacity, to the setting of relays, and to the stability of the system when it is disturbed by unsymmetrical faults. The behavior of a power network, however, depends to a very considerable degree on the rotating machines connected to it and, in general, a solution that ignored them would not be considered satisfactory. This is true whether the operating conditions are symmetrical or unsymmetrical. When the network is unbalanced by faults or by combinations of single-phase loads, the Method of Symmetrical Components offers the best systematic method of analysis that has yet been developed.[1] However, if the analysis concerns a portion of the network which contains no polyphase rotating machines and for which it is sufficiently accurate to assume that the values of the applied potentials are independent of the current supplied to this particular part of the network, it is doubtful if the Method of Symmetrical Components offers a better means of obtaining a solution than the methods that have preceded it. That is, the Method of Symmetrical Components should not be used indiscriminately but should be used only with a proper understanding of its scope and its limitations and the relation it bears to the other methods of applying the fundamental laws of electrical science to the solution of electric circuits.

[1] The systematic methods of dyadic algebra have been successfully applied to the solution of unsymmetrical 3-phase circuits. See A. Pen-Tung Sah, Representation of Polyphase Systems by Multidimensional Vectors, *World Eng. Congress Proc.*, **22**, 3, 1929. A.P.-T. Sah and C. Yen, Impedance Dyadics of Synchronous Machines, *Science Reports*, Tsing Hua Univ. Peiping, China, **3**, 127, 1935. A.P.-T. Sah, Dyadic Algebra Applied to 3-phase Circuits, *A.I.E.E.*, March, 1936, Complex Vectors in 3-phase Circuits, *A.I.E.E.* Dec., 1936.

## CHAPTER I

## GENERAL PRINCIPLES

**1. Resolution into Symmetrical Components.**—Much of the progress in applying the fundamental laws to electric circuits has been due to the development of mathematical tools by which the behavior of circuits can be determined accurately without undue effort. The most widely used of these tools is the Vector Method for solving alternating-current circuits. Since the representation of vectors by complex numbers is an almost indispensable part of this tool, one should be familiar with the fundamental operations of addition and multiplication as applied to complex numbers. Likewise, if one is to master the Method of Symmetrical Components, it is best to understand how, in general, $n$ quantities can be resolved into their symmetrical components and how the operations of addition and multiplication are performed when these quantities are expressed in terms of their symmetrical components. To this end, let us consider $n$ quantities of the same kind, having values $V_a$, $V_b$, $V_c$, . . . $V_h$, . . . $V_k$, where $V_k$ is the last or the $n$th of the quantities. These values may be real, imaginary, complex, constant, or variable. They may represent either fixed or rotating vectors. In our process of resolving these quantities into their symmetrical components we shall not inquire into what the quantities represent. That is, we shall deal with harmonic or even nonharmonic potentials or currents in the same way in which we deal with resistances, inductances, capacitances, or complex impedances. Thus this method of resolution has been successfully applied to the transient currents that are set up by the sudden alteration of the electric circuits of synchronous and induction machines. It is always possible to resolve each of these $n$ quantities into $n$ related components by solving the following simultaneous linear equations:

$$\left.\begin{aligned}
V_a &= V_0 + V_1 + V_2 \;.\;.\;. + V_{n-1} \\
V_b &= b_0V_0 + b_1V_1 + b_2V_2 \;.\;.\;. + b_{n-1}V_{n-1} \\
V_c &= c_0V_0 + c_1V_1 + c_2V_2 \;.\;.\;. + c_{n-1}V_{n-1} \\
&\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \\
V_k &= k_0V_0 + k_1V_1 + k_2V_2 \;.\;.\;. + k_{n-1}V_{n-1}
\end{aligned}\right\} \qquad (1)$$

$V_0$, $V_1$, $V_2$, etc. are the components of the quantity $V_a$, while $c_0V_0$, $c_1V_1$, $c_2V_2$, etc., for example, are the related or corresponding components of the quantity $V_c$. The coefficients $b_0$, $b_1$, etc., may be arbitrarily chosen provided they satisfy Cramer's Rule. By this rule the necessary and sufficient condition that it shall be possible to determine unique values for $V_0, V_1, V_2, \ldots V_{n-1}$, is that the determinant of the coefficients shall *not* vanish.[1] That is:

$$\begin{vmatrix} 1, & 1, & \ldots & 1 \\ b_0, & b_1, & \ldots & b_{n-1} \\ c_0, & c_1 & \ldots & c_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ k_0, & k_1, & \ldots & k_{n-1} \end{vmatrix} \neq 0$$

For any particular case, the only way to determine the values of these coefficients, which will ultimately produce the greatest simplification in the analysis, is by a thorough study of the problem in hand. By such a study Fortescue found that the solution of the problem in which he was interested consisted naturally of two component parts, each of which was the solution of a symmetrical 3-phase circuit. This finally led him to the idea of resolving the quantities of a general $n$-phase system into component systems, each of which was symmetrical. Thus in the following analysis the arbitrary coefficients will be assigned values so that the corresponding components of the original quantities will form symmetrical polyphase systems. In the sense here employed, a symmetrical polyphase system consists in general of $n$ vectors of equal magnitude, the angle between successive vectors being $h\frac{2\pi}{n}$, where $h$ is any integer from 0 to $(n - 1)$, inclusive. It should be noted that the sum of all the vectors in a symmetrical polyphase system is zero, except when the angle between successive vectors is zero, that is to say, when $h = 0$. We shall call the angle $2\pi/n$ the characteristic angle of the system. For example, the characteristic angle of a 3-phase system is 120 degrees, and that of a 4-phase system is 90 degrees. As we shall see, a 2-phase system, being a special case of a 4-phase system, also has the characteristic angle of

[1] Bôcher, "Introduction to Higher Algebra," Art. 16.

90 degrees. Fortescue's choice of the resolution into component systems which are symmetrical was most fortunate, as it has since been shown that this resolution has a much wider field of application than he indicated in his original paper.

It should not be inferred from this, however, that the Fortescue formula for resolving quantities into symmetrical components will always give the most direct solution to the problem in hand. It is the spirit rather than the letter of the resolution which should be followed. This is indicated in the two illustrative problems given in Chap. VIII concerning "split-phase" induction motors.

The zero-order system is the one for which $h = 0$; the first-order system is the one for which $h = 1$; the second-order system is the one for which $h = 2$, etc. By common agreement the zero-order system is called the *zero-sequence system;*[1] the first-order system is called the *positive-sequence system;* the $(n - 1)$-order system is called the *negative-sequence system.*

The values of the arbitrary coefficients that will produce the required resolution are

$$
\begin{array}{lll}
b_0 = 1 & c_0 = 1 & k_0 = 1 \\
b_1 = \epsilon^{-j\frac{2\pi}{n}} & c_1 = \epsilon^{-j(2)\frac{2\pi}{n}} & k_1 = \epsilon^{-j(n-1)\frac{2\pi}{n}} \\
b_2 = \epsilon^{-j\frac{4n}{\pi}} & c_2 = \epsilon^{-j(2)\frac{4\pi}{n}} & k_2 = \epsilon^{-j(n-1)\frac{4\pi}{n}} \\
b_3 = \epsilon^{-j\frac{6\pi}{n}} & c_3 = \epsilon^{-j(2)\frac{6\pi}{n}} & k_3 = \epsilon^{-j(n-1)\frac{6\pi}{n}}
\end{array}
$$

Thus the coefficients are rotating operators which, applied to any vector, turn it clockwise, or negatively, through some multiple of the characteristic angle. In order to simplify the notation as much as possible, represent the operator, $\epsilon^{-j\frac{2\pi}{n}}$, by $\alpha^{-1}$. Thus

$$
\left.
\begin{aligned}
V_a &= V_0 + V_1 + V_2 \ldots + V_{n-1} \\
V_b &= V_0 + \alpha^{-1}V_1 + \alpha^{-2}V_2 \ldots + \alpha^{-(n-1)}V_{n-1} \\
V_c &= V_0 + \alpha^{-2}V_1 + \alpha^{-4}V_2 \ldots + \alpha^{-2(n-1)}V_{n-1} \\
V_d &= V_0 + \alpha^{-3}V_1 + \alpha^{-6}V_2 \ldots + \alpha^{-3(n-1)}V_{n-1} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
V_k &= V_0 + \alpha^{-(n-1)}V_1 + \alpha^{-2(n-1)}V_2 \ldots + \alpha^{-(n-1)(n-1)}V_{n-1}
\end{aligned}
\right\} \quad (2)
$$

In equations (2), and subsequently, letter subscripts will be used to denote the branch or circuit to which the quantity

[1] It has also been called the *uni-phase system.*

belongs, and numbers from 0 to $(n - 1)$ will be used to denote the order of the system. It is very important to realize at the outset that, whatever the nature of the quantities $V_a$, $V_b$, etc., whether their values are real or complex, their symmetrical components are in general complex and are represented by vectors.

Notice that in the zero-order system the components $V_0$ are equal in each of the original quantities, $V_a$, $V_b$, etc.; that in the first-order system the component $\alpha^{-1}V_1$ in $V_b$ *lags* the corresponding component $V_1$ in $V_a$ by the characteristic angle; that in the second-order system the component $\alpha^{-2}V_2$ in $V_b$ *lags* the corresponding component $V_2$ in $V_a$ by twice the characteristic angle; that in the $(n - 1)$-order system the component $\alpha^{-(n-1)}V_{n-1}$ in $V_b$ lags the corresponding component $V_{n-1}$ in $V_a$ by $(n - 1)$ times the characteristic angle. Since lagging by $(n - 1)$ times the characteristic angle is the same as leading by the characteristic angle, the angle between successive components in the $(n - 1)$-order system is the same as the angle between successive components in the first-order system, but the phase orders in the two systems are opposite. It is for this reason that one is called the positive-sequence system and the other the negative-sequence system. Equations (2) are sometimes written in what might be called an angular notation by substituting for $\epsilon^{-j\frac{2\pi}{n}}$ the angle sign $\underline{/-2\pi/n}$. Thus, if we represent the characteristic angle $2\pi/n$ of the system by $\beta$, equations (2) would be written:

$$\left.\begin{aligned}
V_a &= V_0 + V_1 + V_2 \; . \; . \; . + V_{n-1} \\
V_b &= V_0 + V_1\underline{/-\beta} + V_2\underline{/-2\beta} \; . \; . \; . + V_{n-1}\underline{/-(n-1)\beta} \\
V_c &= V_0 + V_1\underline{/-2\beta} + V_2\underline{/-4\beta} \; . \; . \; . + V_{n-1}\underline{/-(n-1)2\beta} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
V_k &= V_0 + V_1\underline{/-(n-1)\beta} + V_2\underline{/-(n-1)2\beta} \; . \; . \; . \\
&\qquad + V_{n-1}\underline{/-(n-1)(n-1)\beta}
\end{aligned}\right\} \quad (2a)$$

With a 3-phase system these equations become

$$\left.\begin{aligned}
V_a &= V_0 + V_1 + V_2 \\
V_b &= V_0 + V_1\underline{/-120^\circ} + V_2\underline{/-240^\circ} \\
V_c &= V_0 + V_1\underline{/-240^\circ} + V_2\underline{/-480^\circ}
\end{aligned}\right\} \quad (2b)$$

The phase order of balanced polyphase systems is usually chosen so that the potentials and currents in the second phase lag the potentials and currents in the first phase by the characteristic angle of the system. In his original paper Fortescue adopted this convention for the first-order or positive-sequence system. Consequently we have followed this precedent and have used the same convention.

It is sometimes desirable to indicate by subscripts not only the order of the system but also the branch to which the component belongs. Thus equations (2) may be written:

$$\left.\begin{aligned} V_a &= V_0 + V_{a1} + V_{a2} \ldots + V_{a(n-1)} \\ V_b &= V_0 + V_{b1} + V_{b2} \ldots + V_{b(n-1)} \\ &\ldots\ldots\ldots\ldots \\ V_l &= V_0 + V_{k1} + V_{l2} \ldots + V_{k(n-1)} \end{aligned}\right\} \quad (3)$$

Since the zero-sequence components are the same in all the branches, there is no need of indicating by letters to which

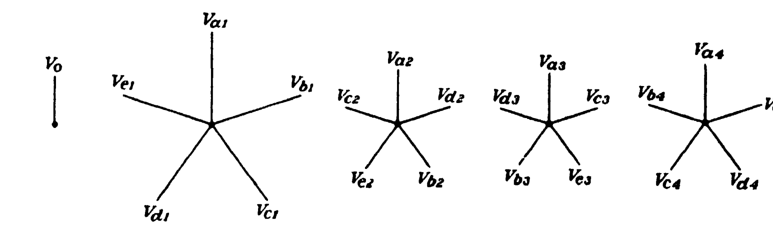



FIG. 1.

branch they belong. $V_{a1}$, $V_{b1}$, . . . $V_{k1}$ form the first-order or positive-sequence system; $V_{a2}$, $V_{b2}$, . . . $V_{k2}$ form the second-order system; $V_{a(n-1)}$, $V_{b(n-1)}$, . . . $V_{k(n-1)}$ form the $(n - 1)$-order or negative-sequence system. The vector diagram, Fig. 1, shows the five symmetrical systems of components of the vectors $V_a$, $V_b$, $V_c$, $V_d$, $V_e$, which are themselves shown in Fig. 2. No significance should be attached to the fact that the components of $V_a$ are shown in phase with one another.

We have here used the vector $V$ to represent any quantity which is to be resolved, such as potential, current, impedance, or admittance. In succeeding chapters we shall follow the standard convention of representing potential by $V$ or $E$, current by $I$,

impedance by $Z$ or $z$, and admittance by $Y$ or $y$. A letter subscript $(a,b,c,d)$ will be used to indicate the part of the electric circuit to which the quantity belongs, and a number subscript (0,1,2,3) will be used to indicate the order of the system to which the quantity belongs.

Thus the subscript 0 always indicates a zero-sequence component Likewise the subscript 1 always indicates a positive-sequence component. In a 3-phase system a subscript 2 indicates a negative-sequence component, whereas in a 4-phase system a subscript 3 indicates a negative-sequence component, while the subscript 2 indicates a second-order component for which at present no other name has been proposed.

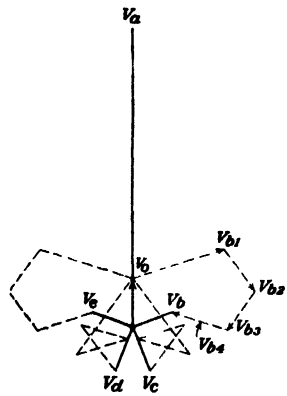



FIG. 2.

**2. Determination of the Symmetrical Components.**—The general systematic method for determining the symmetrical components of the quantities is as follows: It will be noted, equations (2), that the sum of the first-order components is zero, since they are vectors of the same magnitude with equal angular spacing about the origin. This is also true of all the other orders except the zero order, whose sum is $nV_0$. Therefore it follows that

$$V_0 = \frac{V_a + V_b \ . \quad . + V_k}{n} \tag{4}$$

The zero-sequence component is one $n$th the sum of all the quantities. That is to say, it is the average of all the quantities. If $V_b$, equations (2), is multiplied by $\alpha$, $V_c$ by $\alpha^2$, $V_d$ by $\alpha^3$, etc.,

$$\left.\begin{aligned}
V_a &= V_0 + V_1 + V_2 \ . \ . \ . \ + V_{n-1} \\
\alpha V_b &= \alpha V_0 + V_1 + \alpha^{-1}V_2 \ . \ . \ . \ + \alpha^{-(n-2)}V_{n-1} \\
\alpha^2 V_c &= \alpha^2 V_0 + V_1 + \alpha^{-2}V_2 \ . \ . \ . \ + \alpha^{-2(n-2)}V_{n-1} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\alpha^{n-1}V_k &= \alpha^{n-1}V_0 + V_1 + \alpha^{-(n-1)}V_2 \ . \ . \ . \ + \alpha^{-(n-1)(n-2)}V_{n-1}
\end{aligned}\right\} \tag{5}$$

If these equations are added, the sum for each column on the right of the equality signs, except the second column, will vanish, and thus the first-order or positive-sequence component of $V_a$ is

$$V_1 = \frac{V_a + \alpha V_b + \alpha^2 V_c \dots + \alpha^{n-1} V_k}{n} \tag{6}$$

By a similar process each of the other symmetrical components of $V_a$ can be found

$$V_2 = \frac{V_a + \alpha^2 V_b + \alpha^4 V_c + \alpha^6 V_d \dots + \alpha^{2(n-1)} V_k}{n} \tag{7}$$

$$V_3 = \frac{V_a + \alpha^3 V_b + \alpha^6 V_c + \alpha^9 V_d \qquad + \alpha^{3(n-1)} V_k}{n} \tag{8}$$

$$\cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot$$

$$V_{(n-1)} = \frac{V_a + \alpha^{n-1} V_b + \alpha^{2(n-1)} V_c \qquad + \alpha^{(n-1)(n-1)} V_k}{n} \tag{9}$$

Note the symmetry of the results.

Since $\alpha = \epsilon^{j\frac{2\pi}{n}}$ it follows that $\alpha^n = \epsilon^{j2\pi} = \cos 2\pi + j \sin 2\pi = 1$. Furthermore, if $k$ is any integer, $\alpha^{kn} = 1$. Therefore the expression for the negative-sequence component [equation (9)] can also be written:

$$V_{n-1} = \frac{V_a + \alpha^{-1} V_b + \alpha^{-2} V_c \dots + \alpha^{-(n-1)} V_k}{n} \tag{9a}$$

On comparing this with the expression for the positive-sequence component [equation (6)], it will be observed that the two expressions are identical except that the rotating operators $\alpha$, $\alpha^2$, etc., in equation (6) become $\alpha^{-1}$, $\alpha^{-2}$, etc., in equation (9a). Since $\alpha = \cos \beta + j \sin \beta$ and $\alpha^{-1} = \cos \beta - j \sin \beta$, the operators in equation (9a) are the conjugates of the corresponding operators in equation (6). Therefore, if the quantities, $V_a$, $V_b$, etc., that are being analyzed are all real numbers, their positive- and negative-sequence components are conjugate vectors.

With a 3-phase system the characteristic angle is 120 degrees and the general equations for the symmetrical components are

$$V_0 = \tfrac{1}{3}(V_a + V_b + V_c) \tag{4a}$$

$$V_{a1} = \tfrac{1}{3}(V_a + V_b\underline{/120°} + V_c\underline{/240°}) \tag{6a}$$

$$V_{a2} = \tfrac{1}{3}(V_a + V_b\underline{/240°} + V_c\underline{/480°}) \tag{7a}$$

Again it is important to emphasize that, although the quantities may be real numbers, their symmetrical components are in general complex and are represented by vectors. Furthermore, the symmetrical components of $V_a$ depend not only upon all the

$n$ quantities in the system but also upon the order or sequence in which these quantities are given. For example, the symmetrical components of $(V_a, V_b, V_c, \ldots V_{k-1}, V_k)$ are not the same as the symmetrical components of $(V_a, V_c, V_b, \ldots V_{k-1}, V_k)$ except when $V_b$ and $V_c$ are identical. They are, however, the same as the symmetrical components of $V_a$ in the series of quantities $(V_{k-1}, V_k, V_a, V_b, \ldots V_{k-2})$.

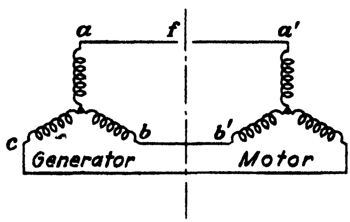


FIG. 3.

Suppose we have two sets of $n$ quantities each, *viz.* $V_a'$, $V_b'$, $V_c'$, $\ldots V_k'$ and $V_a''$, $V_b''$, $V_c''$, $\ldots V_k''$. If $V_a'$ equals $V_a''$, the symmetrical components of $V_a'$ and $V_a''$ are probably not equal. They will be equal *only* when all of the other corresponding quantities in the two sets are identical, *i.e.*, when $V_b'$ equals $V_b''$, $V_c'$ equals $V_c''$, $\ldots$, and $V_k'$ equals $V_k''$. A lack of appreciation of this fact has frequently led to an incorrect analysis. For example, consider the circuit shown in Fig. 3 which indicates that an open-circuit fault at $f$ occurs in the line $aa'$ between the generator and the motor. Since the currents in the other corresponding phases are equal, the currents in the open phases $a$ and $a'$ being zero, the symmetrical components of the generator currents are the same as the symmetrical components of the motor currents. On the other hand, although the terminal potential of the generator $V_{bc}$ equals the terminal potential of the motor $V_{b'c'}$, the symmetrical components of the terminal potentials of the generator and motor may not be the same, inasmuch as the other corresponding terminal potentials of the generator and motor may not be equal. For example, $V_{ab}$ may not equal $V_{a'b'}$. Again, consider the circuit shown in Fig. 4 which indicates that while two 3-phase motors are operating in parallel an open-circuit fault occurs in one of the mains at $f$. It

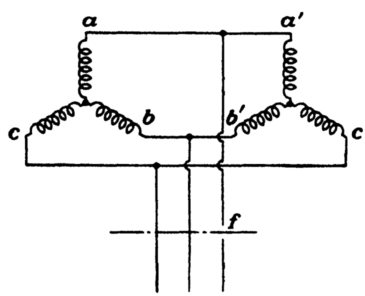


FIG. 4.

will be observed that since the corresponding terminal potentials of the two motors are the same, the symmetrical components of these potentials are equal. That is, $V_{ab1} = V_{a'b'1}$ and $V_{ab2} = V_{a'b'2}$. On the other hand, although the current $I_a$ in one motor is the same as the current $I_{a'}$ in the other motor, the symmetrical components of these currents are not the same, inasmuch as the other currents in the corresponding phases of the two motors are not the same. For example, $I_b$ is not the same as $I_{b'}$.

If all the quantities ($V_a$, $V_b$, . . . $V_k$) are real, the zero-sequence component is also real, but the other symmetrical components are in general complex. For example, consider an unbalanced 3-phase noninductive load. The zero-sequence component of the load resistances is real and thus represents a true resistance, but the positive- and negative-sequence components are complex and indicate impedances consisting of combinations of resistance and reactance each of which may be positive or negative. A negative resistance signifies a source of power whose emf is proportional to and in phase with the current A negative reactance indicates a condenser If the $n$ quantities are equal, all but the zero-sequence components are zero. It is very important to note that two complex quantities are equal to each other only when their real parts are identically equal and their imaginary parts also are identically equal, *i.e.*, in regard to both magnitude and sign. If the impedances of the conductors of a 3-phase transmission line are equal, they are said to be balanced and have neither positive- nor negative-sequence components. On the other hand, if the line potentials of the transmission circuit are equal (in magnitude), they are also said to be balanced, but in this case they have neither zero- nor negative-sequence components, since they themselves form a positive-sequence system. Note the difference in the meaning of "balanced" when applied to impedances and to potentials or currents.

Furthermore, notice that if all the $n$ quantities have the same angle, and if all except $V_a$ equal $m$ times $V_a$, the symmetrical components of $V_a$ are

$$V_0 = V_a \frac{1 + m(n - 1)}{n} \tag{10}$$

$$V_1 = V_2 = \dots = V_{n-1} = V_a \frac{1 - m}{n} \tag{11}$$

Again, if all the $n$ quantities except $V_a$ are zero, each of the symmetrical components of $V_a$ is equal to $V_a/n$. Also, if $V_a$ is zero and each of the remaining quantities is equal to $V$, the symmetrical components of $V_a$ are

$$V_0 = V \frac{n - 1}{n} \tag{12}$$

$$V_1 = V_2 = \dots = V_{n-1} = -\frac{V}{n} \tag{13}$$

The foregoing illustrations show that there is often considerable advantage in electing to determine the symmetrical components of one of the quantities rather than those of another. In the cases just cited it will be found that the symmetrical components of any one of the other quantities, $V_b$, $V_c$, . . . $V_k$, are not so easily obtained as are the symmetrical components of $V_a$.

**3. Addition.**—Let us now consider the fundamental operations of addition and multiplication when applied to two systems of $n$ quantities each. The addition of two such systems can be expressed as follows:

$$(V_a', V_b', V_c', \dots V_k') + (V_a'', V_b'', V_c'', \dots V_k'') = (V_a' + V_a'', V_b' + V_b'', V_c' + V_c'', \dots V_k' + V_k'')$$

This operation is performed when the Kirchoff Laws are applied to an $n$-phase circuit. Thus, if the potentials across the first portion (prime) of the circuit are $V_a'$, $V_b'$, . . . $V_k'$, and those across the second portion (double prime) of the circuit are $V_a''$, $V_b''$, $V_c''$, . . . $V_k''$, as shown in Fig. 5, the resulting potentials across both portions, *i.e.*, across $a'a''$, $b'b''$, $c'c''$, . . . and $k'k''$, are $V_a' + V_a''$, $V_b' + V_b''$, $V_c' + V_c''$, . . . $V_k' + V_k''$. The zero-sequence component of the resulting potential, $V_a' + V_a''$, is

$$\begin{aligned}(V_a' + V_a'')_0 &= \frac{(V_a' + V_a'') + (V_b' + V_b'') \dots + (V_k' + V_k'')}{n} \\ &= \frac{V_a' + V_b' \dots + V_k'}{n} + \frac{V_a'' + V_b'' \dots + V_k''}{n} \\ &= V_{a0}' + V_{a0}'' \end{aligned} \tag{14}$$

That is, the zero-sequence component of the resulting potential is equal to the sum of the zero-sequence components of the individual potentials. Also

$$(V_a'+V_a'')_1 = \frac{(V_a'+V_a'')+\alpha(V_b'+V_b'')\ .\ .\ .\ +\alpha^{n-1}(V_k'+V_k'')}{n}$$

$$= \frac{(V_a' + \alpha V_b'\ .\ .\ .\ + \alpha^{n-1}V_k')}{n} + \frac{(V_a'' + \alpha V_b''\ .\ .\ .\ + \alpha^{n-1}V_k'')}{n}$$

$$= V_{a1}' + V_{a1}'' \qquad (15)$$

That is, the positive-sequence component of the resulting potential is equal to the sum of the positive-sequence components of the individual potentials. It will be observed that this is also

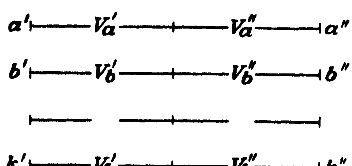



FIG. 5.

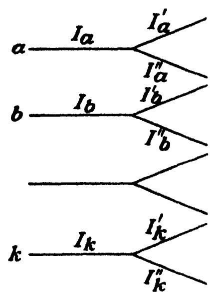



FIG. 6.

true for all the other symmetrical components. Furthermore, if the current in each of the $n$ phases flows into two branches, as indicated in Fig. 6, the symmetrical components of $I_a$ are, respectively, equal to the sum of the corresponding symmetrical components of $I_a'$ and $I_a''$. Let us apply these principles to the concrete case which is illustrated diagrammatically in Fig. 7. Two generators $A$ and $B$ supply power through two unsymmetrical feeders to an unsymmetrical combination of single-phase loads. The impedances of an equivalent $Y$-connected load are $Z_a'$, $Z_b'$, and $Z_c'$. The Kirchhoff equation for potential about the closed loop $naa'n'n$ is

$$V_{na} + V_{aa'} + V_{a'n'} + V_{n'n} = 0 \qquad (16)$$

Similar equations can be written for phases $b$ and $c$. Since in each case the resultant potential is zero, the symmetrical com-

ponents of the resultant potential are also zero. If the principles that are indicated in equations (14) and (15) are applied to equation (16), we can write

$$V_{na0} + V_{aa'0} + V_{a'n'0} + V_{n'n0} = 0 \qquad (17)$$
$$V_{na1} + V_{aa'1} + V_{a'n'1} + V_{n'n1} = 0 \qquad (18)$$

A similar equation can be written for the second-order or negative-sequence components of the potentials. Had the circuit been completely balanced or symmetrical, the zero- and negative-

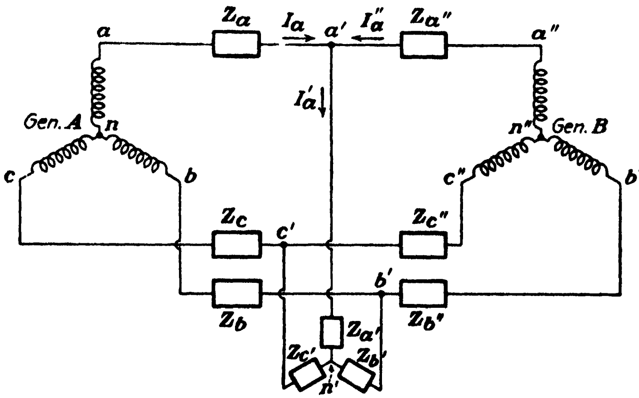



FIG. 7.

sequence components of potential would have all been zero. The Kirchhoff equation for current at the point $a'$ is

$$I_a + I_a'' = I_a' \qquad (19)$$

Similar equations can be written for the other phases at the points $b'$ and $c'$. Again, if the principles that are indicated in equations (14) and (15) are applied to equation (19), we can write

$$I_{a0} + I_{a0}'' = I_{a0}' \qquad (20)$$
$$I_{a1} + I_{a1}'' = I_{a1}' \qquad (21)$$

It also follows that

$$I_{a2} + I_{a2}'' = I_{a2}' \qquad (22)$$

It should be noted of this particular circuit that, since the sum of the phase currents is zero at each generator and at the load, the zero-sequence components of the currents are also zero, and equation (20) vanishes.

*Therefore it should be evident that not only can the Kirchhoff equations for an n-phase network be written in terms of the symmetrical components of the potential and current, but they can be written for each order of the symmetrical components independently.*

Moreover, since these symmetrical components represent symmetrical or balanced circuit conditions in regard to potentials and currents (but not in regard to the circuit constants $Z$ or $Y$), the equations need be written for one phase only, namely, the phase for which the symmetrical components are given. That is, one set of Kirchhoff equations is written for the zero-sequence values of potential and current in phase $a$ as if the circuit were symmetrical in the zero-sequence sense. Another set of Kirchhoff equations is written for the first order or positive-sequence components of potential and current in phase $a$ as if the circuit were symmetrical in the positive-sequence sense. Similar equations are also written for each of the other symmetrical components. These sets of Kirchhoff equations for each symmetrical system of potential and current are, however, related to one another, as we shall presently see, owing to the fact that the component fall in potential of *each* order through impedance involves in general *all* the symmetrical components of the current. Thus, in the illustrative case (Fig. 7) the positive-sequence component $V_{aa'1}$ of the fall in potential $V_{aa'}$ through the impedance $Z_a$ depends in general upon the zero-, positive-, and negative-sequence components of the current $I_a$. In the usual method of applying the Kirchhoff Laws, simultaneous equations are written involving the potentials and currents in all the $n$ phases. On the other hand, when the method of symmetrical components is used, the equations are written for *one* phase only, but they involve each of the $n$ symmetrical components of potential and current in that phase. The number of equations is not reduced, but a certain symmetry of expression is achieved that has proved exceedingly useful.

Furthermore, it is of the utmost importance to realize that *any* relation or principle which applies to a symmetrical polyphase system of potentials or currents can also apply to any one of the symmetrical components of an unsymmetrical system of potentials or currents.

For example, since the zero-sequence components of the line-to-neutral potentials are equal in a vector sense, there can be no

potential between the transmission line conductors in the zero-sequence system of potentials. Therefore it can be assumed that the line conductors are connected in parallel. Thus the manner in which the current supplied by the zero-sequence potential divides between the conductors will depend upon their self- and mutual impedances. In general these currents will not be equal and therefore in the case of 3-phase transmission will contain zero-, positive-, and negative-sequence components. In order that zero-sequence potential shall produce zero-sequence current only, that is to say, in order that the current shall divide equally among three conductors, it is evident that the following relations among their impedances must be true

$$Z_{aa} + Z_{ab} + Z_{ac} = Z_{bb} + Z_{bc} + Z_{ba} = Z_{cc} + Z_{ca} + Z_{cb} \quad (23)$$

The self-impedances are $Z_{aa}$, $Z_{bb}$, and $Z_{cc}$ and the mutual impedances are $Z_{ab}$, $Z_{bc}$, and $Z_{ca}$. It should be noted that the mutual impedances $Z_{ab}$ and $Z_{ba}$, for example, are equal. The foregoing relations among the impedances are true if the self-impedances are equal and the mutual impedances are also equal, but this is not a necessary condition.

Let us consider the positive-sequence components in a 3-phase system. The positive-sequence line potential $V_{ab}$ is $\sqrt{3}$ times and leads by 30 degrees the positive-sequence phase potential $V_{an}$. Similarly, the positive-sequence line current $I_a$ is $\sqrt{3}$ times and lags by 30 degrees the positive sequence $\Delta$ current $I_{ab}$. In general, positive-sequence potentials will produce currents in an unbalanced load that are not equal in magnitude and are not displaced in time-phase by 120 degrees. Thus these currents will contain zero-, positive-, and negative-sequence components if the vector sum of the currents is not zero. The conditions that must be satisfied if positive-sequence potentials are to produce none but positive-sequence currents are readily determined as follows. Write the expression for the fall in potential in each of the three phases in terms of the currents and the self- and mutual impedances, noting that the currents are equal in magnitude and differ in phase by 120 degrees. On equating the ratios $V_a/I_a$, $V_b/I_b$, and $V_c/I_c$, it will be found that

$$\begin{aligned} Z_{aa} + Z_{ab}\underline{/-120^\circ} + Z_{ac}\underline{/120^\circ} &= Z_{bb} + Z_{bc}\underline{/-120^\circ} + Z_{ab}\underline{/120^\circ} \\ &= Z_{cc} + Z_{ac}\underline{/-120^\circ} + Z_{bc}\underline{/120^\circ} \quad (24) \end{aligned}$$

These are the general conditions that must be satisfied if both the applied potentials and the currents are to have positive-sequence values only. These conditions are satisfied if the circuit is balanced, that is, if the self-impedances are equal and the mutual impedances are also equal, but it is not necessary that the circuit should be balanced in order that both the potentials and currents should consist of positive-sequence components only.

We shall further illustrate the application of these principles to many concrete problems in succeeding chapters.

**4. Multiplication.**—The multiplication of two systems, each of $n$ quantities, can be expressed as follows:

$$(V_a', V_b', \ldots V_k')(V_a'', V_b'', \ldots V_k'') = (V_a'V_a'', V_b'V_b'', \ldots V_k'V_k'')$$

For example, this operation is performed when it is desired to find the potentials across the impedances in the $n$ phases of any part of the network due to the currents in them, by multiplying the impedances by their *respective* currents. Thus, if the $n$ impedances are $(V_a', V_b', \ldots V_l')$, and the currents in them are $(V_a'', V_b'', \ldots V_k'')$, the values of the potentials across the impedances are $(V_a'V_a'', V_b'V_b'', \ldots V_k'V_k'')$. That is, in the illustrative case (Fig. 7), if the part of the network under consideration is the feeder which connects the generator $A$ with the load,

$$\begin{aligned} V_a' &= Z_a \\ V_b' &= Z_b \\ V_c' &= Z_c \end{aligned}$$

Likewise

$$\begin{aligned} V_a'' &= I_a \\ V_b'' &= I_b \\ V_c'' &= I_c \end{aligned}$$

We shall now express $V_a'$ and $V_a''$ in terms of their symmetrical components by means of equations (4), (6), (7), (8), and (9). Thus

$$\begin{aligned} V_a' &= V_0' + V_1' + V_2 \ldots + V_{n-1}' \\ V_a'' &= V_0'' + V_1'' + V_2'' \ldots + V_{n-1}'' \end{aligned}$$

The system of the $n$-product terms $(V_a'V_a'', V_b'V_b'', \ldots V_k'V_k'')$ has the same characteristic angle as do both the prime (′) and

double-prime (″) systems, and thus the symmetrical components of $V_a'V_a''$ are vectors which are equal in magnitude to, but differ in phase from, the corresponding components in $V_b'V_b''$ by some multiple of this characteristic angle.

The product $V_a'V_a''$ will contain $n^2$ terms and may be written as a double summation:

$$V_a'V_a'' = \sum_{p=0}^{p=n-1} \sum_{q=0}^{q=n-1} V_p'V_q''$$

where $V_p'$ and $V_q''$ are the $p$th and $q$th order symmetrical components of $V_a'$ and $V_a''$.

Consider one of these terms, $V_p'V_q''$. It is the product of the $p$th-order impedance and the $q$th-order current in phase $a$. In the next phase, $b$, the corresponding product term is $\alpha^{-p}V_p'\alpha^{-q}V_q''$, where $\alpha$ is the characteristic rotating operator for an $n$-phase system. Since this product term in phase $b$ may be written $\alpha^{-(p+q)}V_p'V_q''$, it lags the corresponding term in phase $a$ by $(p + q)$ times the characteristic angle of the system. That is, the component falls in potential in the different phases of any portion of a network, due to the $p$th-order component of the currents in them and the $q$th-order component of the corresponding impedances, form a symmetrical system of potentials in which the component in the $b$ phase lags the corresponding component in the $a$ phase by $(p + q)$ times the characteristic angle. If $(p + q)$ is less than $n$, this product term belongs to the $(p + q)$ order of symmetrical components of the potential drop due to the current and impedance in phase $a$. If $(p + q)$ is equal to or greater than $n$, the order is $(p + q - n)$, since $n$ times the characteristic angle equals $2\pi$. This gives a very simple rule for the multiplication of two systems of symmetrical components.

**Sequence Rule.—The order of the system to which the product of two components belongs is equal to the sum of the orders of the systems to which the two components belong individually.**

In applying this rule it is essential to note that since the symmetrical components of one of the $n$ quantities depend upon the order or sequence in which the $n$ quantities are given,[1] the order in which the $n$ quantities of each factor are written must

[1] See paragraph following equation (7a), p. 32.

be the same as the order in which they appear in the $n$ product terms. That is, in the illustrative case (Fig. 7), if the symmetrical components of the fall in potential $V_{aa'}$ are to be determined from the potentials $V_{aa'}$, $V_{bb'}$, and $V_{cc'}$, the symmetrical components of $Z_a$ must be determined from $Z_a$, $Z_b$, and $Z_c$ and the symmetrical components of $I_a$ must be determined from $I_a$, $I_b$, and $I_c$, and not, for example, from $I_a$, $I_c$, and $I_b$. It is also of interest to note that, for *every* value of $p$ from 0 to $(n - 1)$, inclusive, there is a value of $q$ such that the sum $(p + q)$ indicates a symmetrical component of a given order. Thus it follows that the $n^2$ terms in the product $V_a'V_a''$ form $n$ groups each of which contains $n$ components of the same sequence. For example, in a 5-phase system, if $I$ and $Z$ with the appropriate subscripts represent the symmetrical components of the phase currents and phase impedances in some part of the network, the five symmetrical components, $I_0Z_3$, $I_1Z_2$, $I_2Z_1$, $I_3Z_0$, and $I_4Z_4$, of the impedance drop in this part of the network are all of the third order. Hence each of these component falls in potential in the $b$ phase lags the corresponding component fall in potential in the $a$ phase by 3 times 72 degrees, that is, by 3 times the characteristic angle of a 5-phase system.

Referring again to the illustrative 3-phase case (Fig. 7), apply the Sequence Rule to the symmetrical components of the fall in potential in the feeder between the generator $A$ and the load. Although in this case there is no zero-sequence component in the line current, we shall, for the sake of generality, write the equations as if there were a zero-sequence component. For our present purpose it is necessary to assume that there is no mutual-impedance coupling between the three line impedances $Z_a$, $Z_b$, and $Z_c$. If the symmetrical components of the line impedance $Z_a$ are $Z_{a0}$, $Z_{a1}$, and $Z_{a2}$ and the symmetrical components of the line current $I_a$ are $I_{a0}$, $I_{a1}$, and $I_{a2}$, the fall in line-to-neutral potential $I_aZ_a$ in line conductor $a$ from the generator to the load is

$$I_aZ_a = (I_{a0} + I_{a1} + I_{a2})(Z_{a0} + Z_{a1} + Z_{a2})$$

If this expression for the fall in potential is expanded,

$$\left.\begin{aligned} I_aZ_a = I_{a0}Z_{a0} + I_{a1}Z_{a2} + I_{a2}Z_{a1} \\ + I_{a0}Z_{a1} + I_{a1}Z_{a0} + I_{a2}Z_{a2} \\ + I_{a0}Z_{a2} + I_{a1}Z_{a1} + I_{a2}Z_{a0} \end{aligned}\right\} \quad (25)$$

On referring to equations (2*b*), p. 29, it will be seen that the expression for the fall in potential in the next line conductor *b* can be written:

$$I_bZ_b = (I_{a0} + I_{a1}\underline{/-120^\circ} + I_{a2}\underline{/-240^\circ})$$
$$(Z_{a0} + Z_{a1}\underline{/-120^\circ} + Z_{a2}\underline{/-240^\circ})$$

If this expression for the fall in potential in conductor *b* is expanded,

$$\left.\begin{aligned} I_bZ_b = {} & I_{a0}Z_{a0} + I_{a1}Z_{a2}\underline{/-360^\circ} + I_{a2}Z_{a1}\underline{/-360^\circ} \\ & + I_{a0}Z_{a1}\underline{/-120^\circ} + I_{a1}Z_{a0}\underline{/-120^\circ} + I_{a2}Z_{a2}\underline{/-480^\circ} \\ & + I_{a0}Z_{a2}\underline{/-240^\circ} + I_{a1}Z_{a1}\underline{/-240^\circ} + I_{a2}Z_{a0}\underline{/-240^\circ} \end{aligned}\right\} \quad (26)$$

Note that in each of these nine components the magnitude of the attached angle is 120 degrees multiplied by the sum of the sequence subscripts.

On comparing these expressions for the fall in potential in the *a* and *b* line conductors it will be observed that the terms in the *first* line of $I_bZ_b$ are in phase with the corresponding terms in the *first* line of $I_aZ_a$; that the terms in the *second* line of $I_bZ_b$ lag the corresponding terms in the *second* line of $I_aZ_a$ by 120 degrees; that the terms in the *third* line of $I_bZ_b$ lag the corresponding terms in the *third* line of $I_aZ_a$ by 240 degrees. The fall in potential in the third line conductor *c* is

$$I_cZ_c = (I_{a0} + I_{a1}\underline{/-240^\circ} + I_{a2}\underline{/-480^\circ})$$
$$(Z_{a0} + Z_{a1}\underline{/-240^\circ} + Z_{a2}\underline{/-480^\circ})$$

On expanding this expression for the fall in potential in conductor *c*

$$\left.\begin{aligned} I_cZ_c = {} & I_{a0}Z_{a0} + I_{a1}Z_{a2}\underline{/-720^\circ} + I_{a2}Z_{a1}\underline{/-720^\circ} \\ & + I_{a0}Z_{a1}\underline{/-240^\circ} + I_{a1}Z_{a0}\underline{/-240^\circ} + I_{a2}Z_{a2}\underline{/-960^\circ} \\ & + I_{a0}Z_{a2}\underline{/-480^\circ} + I_{a1}Z_{a1}\underline{/-480^\circ} + I_{a2}Z_{a0}\underline{/-480^\circ} \end{aligned}\right\} \quad (27)$$

It will be observed that the individual terms in this expression for $I_cZ_c$ have the same phase relation with respect to the corresponding terms in $I_bZ_b$ that the same terms in $I_bZ_b$ have with respect to the corresponding terms in $I_aZ_a$.

Thus, the first line in each of these expressions for the fall in potential consists of *zero-sequence* terms, the second line consists

of *first order* or positive-sequence terms, and the third line consists of *second order* or negative-sequence terms. Furthermore, observe that the sum of the subscripts for each product term in the *first* line is either 0 or 3, in the *second* line is either 1 or 4, and in the *third* line is 2. This is clearly in accordance with the Sequence Rule.

We see in the operation of the Sequence Rule the crowning evidence of the power of the method of resolving quantities into their symmetrical components. It has been shown that owing to the manner in which the symmetrical components are obtained, the two Kirchhoff Laws can be applied to each order of the components of potential and current independently. Furthermore, it has been shown that, if the circuit impedances are similarly resolved into symmetrical components, the products of current and impedance in the different phases, when expressed in terms of their symmetrical components, form symmetrical systems of potential which can then be written into the Kirchhoff equations.

**5. Conjugate System.**—The vector volt-ampere input supplied to an alternating-current circuit can be simply expressed as $\underset{\wedge}{E}I$, where the vector $E$ represents the potential drop across the circuit, and the vector $I$ the current through the circuit. The inverted angle sign, $\wedge$, beneath $E$ indicates that the conjugate of $E$ with respect to the axis of reals is to be taken. The real part of the vector volt-ampere input, $|E| \cdot |I| \cos\theta$, is the power (watts) supplied to the circuit; the imaginary part, $|E| \cdot |I| \sin\theta$, is the reactive power (vars). The reactive power is positive when the current leads the potential, that is, when the load is condensive. In order to apply the Sequence Rule to this product $\underset{\wedge}{E}I$, it is first necessary to establish the order of the symmetrical components of $\underset{\wedge}{E}$.

Let the order of the phases be $a, b, c, \ldots k$, and let $E_{ap}$ be the $p$th-order component of $E_a$. Then the $p$th-order component of $E_b$ lags $E_{ap}$ by an angle $p\frac{2\pi}{n}$, or

$$E_{bp} = E_{ap}\epsilon^{-jp\frac{2\pi}{n}}$$

If the conjugates of both sides are taken,

$$\underset{\wedge}{E}_{bp} = \underset{\wedge}{E}_{ap}\epsilon^{jp\frac{2\pi}{n}}$$

That is, the conjugate of the $p$th-order component in $E_b$ *leads* the conjugate of the $p$th-order component in $E_a$ by the angle $p\frac{2\pi}{n}\cdot$ *Leading* by an angle $p\frac{2\pi}{n}$ is equivalent to *lagging* by an angle $\left(2\pi - p\frac{2\pi}{n}\right)$, or $(n - p)\frac{2\pi}{n}\cdot$ Therefore the conjugates of the $p$th-order components of an $n$-phase circuit form a system of symmetrical components of the $(n - p)$ order, and must be treated as such in applying the Sequence Rule.

For example, consider the positive-sequence vectors $V_{a1}$, $V_{b1}$, and $V_{c1}$ shown in Fig. 8. It will be observed that the conjugates of these vectors form another symmetrical system in which $\underset{\wedge}{V}_{a1}$ lags $\underset{\wedge}{V}_{b1}$ by 120 degrees. Thus the conjugate vectors form a negative-sequence system.

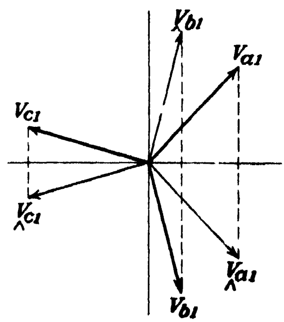


FIG. 8.

In these fundamental principles we have all the mechanism that is necessary for solving a wide variety of problems which involve *unsymmetrical* polyphase circuits. The application of these principles to various concrete problems is given in the following chapters.

The Method of Symmetrical Components is not designed to apply to the solution of problems in which the circuit constants are *nonlinear*. That is, when the method is applied, one must be content to assume that the circuit constants do not depend upon either the magnitudes of the applied potentials or the magnitudes of the resulting currents.

# CHAPTER II

## THREE-PHASE STATIC CIRCUITS

We shall now consider the application of the principles of resolution developed in Chapter I to the analysis of 3-phase static circuits both when there is and when there is not a neutral conductor. As here used, a *static* circuit denotes either a circuit without rotating parts or a circuit which, for the purposes of the analysis, can be replaced by a circuit without moving parts. For example, in the analysis of a power distribution system, a connected *single-phase motor* load can often be represented by an impedance $R + jX$, whose resistance and reactance are determined by the power input to the motor and the power factor at which it operates. Since, however, the values of $R$ and $X$ depend also upon the potential at the motor terminals, it may be desirable to make a preliminary solution in order to determine the approximate value of this potential.

It should be understood at the outset that there may be little or no advantage gained by using the Method of Symmetrical Components in the solution of unbalanced static circuits. However, if portions of the unbalanced circuits are polyphase *rotating* machines, the method is invaluable and has no equal. In commercial networks the polyphase rotating machines are almost invariably symmetrical and, per se, contribute nothing to the unsymmetrical conditions in potential and current that may be due to unbalanced arrangements of the distribution circuits or attached loads. On the contrary, where such unsymmetry in the potential or current exists, the *addition* of a *symmetrical* rotating machine to the circuit will reduce the amount of the unsymmetry (see, for example, Art. 88, page 243). Since the unsymmetry that does exist in the potential and current is usually due to the unsymmetry in the *static* portions of the network, it becomes essential to understand how the method of symmetrical components is applied in the analysis of unbalanced static circuits.

In this and in the following chapters it is necessary to note that the $V$ or $E$, the $I$, the $Z$, and the $Y$, which represent, respectively, the potential or emf, the current, the impedance, and the admittance are *vector quantities.* The value of a vector quantity must be given so as to indicate both its *magnitude* and its *phase.* We shall do this by using either the angular or the complex notation. The magnitude of any of these quantities, the potential $V$ for example, will be denoted thus $|V|$. In all the fundamental algebraic processes of addition, subtraction, multiplication, and division these vector quantities are treated as complex numbers.

**6. Resolution of Potentials.**—Let $V_a$, $V_b$, and $V_c$ represent the 3-phase potentials whose symmetrical components we wish to determine. They may be, for example:

Line potentials.

Potential drops in the line conductors from generator to some load.

Potentials from the line terminals to the neutral of a $Y$-connected load.

Potentials from the line conductors to ground.

Potentials from the terminals to the neutral of a 3-phase, $Y$-connected generator.

In this last case the armature windings of the generator are shown diagrammatically in Fig. 9. Although it is not a static circuit, the generator has been chosen for analysis, since the physical significance of the resolution of its armature emfs can be clearly illustrated.

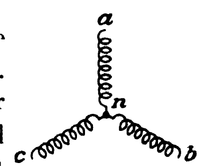


FIG. 9.

The neutral point is $n$ and the terminals of the phases are $a$, $b$, and $c$. Assume that the windings of this generator are *not* symmetrically arranged. Thus the phase potentials $V_{an}$, $V_{bn}$, and $V_{cn}$ are neither equal in magnitude nor displaced in time-phase by 120 degrees, and therefore their sum is in general not zero. Double subscripts are a convenient way of indicating the *direction* in which the potential is taken. Thus $V_{ab}$ is the potential from $a$ to $b$. It is important to note that $V_{ab} = -V_{ba}$. In the following equations the subscript $n$ has been omitted from these potentials in order to simplify the notation. Hereafter this will be done in all cases, and it should be noted that, when $V_a$ represents the phase potential of a star-connected circuit, it is understood that it is the potential from the terminal $a$ to the

neutral $n$ of the star. In general, $V_a$ may be either the rise or the fall in potential from $a$ to $n$. Whenever it is essential, care should be taken to indicate whether it is the *rise* or the *fall*.

The characteristic angle, $\beta$, of a 3-phase system is 120 degrees, and the symmetrical components of the three generator potentials are determined by [see equations (2b), page 29]:

$$\begin{aligned} V_a &= V_0 + V_1 + V_2 \\ V_b &= V_0 + V_1\underline{/-120^\circ} + V_2\underline{/-240^\circ} \\ V_c &= V_0 + V_1\underline{/-240^\circ} + V_2\underline{/-480^\circ} \end{aligned}$$

Since 480 degrees is equivalent to 120 degrees and, furthermore, since a lag angle of 240 degrees is the same as a lead angle of 120 degrees and since a lead angle of 240 degrees is the same as a lag angle of 120 degrees, these equations may be rewritten in their more common form

$$\left.\begin{aligned} V_a &= V_0 + V_1 + V_2 \\ V_b &= V_0 + V_1\underline{/-120^\circ} + V_2\underline{/120^\circ} \\ V_c &= V_0 + V_1\underline{/-240^\circ} + V_2\underline{/240^\circ} \end{aligned}\right\} \qquad (28)$$

Thus each of the phase potentials, $V_a$, $V_b$, and $V_c$, consists of three components: a zero-sequence component, $V_0$, which is the same both in magnitude and in phase in each of the phase potentials; a positive-sequence component which is one of a symmetrical system in which the component in $V_b$ *lags* the corresponding component in $V_a$ by 120 degrees; and a negative-sequence component which is one of a symmetrical system in which the component in $V_b$ *leads* the corresponding component in $V_a$ by 120 degrees. This resolution of the actual phase potentials into their symmetrical components can be represented diagrammatically as in Fig. 10.

In Fig. 10, $G_0$ represents a single-phase generator whose terminal potential between $n'$ and $n$ is equal to the zero-sequence component, $V_0$, and $G_2$ represents a symmetrical 3-phase generator whose terminals are $a'$, $b'$, and $c'$ and whose neutral point is $n'$. The phase potential of this generator, $V_{b'n'}$, leads the phase potential, $V_{a'n'}$, by 120 degrees. Thus $G_2$ generates the negative-sequence emfs. The windings of $G_2$ are connected in series with those of the positive-sequence generator $G_1$, which also has

symmetrically arranged armature windings. In $G_1$ the phase potential $V_{bb'}$ lags the phase potential $V_{aa'}$ by 120 degrees.

That is, insofar as the generation of phase emfs is concerned, the actual unsymmetrical generator is equivalent to three generators whose armature windings are connected *in series.* The first is a single-phase generator, while the second and third are

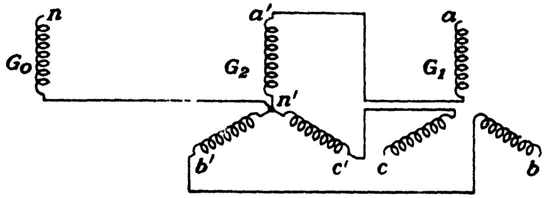



FIG. 10.

symmetrical 3-phase generators whose armature windings are connected so that their emfs have opposite phase orders. This difference in the phase orders is indicated by the interconnection of the phases in Fig. 10. If the armature of the actual generator is symmetrically wound, neither $G_0$ nor $G_2$ are present and the phase potentials are symmetrical, or balanced. It is very important to notice that the zero-sequence emf which is generated in

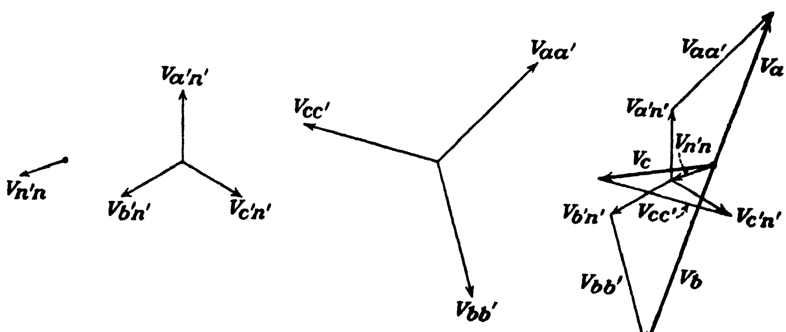



FIG. 11.

$G_0$ has no effect whatsoever on the potentials between the armature terminals, $a$, $b$, and $c$. It affects only the potential at the neutral point $n$. The emfs of these generators and the resulting phase potentials $V_a$, $V_b$, and $V_c$ are shown in Fig. 11.

If a load is connected across the terminals of this unsymmetrical generator, the currents that the generator supplies can be determined by means of the superposition principle, provided the

load will permit its use. In general the superposition principle can be applied if the load has linear characteristics. A load has linear characteristics if, when sinusoidal potentials are applied to it, the currents are also sinusoidal and of the same frequency as the potentials, and, furthermore, if, when the magnitude of each of the applied potentials is increased or diminished by the same relative amount, the magnitude of each of the currents will change in this same proportion. One of the most common causes of the nonlinearity of circuits is magnetic saturation.

Each of the currents supplied to the load will in general consist of three components. The first component will be the current that the zero-sequence generator will supply when it acts alone, both the positive- and negative-sequence generators being short-circuited at their terminals. It will then be observed that this generator $G_0$ can supply no current except when the neutral point $n$ is in some way connected to the load. For example, if the neutral of the generator is grounded, this connection may be through the capacitance of the transmission line conductors to ground. The second component will be the current that the negative-sequence generator $G_2$ will supply when both the zero-sequence and the positive-sequence generators are short-circuited. Similarly, the third component will be the current that the positive-sequence generator $G_1$ will supply when both the zero-sequence and negative-sequence generators are short-circuited. The principle of superposition when thus applied has become a most powerful tool for solving unsymmetrical polyphase circuits.

In any laboratory problem, mark the lines $a$, $b$, and $c$ so that the line potential $V_{bc}$ lags the line potential $V_{ab}$ by a positive angle of less than 180 degrees. The positive-sequence component of the line potentials will then be greater than the negative-sequence component.

It is necessary to know the phase order of the applied potentials, since the values of the line currents taken by any unbalanced load will depend upon this phase order. It follows that the phase order of the applied potentials can be determined by measuring the currents or potentials which result when an unbalanced load is connected to the source. One of the simplest methods of determining the phase order of the potentials is to connect across two of the line terminals, $a,b$ (Fig. 12), a resistance in series with a highly inductive circuit of approximately the

same ohm value as the resistance. A voltmeter is then connected between the third line, $c$, and the junction of the resistance and the inductive circuit as indicated in Fig. 12. If the line potentials are approximately balanced, the line potential $V_{ab}$ *leads* the potential $V_{bc}$ if the voltmeter reading is less than the line potential. If the voltmeter reading is greater than the line potential, $V_{ab}$ *lags* $V_{bc}$. If the phase order of the potentials is not correctly determined, the computed values of the currents will not agree with their measured values.

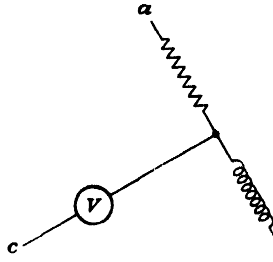



FIG. 12.

**7. Resolution of Line-to-neutral Potentials.**—We shall now determine the magnitudes of the symmetrical components of the line-to-neutral potential by the method indicated in equations (4a), (6a), and (7a). Thus

$$\left.\begin{aligned} V_0 &= \tfrac{1}{3}(V_a + V_b + V_c) \\ V_{a1} &= \tfrac{1}{3}(V_a + V_b\underline{/120^\circ} + V_c\underline{/240^\circ}) \\ V_{a2} &= \tfrac{1}{3}(V_a + V_b\underline{/240^\circ} + V_c\underline{/480^\circ}) \end{aligned}\right\} \qquad (29)$$

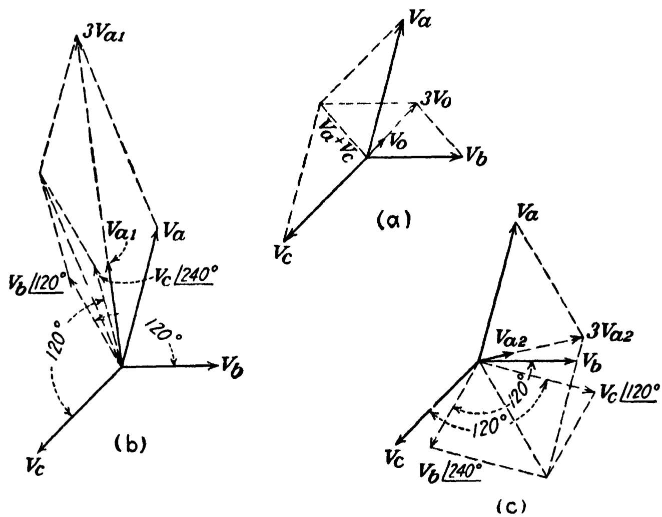



FIG. 13.

The positive- and negative-sequence components are written $V_{a1}$ and $V_{a2}$, rather than $V_1$ and $V_2$, to emphasize the fact that they are the components of the phase potential $V_a$. Note that the

angle operators $\underline{/120°}$ and $\underline{/480°}$ are both equal to $-0.5 + j0.866$, and that the angle operator $\underline{/240°}$ is equal to $-0.5 - j0.866$.

The operations can be performed *graphically,* as indicated in Fig. 13*a*, *b*, and *c*.

The operations can also be performed *analytically,* as in the following simple illustration:

$$\begin{aligned}
\text{Let}\quad V_a &= 100 + j0 \\
V_b &= 0 - j100 \\
V_c &= 0 + j100 \\
\text{Then}\quad V_0 &= \tfrac{1}{3}(100 - j100 + j100) \\
&= 33.3 + j0 \\
V_{a1} &= \tfrac{1}{3}[100 + (-j100)(-0.5 + j0.866) \\
&\qquad + (j100)(-0.5 - j0.866)] \\
&= 91.1 + j0 \\
V_{a2} &= \tfrac{1}{3}[100 + (-j100)(-0.5 - j0.866) \\
&\qquad + (j100)(-0.5 + j0.866)] \\
&= -24.4 + j0
\end{aligned}$$

The preceding operations can be somewhat simplified. Observe that, if the rotating operators are separated into their real and imaginary parts, the positive- and negative-sequence components can be written

$$\begin{aligned}
V_{a1} &= \tfrac{1}{3}[V_a - 0.5(V_b + V_c) + j0.866(V_b - V_c)] \\
V_{a2} &= \tfrac{1}{3}[V_a - 0.5(V_b + V_c) - j0.866(V_b - V_c)]
\end{aligned}$$

By introducing $V_0$, the expressions for the positive- and negative-sequence components are still further simplified

$$\left.\begin{aligned}
V_{a1} &= 0.5(V_a - V_0) + j0.289(V_b - V_c) \\
V_{a2} &= 0.5(V_a - V_0) - j0.289(V_b - V_c)
\end{aligned}\right\} \tag{30}$$

This procedure is particularly useful in computing the positive- and negative-sequence components of impedances. If in the numerical example the positive- and negative-sequence components of potential are computed by this last method,

$$\begin{aligned}
V_{a1} &= 0.5[100 + j0 - (33.3 + j0)] \\
&\qquad + j0.289[0 - j100 - (0 + j100)] \\
&= 91.1 + j0 \\
V_{a2} &= 0.5[100 + j0 - (33.3 + j0)] \\
&\qquad - j0.289[0 - j100 - (0 + j100)] \\
&= -24.4 + j0
\end{aligned}$$

**8. Resolution of Line Potentials.**—Since the sum of the line potentials of a 3-phase circuit is always zero, they can have no zero-sequence component [equation (4), page 31]. Let the line potentials be $V_{ab}$, $V_{bc}$, and $V_{ca}$. Substitute these values for $V_a$, $V_b$, and $V_c$ in equations (29). Since $V_{ca}$ equals $(-V_{ab} - V_{bc})$, the equation for the positive-sequence component, $V_1$, of $V_{ab}$ is

$$\begin{aligned} V_1 &= \tfrac{1}{3}[V_{ab}(1 - \underline{/240^\circ}) + V_{bc}(\underline{/120^\circ} - \underline{/240^\circ})] \\ &= \tfrac{1}{3}(V_{ab}\sqrt{3}\underline{/30^\circ} + V_{bc}\sqrt{3}\underline{/90^\circ}) \\ &= \frac{\underline{/30^\circ}}{\sqrt{3}}(V_{ab} + V_{bc}\underline{/60^\circ}) \end{aligned} \tag{31}$$

Similarly, the negative-sequence component, $V_2$, of $V_{ab}$ is:

$$V_2 = \frac{\underline{/-30^\circ}}{\sqrt{3}}(V_{ab} + V_{bc}\underline{/-60^\circ}) \tag{32}$$

These two equations for the positive- and negative-sequence components lend themselves to the simple geometrical construction shown in Figs. 14*a*, *b*. All that is necessary is to construct the two equilateral triangles having $V_{bc}$ for a base, as indicated. Numerous other constructions have been devised for obtaining these components, but none of them is simpler than this.

Since the positive-sequence potential to neutral $V_{a1}$ is $1/\sqrt{3}$ of and *lags* by 30 degrees (see Fig. 15*a*) the positive-sequence component of the line potential $V_{ab}$,

$$V_{a1} = \tfrac{1}{3}(V_{ab} + V_{bc}\underline{/60^\circ}) \tag{33}$$

and thus is obtained readily by the construction shown in Fig. 14*a*. Since the negative-sequence potential $V_{a2}$ is $1/\sqrt{3}$ of and *leads* by 30 degrees (see Fig. 15*b*) the negative-sequence component of the line potential $V_{ab}$,

$$V_{a2} = \tfrac{1}{3}(V_{ab} + V_{bc}\underline{/-60^\circ}) \tag{34}$$

This construction is shown in Fig. 14*b*.

The positive- and negative-sequence components of the line potentials can also be computed in the following manner, which is due to A. E. Kennelly.

If the values of the line potentials $V_{ab}$, $V_{bc}$, and $V_{ca}$ are $a$, $b$, and $c$, respectively, the length of the line $gh$ in Fig. 14 is

$$gh = \sqrt{a^2 + b^2 - 2ab\cos(\theta \pm 60°)}$$

The positive sign is used in Fig. 14a and the negative sign in Fig. 14b. This readily reduces to

$$gh = \sqrt{\frac{a^2 + b^2 + c^2}{2} \pm \sqrt{3}\, ab \sin\theta}$$

Since $ab \sin\theta$ is twice the area of the potential triangle, it can be reduced to an expression in terms of the sides alone. Thus,

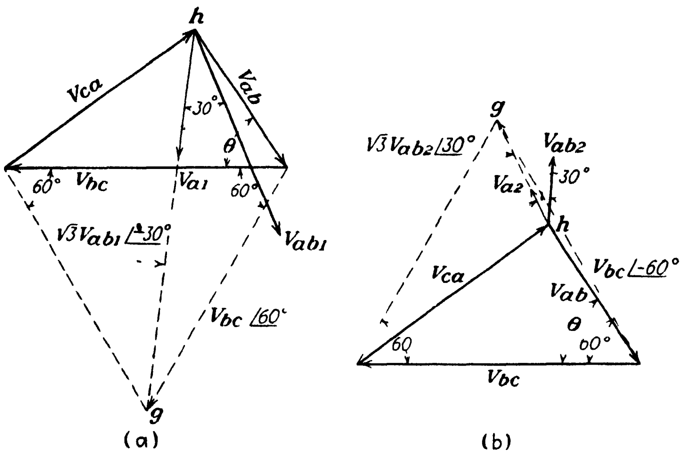



FIG. 14

if $s$ is one-half the perimeter of the triangle, $s = \frac{a + b + c}{2}$ and the area is $\sqrt{s(s-a)(s-b)(s-c)}$. Thus the magnitude of the positive-sequence component of the line potentials is

$$V_1 = \sqrt{\frac{a^2 + b^2 + c^2}{6} + 2\sqrt{\frac{s(s-a)(s-b)(s-c)}{3}}} \tag{35}$$

Similarly, the magnitude of the negative-sequence component is

$$V_2 = \sqrt{\frac{a^2 + b^2 + c^2}{6} - 2\sqrt{\frac{s(s-a)(s-b)(s-c)}{3}}} \tag{36}$$

**9. Unbalance Factor.**—The ratio of the magnitude of the zero-sequence component of the phase potential to the magnitude of the positive-sequence component of the phase potential is called the "zero-sequence unbalance factor." The ratio of the negative-sequence component of the phase or the line potential to the positive-sequence component of the phase or the line potential is called the "negative-sequence unbalance factor."

**10. Relation between Line Potentials and Potentials to Neutral.**—It is frequently desirable to express the line potentials in terms of the phase potentials to neutral. The positive- and

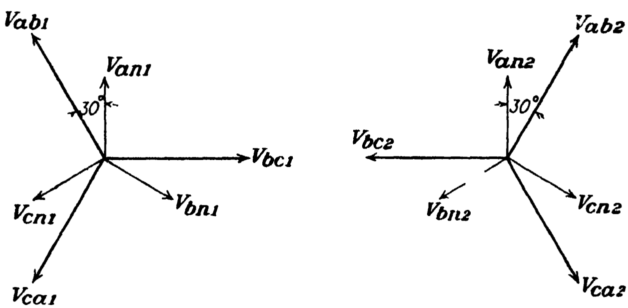



FIG 15

negative-sequence vector relations are shown in Fig. 15, *a* and *b*. Notice that the positive-sequence component of the line potential $V_{bc}$ is equal to the positive-sequence component of the phase potential $V_{an}$ multiplied by $\sqrt{3}$ and rotated 90 degrees negatively. That is

$$V_{bc1} = -j\sqrt{3}V_{a1} \tag{37}$$

Similarly

$$V_{bc2} = j\sqrt{3}V_{a2} \tag{38}$$

As an example of the use of these relations, consider a 3-phase circuit on which there is a line-to-line short circuit between the conductors *b* and *c*.

$$V_{bc} = V_{bc1} + V_{bc2} = 0$$

or

$$-j\sqrt{3}V_{a1} + j\sqrt{3}V_{a2} = 0$$

Therefore

$$V_{a1} = V_{a2}$$

Let us now assume that, while $V_{bc}$ is zero, $V_{ab}$ is $V$ and $V_{ca}$ is $-V$. The positive-sequence component of $V_{bc}$ is

$$V_{bc1} = \tfrac{1}{3}(0 + (-V)\underline{/120°} + V\underline{/240°})$$
$$= -j\frac{V}{\sqrt{3}}$$

The negative-sequence component of $V_{bc}$ is

$$V_{bc2} = \tfrac{1}{3}(0 + (-V)\underline{/-120°} + V\underline{/-240°})$$
$$= j\frac{V}{\sqrt{3}}$$

If the relations expressed in equations (37) and (38) are introduced, it will be observed that the positive- and negative-

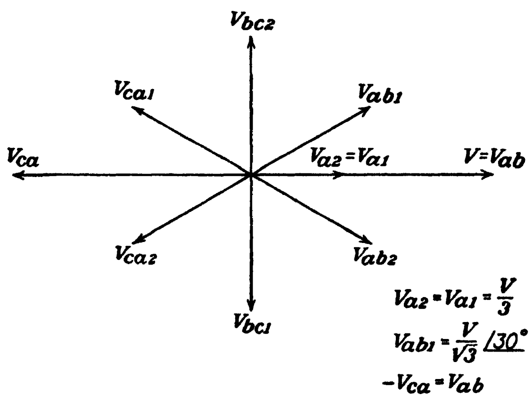



FIG. 16.

sequence components of the line-to-neutral potential $V_a$ are each one-third of the line potential $V_{ab}$.

These relations are shown in the vector diagram, Fig. 16.

It is of special interest to note that the line potential $V_{bc}$, although it is zero, consists of a positive- and a negative-sequence component which are in phase opposition and each of which is equal in magnitude to $V_{ab}/\sqrt{3}$.

**11. Measurement of Symmetrical Components of Potential.**[1]— The symmetrical components of potential and current can be measured either by means of specially designed instruments or

[1] HAYWARD, A. P., Measurement of Symmetrical Components. *Elec. J.*, June, 1931.

by using standard types of instruments connected to the circuit through the proper network. The latter method is in general more satisfactory.

The zero-sequence component of potential can be measured by connecting a voltmeter in series with the secondaries, also in series, of three potential transformers whose primaries are connected between the lines and neutral, Fig. 17. The voltmeter is calibrated to register one-third the sum of the line-to-neutral potentials. That is, the voltmeter indicates the zero-sequence component of $V_{an}$, $V_{bn}$, and $V_{cn}$.

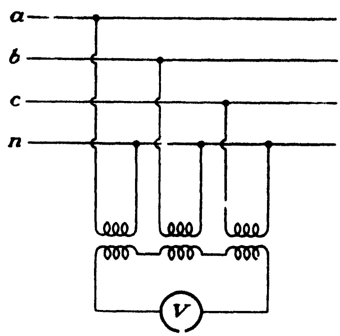
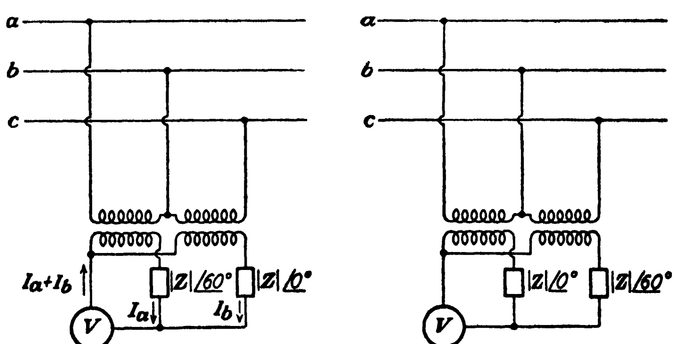


FIG. 17.

The positive-sequence component of line-to-neutral potential can be measured by connecting the voltmeter to two potential transformers through the network shown in Fig. 18. A voltmeter of the dynamometer type consists of a fixed coil and a moving coil connected in series with a high resistance. In



Positive sequence

Negative sequence.

FIG. 18.

Fig. 18 the circle enclosing the $V$ represents the fixed and moving coils; the impedances $|Z|\underline{/0°}$ and $|Z|\underline{/60°}$ replace the high resistance. These impedances have the same ohm value but have angles as indicated. If the impedance of the fixed and moving coils is small as compared with the impedance $Z$, the

current in the voltmeter (Fig. 18*a*) is

$$\frac{V_{ab}}{|Z|\underline{/60^\circ}} + \frac{V_{bc}}{|Z|\underline{/0^\circ}}.$$

It should be noticed that equation (31) cannot be realized without the use of a static condenser. That is, a condenser would be necessary to produce a current leading the potential $V_{bc}$ by 60 degrees. Nevertheless, since

$$\frac{1}{|Z|}(V_{ab}\underline{/-60^\circ} + V_{bc}) = \frac{\underline{/-60^\circ}}{|Z|}(V_{ab} + V_{bc}\underline{/60^\circ}) = \frac{3}{|Z|}V_{an1}\underline{/-60^\circ}$$
$$= \frac{-3V_{cn1}}{|Z|}$$

the current in the voltmeter, Fig. 18*a*, is proportional to, and in phase with, the positive-sequence component of the line-to-neutral potential $V_{cn}$. The voltmeter current is either in phase with or in opposition to the line-to-neutral potential $V_{cn1}$, depending upon which direction is assumed to be positive for the current in the voltmeter coil. Thus the voltmeter can be calibrated to indicate the magnitude of the positive-sequence component of either the line potential or the line-to-neutral potential.

The negative-sequence component of line-to-neutral potential can be measured by using the same network as is used for the measurement of the positive-sequence component except that the impedances are interchanged, as indicated in Fig. 18*b*. The current in the voltmeter is now

$$\frac{V_{ab}}{|Z|\underline{/0^\circ}} + \frac{V_{bc}}{|Z|\underline{/60^\circ}}$$

or

$$\frac{1}{|Z|}(V_{ab} + V_{bc}\underline{/-60^\circ})$$

Compare with equation (32), page 53. In this case the voltmeter current is proportional to, and in phase with, the negative-sequence component of the line-to-neutral potential $V_{an}$.

**11a. Correction for the Impedance of the Voltmeter Coil.**—If the impedance of the voltmeter in Fig. 18 is assumed to be $z$, and if the impedances $|Z|\underline{/60^\circ}$ and $|Z|\underline{/0^\circ}$ are replaced by $Z_a$ and $Z_b$, the following equations can be written

$$V_{ab} = I_aZ_a + (I_a + I_b)z$$
$$V_{bc} = I_bZ_b + (I_a + I_b)z$$

The transformers are assumed for convenience to have a ratio of transformation unity. The solution for the current $(I_a + I_b)$ in the voltmeter is

$$I_a + I_b = \frac{V_{ab}Z_b + V_{bc}Z_a}{Z_aZ_b + z(Z_a + Z_b)} = \frac{\dfrac{V_{ab}}{Z_a} + \dfrac{V_{bc}}{Z_b}}{1 + z\dfrac{Z_a + Z_b}{Z_aZ_b}} \tag{39}$$

It will be observed that the correction depends upon the ratio of the voltmeter impedance to the parallel impedance of the two branches of the network. If $Z_a$ and $Z_b$ have their original values of $|Z|\underline{/60^\circ}$ and $|Z|\underline{/0^\circ}$, the denominator in the expression equation (39) for the voltmeter current becomes

$$1 + z\frac{|Z|\underline{/60^\circ} + |Z|\underline{/0^\circ}}{|Z|\underline{/60^\circ}\,|Z|\underline{/0^\circ}} = 1 + \frac{z\sqrt{3}\underline{/-30^\circ}}{|Z|}$$

If the angle of the impedance of the voltmeter is 30 degrees, the denominator becomes a numeric of value $1 + \sqrt{3}\left|\dfrac{z}{Z}\right|$. If desirable, a calibrating impedance can be connected in series with the voltmeter, in which case the impedance $z$ should be taken as the sum of this added impedance and the impedance of the voltmeter. When the angle of this entire impedance is 30 degrees, the current in the voltmeter is in phase with the positive-sequence component $V_{cn1}$ of the phase potential. It should be noted that, if $Z_a$ and $Z_b$ are interchanged, as they are when measuring the negative-sequence potential, the term $\dfrac{Z_a + Z_b}{Z_aZ_b}$ is not altered. That is, the impedance of the voltmeter has the same effect in the measurement of both positive- and negative-sequence potentials.

It is thus possible to correct the error in phase which may be introduced by the impedance of the measuring instruments. It is important to note, however, that the indications of the

instruments can be correct only at one frequency. *Thus all of the instruments used to measure positive- and negative-sequence quantities are subject to frequency errors.*

The effect of the leakage impedances of the potential transformers can be accounted for by considering that these impedances are portions of $Z_a$ and $Z_b$.

**12. Sequence Filter.**—The Δ-connected combination of impedances indicated in Fig. 19 has interesting and useful properties. Assume that positive-sequence potentials are applied between the neutral *n* and the points *a*, *b*, and *c*. Since the potentials $V_{ab}$, $V_{bn}$, and $V_{na}$ are equal in magnitude to the potentials $V_{bc}$, $V_{cn}$, and $V_{nb}$, and since the impedances $Z_{af}$, $Z_{fb}$, and $Z_{fn}$ are equal to the impedances $Z_{bd}$, $Z_{dc}$, and $Z_{dn}$, it is evident that the currents $I_{fn}$ and $I_{dn}$ are equal in magnitude. Furthermore, since the potential $V_{ab}$ leads the potential $V_{bc}$ by 120 degrees, the current $I_{fn}$ leads the current $I_{dn}$ by 120 degrees. Similarly, it can be shown that the current $I_{dn}$ equals in magnitude and leads by 120 degrees the current $I_{en}$. Therefore, if positive-sequence potentials are applied at the terminals *a*, *b*, and *c*, positive-sequence currents *only* will flow in the *equal* load impedances that are connected between the points *d*, *e*, *f*, and the neutral *n*. Likewise, if negative-sequence potentials are applied at the terminals *a*, *b*, and *c*, negative-sequence currents *only* will flow in the *equal* impedances that are connected between the points *d*, *e*, *f*, and the neutral *n*. There is no interrelation whatsoever between the positive and negative sequences. Hence for *all* values of the load impedance *Z*, there is no relation between the negative-sequence potentials at the points *d*, *e*, and *f* and the positive-sequence potentials applied at the points *a*, *b* and *c*. Assume that no load is connected to the points *d*, *e*, and *f*, or that *Z* is infinitely great, and furthermore assume that the impedances $Z_{af}$ and $Z_{bf}$ are equal in magnitude but have angles differing by 60 degrees.

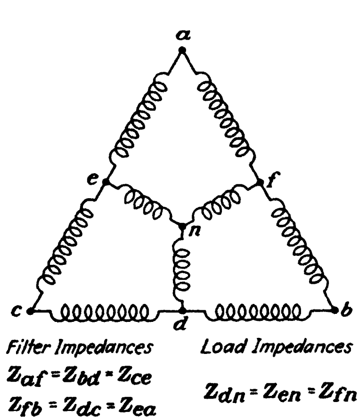



FIG. 19.

First, let $Z_{af} = Z_{bf}\underline{/60°}$. It follows that

$$V_{af} = V_{fb}\underline{/60°}$$

Therefore,

$$V_{af} = \frac{V_{ab}}{\sqrt{3}}\underline{/+30°}$$

The potential $V_{af}$ is the same whether $V_{ab}$ is one of a positive-sequence system of potentials or one of a negative-sequence system of potentials. If positive-sequence potentials are applied at the points $a$, $b$, and $c$, the potential $V_{an1}$ to the neutral $n$ is

$$V_{an1} = \frac{V_{ab1}}{\sqrt{3}}\underline{/-30°}$$

Thus the positive-sequence potential $V_{fn1}$ between $f$ and $n$ is

$$V_{fn1} = -j\frac{V_{ab1}}{\sqrt{3}}$$

If negative-sequence potentials are applied at the points $a$, $b$, and $c$, the potential $V_{an2}$ to the neutral $n$ is

$$V_{an2} = \frac{V_{ab2}}{\sqrt{3}}\underline{/+30°}$$

Thus the *negative-sequence potential* between the neutral $n$ and the point $f$ is *zero* and will remain zero for *all* values of *positive-* or *negative*-sequence potentials applied at the points $a$, $b$, and $c$ and for all values of the load impedance $Z$.

Second, let $Z_{af}\underline{/60°} = Z_{bf}$. In this case

$$V_{af}\underline{/60°} = V_{fb}$$

Therefore

$$V_{af} = \frac{V_{ab}}{\sqrt{3}}\underline{/-30°}$$

It will be observed that the *positive-sequence potential* between the neutral $n$ and the point $f$ is zero; also this potential will remain zero for all values of positive- or negative-sequence potentials applied at the points $a$, $b$, and $c$ and for all values of the load impedance $Z$.

That is to say, the circuit of Fig. 19 can be adjusted to act as a sequence filter if the load connected to the points *d*, *e*, and *f* is balanced. The circuit can be adjusted to pass either positive- or negative-sequence potentials, so that the potentials applied to the load are always balanced. Even in the extreme case in which *single-phase potentials* $V_{ab} = V_{bc} = 0.5V_{ac}$ are applied to the circuit, the potentials at *d*, *e*, and *f* are balanced if the load connected to these points is balanced. In this manner, *balanced* 3-*phase* currents can be obtained from a *single-phase* source through the medium of a *static circuit*.

**13. Resolution of Currents.**—In general, when there is a neutral connection, the sum of the three line currents is not zero

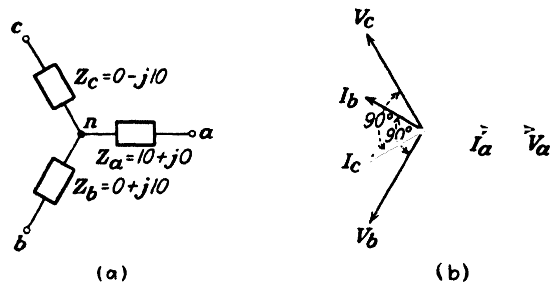



FIG. 20.

but is equal to the current in the neutral conductor or the current to ground in the case of a line fault. In either case the zero-sequence current is one-third of the neutral or the fault current. The zero-sequence component of the line currents and also the positive- and negative-sequence components of the line currents are determined in the same way in which the components of the phase potentials were determined. Let the vectors $I_a$, $I_b$, and $I_c$ represent the line currents flowing toward the neutral point. This neutral point may be either that of the generator which is supplying the load or that of the load itself. In every problem, however, it is essential to indicate clearly the neutral point toward which the current is flowing. A convenient way to indicate this is to place an arrow on the circuit diagram directed toward the particular neutral point chosen.

$$\left.\begin{aligned} I_0 &= \tfrac{1}{3}(I_a + I_b + I_c) \\ I_{a1} &= \tfrac{1}{3}(I_a + I_b\underline{/120^\circ} + I_c\underline{/240^\circ}) \\ I_{a2} &= \tfrac{1}{3}(I_a + I_b\underline{/-120^\circ} + I_c\underline{/-240^\circ}) \end{aligned}\right\} \quad (40)$$

As an illustration of this method [equation (40)] of obtaining the symmetrical components, consider the circuit shown in Fig. 20a, in which three impedances are connected between the lines and neutral conductor of a circuit that has balanced potentials of 100 volts per phase, *i.e.*, from line to neutral. The impedances are

$$Z_a = 10 + j0, \qquad Z_b = 0 + j10, \qquad Z_c = 0 - j10$$

The vector diagram showing the potentials to neutral and the line currents is given in Fig. 20b in which $V_a$ represents the fall in potential from $a$ to $n$, and $I_a$ the current from $a$ to $n$. The vector expressions for the line currents are

$$\begin{aligned} I_a &= 10 + j0 \\ I_b &= -8.66 + j5 \\ I_c &= -8.66 - j5 \end{aligned}$$

Therefore

$$\begin{aligned} I_0 &= \tfrac{1}{3}[10 + j0 - 8.66 + j5 - 8.66 - j5] \\ &= -2.44 + j0 \\ I_{a1} &= \tfrac{1}{3}[10 + j0 + (-8.66 + j5)(-0.5 + j0.866) \\ &\qquad\qquad + (-8.66 - j5)(-0.5 - j0.866)] \\ &= 3.33 + j0 \\ I_{a2} &= \tfrac{1}{3}[10 + j0 + (-8.66 + j5)(-0.5 - j0.866) \\ &\qquad\qquad + (-8.66 - j5)(-0.5 + j0.866)] \\ &= 9.11 + j0 \end{aligned}$$

The load is so badly unbalanced that the negative-sequence current is nearly three times the positive-sequence current. That is, the negative-sequence unbalance factor is 2.74. The zero-sequence unbalance factor is 0.733.

When there is neither neutral connection nor line fault, the sum of the line currents is zero and there is no zero-sequence component. In this case the positive- and negative-sequence components can be obtained by the graphical construction shown in Figs. 14a, b. These components can be obtained analytically by equations similar to (31) and (32). Thus

$$I_{a1} = \frac{\underline{/30^\circ}}{\sqrt{3}}(I_a + I_b\underline{/60^\circ}) \tag{41}$$

$$I_{a2} = \frac{\underline{/-30^\circ}}{\sqrt{3}}(I_a + I_b\underline{/-60^\circ}) \tag{42}$$

The magnitudes of the components can be computed by means of equations (35) and (36), page 54, in which $a$, $b$, and $c$ represent the ampere values of the line currents.

For a Δ-connected load composed, for example, of three single-phase loads connected between the line conductors, the following relations are useful

$$I_{a1} = j\sqrt{3}I_{bc1} \tag{43}$$

$$I_{a2} = -j\sqrt{3}I_{bc2} \tag{44}$$

These are similar to the relations between the line and phase potentials expressed in equations (37) and (38).

As a special case of an unbalanced load, consider a single-phase load connected across lines $b$ and $c$ and taking a current $I$, whence the currents in lines $b$ and $c$ are equal in magnitude and opposite in time-phase. Although the current in line $a$ is zero, its symmetrical components are

$$I_{a1} = \frac{1}{3}(0 + I\underline{/120^\circ} - I\underline{/240^\circ}) = j\frac{I}{\sqrt{3}} \tag{45}$$

$$I_{a2} = \frac{1}{3}(0 + I\underline{/-120^\circ} - I\underline{/-240^\circ}) = -j\frac{I}{\sqrt{3}} \tag{46}$$

The vector diagram is given in Fig. 21.

This case is interesting in that the current in line $a$, although zero, is considered to consist of positive- and negative-sequence components which are equal in magnitude and are in time-phase opposition. This conception forms the basis of the analysis of the single-phase induction motor (see page 235).

It is interesting to note how the analysis of the current taken by a single-phase load can be made from two different points of view. Consider a single-phase load which takes a current $I$ and is connected to the lines $b$-$c$. First, it can be assumed that the load current is part of a Δ system of currents where $I_{ab} = 0$, $I_{bc} = I$, and $I_{ca} = 0$. In this case the zero-, positive-, and negative-sequence components of $I_{bc}$ are each $I/3$. The positive- and negative-sequence components of the line current in the $a$ conductor are, by equations (43) and (44),

$$I_{a1} = j\frac{I}{\sqrt{3}}$$

$$I_{a2} = -j\frac{I}{\sqrt{3}}$$

Second, it can be assumed that the load current is part of a $Y$ system of currents where $I_a = 0$, $I_b = I$, and $I_c = -I$. In this case the positive and negative components of $I_a$ are computed by means of equations (45) and (46).

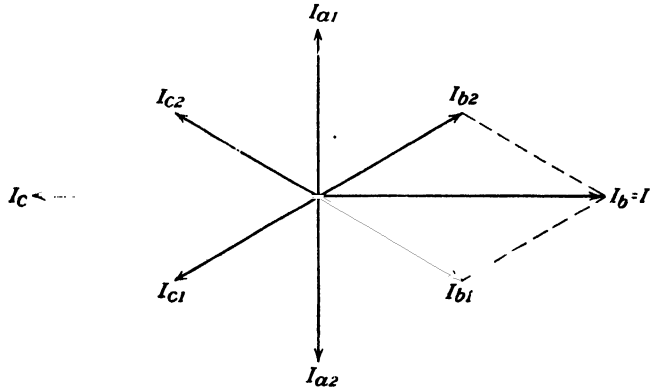


FIG. 21.

**14. Measurement of Symmetrical Components of Line Currents.**—Zero-sequence component of line current is measured by connecting an ammeter across the paralleled secondaries of three current transformers, Fig. 22. The ammeter is calibrated to register one-third the sum of the three line currents, that is to say, it registers the zero-sequence component of the line currents. In a laboratory experiment the zero-sequence component of the line currents can be measured by inserting an ammeter in the neutral conductor. The zero-sequence component is one-third the ammeter reading.

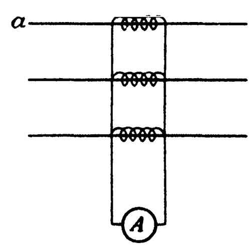


FIG. 22.

If there is no zero-sequence component in the line currents, the positive-sequence component of line current can be measured by connecting an ammeter to two current transformers through the network shown in Fig. 23.

The solution for the current $I$ in the ammeter is obtained from the following equation:

$$(I_a - I)|Z|\underline{/0°} + (I_b - I)|Z|\underline{/60°} = Ir$$

$$I_a + I_b\underline{/60°} = I\left(1 + \underline{/60°} + \frac{r}{|Z|}\right) \tag{47}$$

If the resistance $r$ of the ammeter is negligible as compared with the impedance $Z$, of either branch of the network, the current $I$ in the ammeter is

$$I = \frac{\underline{/-30°}}{\sqrt{3}}(I_a + I_b\underline{/60°})$$

Compare with equation (41). This current $I$ is equal in magnitude to, but in phase opposition with, the positive-sequence component of the line current $I_c$. The ammeter current is subject to such errors as exist in the ratios of transformation of the two current transformers and to the effect of the ratio $r/|Z|$ in equation (47). If $r/|Z|$ is not negligible, both the magnitude and the phase of the ammeter current $I$ will be altered. The error in the *phase* of the current can be corrected by connecting in series with the ammeter an impedance of such a value that the phase angle of the combination of ammeter and added impedance is 30 degrees. If the ratio of the impedance of this combination to the impedance $Z$ of one branch of the network is $k$, the expression for the ammeter current becomes

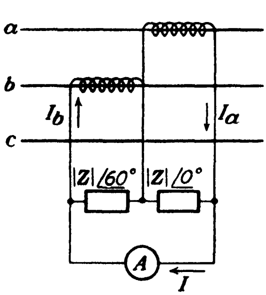


FIG. 23.

$$I = \frac{\underline{/-30°}}{(\sqrt{3} + k)}(I_a + I_b\underline{/60°})$$

Thus the error in the *angle* is corrected. This correction may be important in power measurements. The effect of the *ratio* $k$ can be taken into account by proper calibration.

If there is no zero-sequence component in the line currents, the negative-sequence component of the line current is measured by the ammeter if the two network impedances (Fig. 23) are interchanged. In this case the current in the ammeter is

$$I = \frac{\underline{/-30°}}{\sqrt{3}}(I_b + I_a\underline{/60°})$$

That is,

$$I = \frac{/-30^\circ}{\sqrt{3}}(I_a + I_b/-60^\circ)/60^\circ$$

Compare with equation (42). Thus the current $I$ in the ammeter is equal in magnitude to, and in phase opposition with, the negative-sequence component of the line current $I_c$.

If there is a zero-sequence component in the line currents, the arrangement shown in Fig. 24 can be used to measure the positive-sequence current. Four current transformers are here used. The equation that determines the ammeter current $I$ is

$$(I_b - I_a - I)|Z| + (-I_b + I_c - I)|Z|/60^\circ = Ir$$

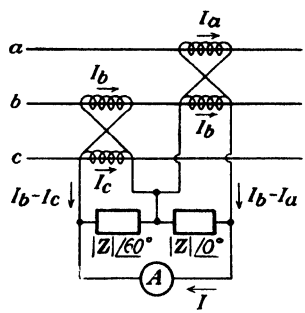


FIG. 24.

This is readily reduced to

$$(-I_a + I_b/-60^\circ + I_c/60^\circ) = I\left(1 + /60^\circ + \frac{r}{|Z|}\right)$$

If the resistance $r$ of the ammeter is small compared with the impedance of each branch of the network, the current $I$ in the ammeter is

$$I = -\frac{/-30^\circ}{\sqrt{3}}(I_a + I_b/120^\circ + I_c/-120^\circ)$$
$$= -\sqrt{3}/-30^\circ I_{a1} = \sqrt{3}I_{a1}/150^\circ$$

Thus the current $I$ in the ammeter is proportional to, but leads by 150 degrees, the positive-sequence component of the current in line $a$.

If the *two network impedances are interchanged*, the current $I$ in the ammeter is

$$I = \sqrt{3}I_{c2}/-30^\circ$$

That is, the current $I$ in the ammeter is now proportional to, but lags by 30 degrees, the negative-sequence component of the current in the line $c$.

**15. Resolution of Impedances.**—Three self-impedances can be resolved into their symmetrical components by the same process that is used for resolving potentials and currents. Consider the three impedances

$$\begin{aligned} Z_a &= 10 + j0 \\ Z_b &= 0 + j10 \\ Z_c &= 0 - j10 \end{aligned}$$

The zero-sequence component of $Z_a$, $Z_b$, or $Z_c$ is

$$\begin{aligned} Z_0 &= \tfrac{1}{3}(10 + j0 + 0 + j10 + 0 - j10) \\ &= 3.33 + j0 \end{aligned}$$

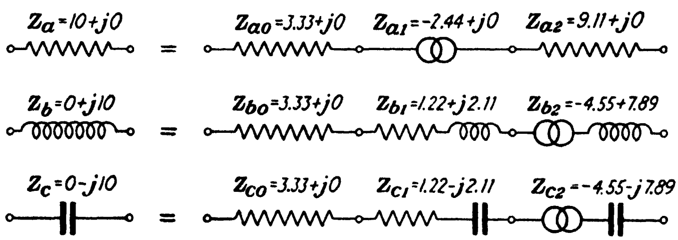



FIG. 25.

The positive-sequence component of $Z_a$ is

$$\begin{aligned} Z_{a1} &= \tfrac{1}{3}[(10 + j0) + (0 + j10)(-0.5 + j0.866) \\ &\qquad + (0 - j10)(-0.5 - j0.866)] \\ &= -2.44 + j0 \end{aligned}$$

The negative-sequence component of $Z_a$ is

$$\begin{aligned} Z_{a2} &= \tfrac{1}{3}[(10 + j0) + (0 + j10)(-0.5 - j0.866) \\ &\qquad + (0 - j10)(-0.5 + j0.866)] \\ &= 9.11 + j0 \end{aligned}$$

Notice that the sum of these three components of $Z_a$ is $(10 + j0)$, as it should be.

That the three actual impedances are equivalent to their symmetrical components is shown diagrammatically in Fig. 25.

A negative resistance indicates a *rise* in emf in phase with, and proportional to, the current in that resistance. The symbol -ↀ- is used to represent this resistance. Notice that $Z_{a1}$, $Z_{b1}$, $Z_{c1}$ are impedances having equal *ohm values* but with resistances and reactances so proportioned that their *vector values* are displaced

120 degrees from one another, with $Z_{b1}$ *lagging* $Z_{a1}$. Similarly, $Z_{a2}$, $Z_{b2}$, $Z_{c2}$, have equal ohm values, but the resistances and reactances are so proportioned that their vector values are displaced 120 degrees from one another, with $Z_{b2}$ *leading* $Z_{a2}$.

**16. Resolution of Admittances.**—Three admittances can also be resolved into symmetrical components by the same process that is used in resolving impedances. Consider the three admittances

$$y_a = 2 + j0$$
$$y_b = 5 + j0$$
$$y_c = 2 + j0$$

The zero-sequence component of each admittance is

$$y_0 = 3 + j0$$

The positive-sequence component of $y_b$ is

$$\begin{aligned} y_{b1} &= \tfrac{1}{3}[5 + 2(-0.5 + j0.866) + 2(-0.5 - j0.866)] \\ &= 1 + j0 \end{aligned}$$

The negative-sequence component of $y_b$ is

$$\begin{aligned} y_{b2} &= \tfrac{1}{3}[5 + 2(-0.5 - j0.866) + 2(-0.5 + j0.866)] \\ &= 1 + j0 \end{aligned}$$

The reason for choosing to resolve $y_b$ rather than either of the other admittances is that the necessary computation is much simpler. The symmetrical components of these three admittances are shown diagrammatically in Fig. 26.

Note that the symmetrical components of $y_a$ and $y_c$ are complex, although $y_a$ and $y_c$ are real. Furthermore, it is important to realize that, although $y_a$ is the reciprocal of $Z_a$, it is not in general true that the symmetrical components of $y_a$ are equal to the reciprocals of the corresponding symmetrical components of $Z_a$. That is, $y_{a0}$ does not equal $1/Z_{a0}$, except when the circuit is balanced. If the circuit is balanced the positive- and negative-sequence components of the admittances and impedances are zero.

It is sometimes convenient to replace a Δ-connected load by its equivalent $Y$-connected load. If $Z_{ab}$, $Z_{bc}$, and $Z_{ca}$ are the impedances of the Δ-connected load and $Z_a$, $Z_b$, and $Z_c$ are the

impedances of the equivalent $Y$-connected load, the well-known relations between these impedances are

$$Z_a = \frac{Z_{ca} \cdot Z_{ab}}{Z_{ab} + Z_{bc} + Z_{ca}}$$
$$Z_b = \frac{Z_{ab} \cdot Z_{bc}}{Z_{ab} + Z_{bc} + Z_{ca}}$$
$$Z_c = \frac{Z_{bc} \cdot Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}}$$

where, for example, $Z_a$, $Z_{ca}$, and $Z_{ab}$ have a common terminal. If the symmetrical components of $Z_{ab}$, $Z_{bc}$, and $Z_{ca}$ are substituted

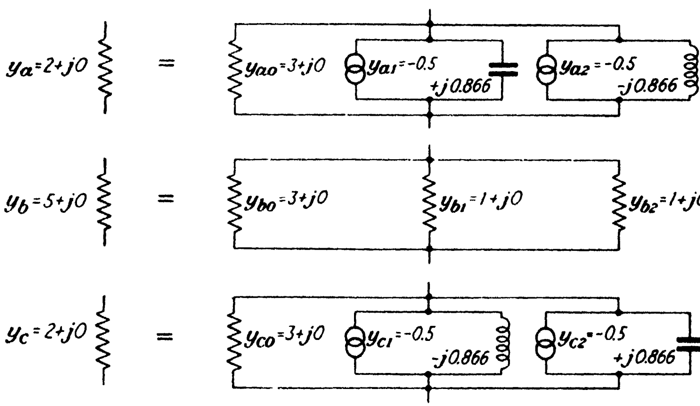


FIG. 26.

in these expressions for the equivalent $Y$-connected impedances and if the symmetrical components of $Z_a$, $Z_b$, and $Z_c$ are then determined by means of the general equations (4$a$), (6$a$), and (7$a$), page 32, the resulting relations between the symmetrical components of the equivalent $Y$- and $\Delta$-connected circuits are

$$\begin{aligned} Z_{a0} &= \frac{Z_{bc0}^2 - Z_{bc1}Z_{bc2}}{3Z_{bc0}} \\ Z_{a1} &= \frac{Z_{bc2}^2 - Z_{bc0}Z_{bc1}}{3Z_{bc0}} \\ Z_{a2} &= \frac{Z_{bc1}^2 - Z_{bc0}Z_{bc2}}{3Z_{bc0}} \end{aligned} \qquad (48)$$

Note how the Sequence Rule applies in these equations.

**17. Resolution of Mutual Reactance.**—Finally, the mutual reactances that may exist among the three phases can also be resolved into symmetrical components by the same process that is used in resolving the self-impedances. Let the mutual reactances between the phases $a$ and $b$, $b$ and $c$, and $c$ and $a$ be $jx_{ab}$, $jx_{bc}$, and $jx_{ca}$. If $x_{ab} = 3$, $x_{bc} = 6$, and $x_{ca} = 9$, the zero-sequence component of $jx_{ab}$ is

$$\begin{aligned} jx_0 &= \tfrac{1}{3}j(3 + 6 + 9) \\ &= j6 \end{aligned}$$

The positive-sequence component $Z_1$ of $jx_{ab}$ is

$$\begin{aligned} Z_1 &= \tfrac{1}{3}j[3 + 6(-0.5 + j0.866) + 9(-0.5 - j0.866)] \\ &= 0.866 - j1\ 5 \end{aligned}$$

The negative-sequence component $Z_2$ of $jx_{ab}$ is

$$\begin{aligned} Z_2 &= \tfrac{1}{3}j[3 + 6(-0.5 - j0.866) + 9(-0.5 + j0.866)] \\ &= -0.866 - j1.5 \end{aligned}$$

The positive- and negative-sequence components of mutual reactances are, in general, complex quantities and should be referred to as impedances.

*In the resolution of quantities into their symmetrical components it is important to note that a system of potentials or currents is considered to be balanced when all but the positive-sequence components are zero; whereas a system of impedances or admittances is balanced when all but the zero-sequence components are zero.*

**18. Vector Volt-amperes.**—Having shown how we may resolve the potential, the current, and the circuit constants into their symmetrical components, we can now discuss those operations which involve the *product* of two vector quantities in terms of their symmetrical components. Let us first consider the vector volt-ampere input. This is a complex quantity whose real part is the average power, $|E||I|\cos\theta$, and whose imaginary part is the reactive power, $|E||I|\sin\theta$, where $|E|$ and $|I|$ are the magnitudes of the potential and current vectors $E$ and $I$. The expression for the vector volt-ampere input in terms of vector quantities is $\underline{E}I$, where the inverted angle sign beneath the potential vector, $E$, indicates the conjugate of $E$ with respect to the horizontal axis, or axis of reals. Let the emf impressed on the circuit at any time, $t$, be represented by the vector

$E = |E|\underline{/\omega t}$ and let the current in the circuit be represented by the vector $I = |I|\underline{/\omega t + \theta}$. Since

$$\underset{\smile}{E} = |E|\underline{/-\omega t}$$

it follows that

$$\begin{aligned} \underset{\smile}{E}I &= |E||I|(\underline{/-\omega t} + \underline{/\omega t + \theta}) \\ &= |E||I|\underline{/\theta} \\ &= |E||I|(\cos\theta + j\sin\theta) \end{aligned} \tag{49}$$

When the vector volt-ampere input is taken as $\underset{\smile}{E}I$, the reactive power is positive if the current *leads* the potential. Vector volt-ampere input has the important property that the total vector volt-ampere input to any network equals the vector sum of the vector volt-ampere inputs to all the branches of the network.

The vector volt-ampere input to phase $a$ of a 3-phase, $Y$-connected circuit is $\underset{\smile}{E}_a I_a$, where $E_a$ represents the potential from the terminal $a$ to the neutral $n$, and $I$ represents the current in the same direction. If both $E_a$ and $I_a$ are expressed in terms of their symmetrical components, this becomes:

$$\underset{\smile}{E}_a I_a = (\underset{\smile}{E}_0 + \underset{\smile}{E}_1 + \underset{\smile}{E}_2)(I_0 + I_1 + I_2)$$

The subscript $a$ has been omitted from the symmetrical components in order to simplify the notation. On performing the multiplication,

$$\begin{aligned} \underset{\smile}{E}_a I_a = \underset{\smile}{E}_0 I_0 + \underset{\smile}{E}_1 I_1 + \underset{\smile}{E}_2 I_2 & \\ + \underset{\smile}{E}_0 I_1 + \underset{\smile}{E}_1 I_2 + \underset{\smile}{E}_2 I_0 & \\ + \underset{\smile}{E}_0 I_2 + \underset{\smile}{E}_1 I_0 + \underset{\smile}{E}_2 I_1 & \end{aligned} \tag{50}$$

The reason for arranging the nine product terms in three lines, as shown, is this. Refer to the Sequence Rule and the Rule for Conjugates (pages 41, 45). Since the conjugate of a first order, or positive-sequence quantity, is a second order, or negative-sequence quantity, and the conjugate of a second order or negative-sequence quantity is a first order or positive-sequence quantity, it will be observed that the sums of the order subscripts in the terms of the first line are 0, 3, and 3. In a 3-phase system the order subscript 3 is equivalent to the subscript 0. Thus the terms in the first line are zero-sequence quantities. In the terms of the second line the sums of the order subscripts are

1, 4, and 1. Since the subscript 4 is equivalent to the subscript 1 in a 3-phase system, the terms in the second line are positive-sequence quantities. In the third line each of the sums of the order subscripts is 2 and thus the terms in this line are negative-sequence quantities. The expression for the vector volt-ampere input in phase $b$, written in terms of the symmetrical components of phase $a$, is

$$\underline{E}_b I_b = \underline{E}_0 I_0 + \underline{E}_1 I_1 + \underline{E}_2 I_2 + (\underline{E}_0 I_1 + \underline{E}_1 I_2 + \underline{E}_2 I_0)\underline{/-120^\circ} + (\underline{E}_0 I_2 + \underline{E}_1 I_0 + \underline{E}_2 I_1)\underline{/120^\circ} \quad (51)$$

A similar expression can be written for the vector volt-ampere input of phase $c$ by interchanging the angles $\underline{/-120^\circ}$ and $\underline{/120^\circ}$ in equation (51). If the load is balanced, all the components of potential and current except $E_1$ and $I_1$ are zero, and the vector volt-ampere input of each phase equals $\underline{E}_1 I_1$. With the unbalanced loads the vector volt-ampere input of the phases is, in general, different. Since the sum of positive-sequence quantities or the sum of negative-sequence quantities for all *three* phases is zero, the *total* vector volt-ampere input for any 3-phase load, whether balanced or unbalanced, is

$$\text{Total } (\underline{E}I) = 3(\underline{E}_0 I_0 + \underline{E}_1 I_1 + \underline{E}_2 I_2) \quad (52)$$

Notice that the only terms which contribute to the *total* vector volt-amperes are those in which the sequence orders of the potential and current are the same. Thus positive-sequence potential and negative-sequence current produce no power, either active or reactive, in the circuit as a whole, although they do contribute to the power of each of the individual phases. That is to say, they contribute to the *unbalancing* of the circuit by increasing the power in one phase above the average power and decreasing the power in another phase below the average power, *i.e.*, above and below the zero-sequence component of power.

Let us assume a single-phase load, which requires a current $I$, connected to a distribution circuit between line conductors $b$ and $c$ at a point where the line potentials $V$ are balanced. If the line potentials are balanced and the currents drawn from the circuit are unbalanced, the positive-sequence components of these currents load the phases equally, whereas the negative-sequence components receive power from one phase and return

power to another phase, so that the total power they receive from the circuit is zero. With a single-phase load requiring a current $I$, the total vector volt-ampere input is $\underline{V}I$. If, for example, the power factor of this load is 0.866 (lagging), it will be observed on constructing a vector diagram that, of the total load of $0.866|V||I|$ watts, phase $a$ takes none, phase $b$ takes $\frac{V}{\sqrt{3}}I$ or two-thirds and phase $c$ takes $\frac{V}{\sqrt{3}}I\cos 60°$ or one-third. Thus, if half the load on phase $b$ can be shifted to phase $a$, the entire load will be equally divided among the three phases.

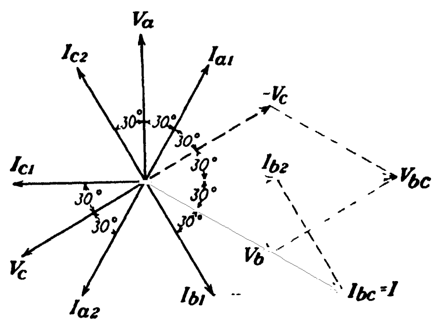



FIG. 27.

If the single-phase current is resolved into its positive- and negative-sequence components as indicated in Fig. 27, the power due to the positive-sequence component of the currents, which is the same in each phase, is $0.866|I/\sqrt{3}||V/\sqrt{3}|$ or one-third the total single-phase power. The power due to the negative-sequence component of the current is $-0.866|I/\sqrt{3}||V/\sqrt{3}|$ for phase $a$; $+0.866|I/\sqrt{3}||V/\sqrt{3}|$ for phase $b$; zero for phase $c$, since the phase angle between $I_{c2}$ and $V_c$ is 90 degrees. This clearly indicates the role of the negative-sequence current in unbalancing the load.

A special and important case is that in which the power is expended in balanced resistances. Since $E_a = RI_a$, $E_b = RI_b$, $E_c = RI_c$, it at once follows that the *total* copper loss is

$$3R(I_0^2 + I_1^2 + I_2^2).$$

It is evident that this loss is not equally divided among the phases and thus one phase will have more heat generated in it than another.

**19. Measurement of Symmetrical Components of Power.**—The symmetrical components of power can be measured by means of a wattmeter of the dynamometer type, if the instrument is combined with the proper networks which insure that the currents in the coils of the wattmeter are not only of the proper sequence but also have the correct relative phase with respect to each other.

The power due to the zero-sequence components of potential and current can be measured by connecting the current coil of the wattmeter in *series* with the zero-sequence ammeter (Fig. 22) and the potential coil (with the usual series resistance) in *parallel* with the zero-sequence voltmeter (Fig. 17). The wattmeter can thus be calibrated to indicate the total power in the three phases due to the zero-sequence components of current and potential.

By definition the zero-sequence components of potential are the same both in magnitude and in time-phase in each of the phase of the circuit. This is also true of the zero-sequence components of current. On the other hand, the positive-sequence components of potential in the several phases of the circuit differ in time-phase by 120 degrees. Therefore, in measuring the power due to the positive-sequence components of potential and current, it is essential that the wattmeter be activated by the components of potential and current in the *same* phase of the circuit. The voltmeter in Fig. 18 indicates the positive-sequence component $V_{cn1}$ of the phase potential. When the system is ungrounded or, in general, when there is no zero-sequence component of current, the ammeter in Fig. 23 indicates the positive-sequence component $I_{c1}$ of the line current. Therefore, if the potential coil (*without* the usual series resistance) of the wattmeter is connected in series with the voltmeter coil, as in Fig. 18, and the current coil of the wattmeter is connected in series with the ammeter, as in Fig. 23, the wattmeter is activated by positive-sequence potential and current in the same phase of the circuit. The wattmeter can then be calibrated to indicate the total power in the three phases of the circuit due to the positive-sequence components of potential and current.

If, in Fig. 18 and also in Fig. 23, the network impedances $|Z|\underline{/0°}$ and $|Z|\underline{/60°}$ are interchanged for the purpose of measuring the negative-sequence potential and current, it has been shown (page 58) that the voltmeter indicates the negative-

sequence potential $V_{an2}$, whereas the ammeter indicates the negative-sequence current $I_{c2}$. A wattmeter whose coils were connected in series with this voltmeter and ammeter would be activated by potential and current in different phases of the circuit and thus would not indicate correctly the power due to the negative-sequence potential and current. The proper phase relation between the potential and current can be obtained if the current transformers are inserted in lines *b* and *c* as in Fig. 28 instead of in lines *a* and *b* as in Fig. 23. Advancing the position of the current transformers from *a* to *b* and from *b* to *c* advances the current in the negative-sequence ammeter from $I_{c2}$ to $I_{a2}$.

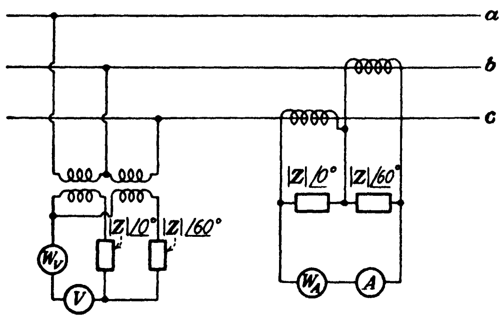



FIG. 28.

Thus, with the potential and current transformers arranged as shown in Fig. 28, the currents in the coils $W_V$ and $W_A$ of the wattmeter are proportional to the negative-sequence potential and current in phase *a* of the circuit. The wattmeter can then be calibrated to indicate the total power due to the negative-sequence potential and current in the circuit. In this measurement it is also assumed that there is no zero-sequence current.

When there is a zero-sequence component in the line currents, it is necessary to use four current transformers arranged as shown in Fig. 24. It has been shown (page 67) that the ammeter then indicates $\sqrt{3}I_{a1}\angle 150°$. If the positive-sequence component of potential is obtained as shown in Fig. 18, the voltmeter indicates $3V_{cn1}$ or $3V_{an1}\angle 120°$. On account of the angular difference of 30 degrees which appears in these expressions for current and potential, a wattmeter whose current coil replaces the ammeter of Fig. 24 and whose potential coil replaces the voltmeter of Fig. 18 does *not* indicate the power due to the posi-

tive-sequence components of potential and current. If the wattmeter is to indicate correctly the power due to the positive-sequence component of potential and current, the potential transformers should be connected in a manner analogous to that in which the current transformers are connected in Fig. 24. This method of connecting the transformers is shown in Fig. 29. The secondaries of the transformers are connected in series

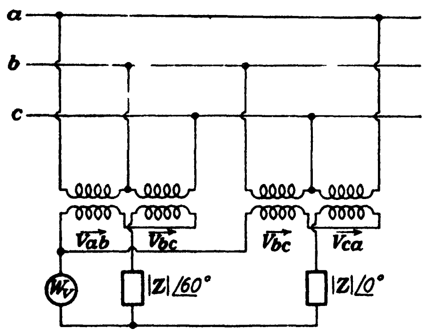



FIG. 29.

opposition. If the voltmeter impedance is negligible as compared with $|Z|$, the current through the voltmeter, assuming that it replaces the potential coil $W_V$ of the wattmeter, is proportional to

$$\frac{(V_{bc} - V_{ab})}{|Z|\underline{/60^\circ}} + \frac{(V_{ca} - V_{bc})}{|Z|\underline{/0^\circ}}$$

This expression is readily reduced to $\frac{3}{|Z|}V_{ca1}$, which in turn equals

$$\frac{3}{|Z|}\sqrt{3}V_{an1}\underline{/150^\circ}$$

That is, the current in the voltmeter is proportional to, and leads the positive-sequence component of $V_{an}$ by 150 degrees. This corresponds to the ammeter current, Fig. 24, which leads the positive-sequence current in line $a$ by 150 degrees. Thus if the potential coil of the wattmeter is inserted in the network, as shown in Fig. 29, and the current coil is inserted in series with the ammeter, as in Fig. 24, the wattmeter can be calibrated to indicate correctly the power due to the positive-sequence components of potential and current.

The arrangement of the networks and instruments for measuring the power due to the negative-sequence components of potential and current, when there *is* a zero-sequence component of current, is shown in Fig. 30. The current coil of the wattmeter is connected in series with the ammeter, as in Fig. 24, except that the network impedances in Fig. 24 are interchanged. It has been shown (page 67) that the ammeter then indicates $\sqrt{3}I_{c2}\underline{/-30°}$ if the impedance of the instrument coils is negligible. If the impedance of the voltmeter and potential coil is negligible

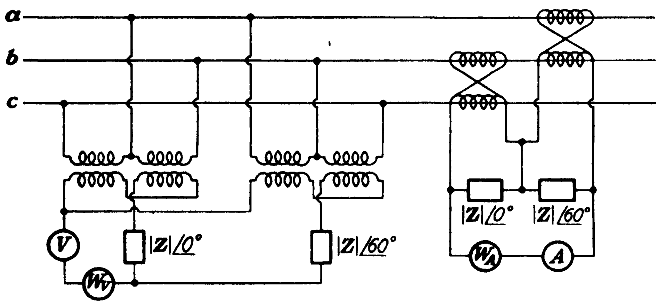



FIG. 30.

as compared with $|Z|$, the current through them is proportional to

$$\frac{V_{ca} - V_{ab}}{|Z|\underline{/0°}} + \frac{V_{ab} - V_{bc}}{|Z|\underline{/60°}}$$

This expression is readily reduced to $\frac{3\sqrt{3}}{|Z|}V_{cn2}\underline{/-30°}$. Thus the currents in the potential and current coils of the wattmeter have the proper phase relation for measuring the power due to the negative-sequence components of potential and current.

**20. Power Pulsation.**—If the emf impressed on a circuit is $e = \sqrt{2}|E| \sin \omega t$ and the current is $i = \sqrt{2}|I| \sin(\omega t + \theta)$, the instantaneous value of the power is

$$p = |E||I|[\cos \theta - \cos(2\omega t + \theta)].$$

That is, the instantaneous power consists of a constant term, $|E||I| \cos \theta$, which is the average power of the circuit, and a sinusoidal term of double frequency, $|E||I| \cos(2\omega t + \theta)$, which represents a pulsation of power into and out of the circuit.

Note that the *maximum* value of this power pulsation always equals the product of the emf and current, that is, it equals the *magnitude* of the vector volt-ampere input independently of the power factor. Also note that the phase $(2\omega t + \theta)$ of the pulsating power is the sum of the phase $(\omega t)$ of the emf and the phase $(\omega t + \theta)$ of the current. In the same sense in which a sinusoidal potential or current is represented by a rotating vector of fundamental frequency, the pulsating power can be represented by a *rotating* vector of double frequency. The length of this vector equals the *maximum* value of the power pulsation. The magnitude of this pulsating power vector is the same as that of the vector representing the vector volt-ampere input. It is important to note, however, that, whereas the power pulsation is a rotating vector of double frequency, the vector volt-ampere input is a fixed or nonrotating vector. Since the pulsating power is the *sinusoidal* portion of the instantaneous power, we shall represent it by the vector $S$. Note that the angle which $S$ makes with the reference axis is the sum of the angles which the emf and current vectors make with the same axis. From this it follows that $S = EI$. If the instantaneous value of the pulsating power is to be obtained by projecting $S$ on the same axis on which $E$ and $I$ are projected to obtain their instantaneous values, we should write $S = -jEI$. This product $EI$ is in general a complex quantity, that is, the pulsating power is represented by a vector which equals the complex product of the vectors representing the potential and current. The total pulsating power in the several branches of a circuit is then the *vector* sum of the pulsating powers in the individual branches. When $E$ and $I$ are the potential and current in phase $a$, and are expressed in terms of their symmetrical components, the pulsating power in phase $a$ is

$$\begin{aligned} S_a = E_0I_0 + E_1I_2 + E_2I_1 \\ + E_0I_1 + E_1I_0 + E_2I_2 \\ + E_0I_2 + E_1I_1 + E_2I_0 \end{aligned} \tag{53}$$

On applying the Sequence Rule, it will be observed that the first line consists of zero-sequence terms, the second line of positive-sequence terms, and the third line of negative-sequence terms. The total power pulsation in all three phases is

$$\text{Total } S = 3(E_0I_0 + E_1I_2 + E_2I_1) \tag{54}$$

Note that, except for the product of the zero-sequence terms, the power pulsation is due to the action of potentials and currents of different phase orders. For a balanced load, all the components except $E_1$ and $I_1$ are zero and the pulsating power is zero. That is, energy is supplied to a balanced polyphase load at a constant rate.

When the power supplied to a rotating machine is pulsating, particular care must be taken in mounting the machine in order to minimize the vibration and the attendant noise produced by the power pulsation.

As a simple illustration of the method of determining the vector volt-ampere input and the pulsating power, consider the circuit given in Fig. 20. We shall focus our attention on phase $a$ and omit writing the subscript $a$, thus simplifying our notation. Since the potentials applied to the load are balanced,

$$\begin{aligned} E_0 &= 0 \\ E_1 &= 100 + j0 \\ E_2 &= 0 \end{aligned}$$

The zero-, positive-, and negative-sequence components of the current in phase $a$ were found to be (page 63).

$$\begin{aligned} I_0 &= -2.44 + j0 \\ I_1 &= 3.33 + j0 \\ I_2 &= 9.11 + j0 \end{aligned}$$

The vector volt-ampere input to phase $a$ is thus [equations (50), page 72]

$$\begin{aligned} \underset{\smile}{E}_a I_a = {} & 0 + 100 \times 3.33 + 0 \\ & 0 + 100 \times 9.11 + 0 \\ & 0 + 100 \times (-2.44) + 0 \\ = {} & 1000 + j0 \end{aligned}$$

In phase $a$ the active (average) power is 1000 *watts*, while the reactive power is zero.

The vector volt-ampere input to phase $b$ is [see equations (51), page 73]

$$\begin{aligned} \underset{\smile}{E}_b I_b = {} & 100 \times 3.33 + 100 \times 9.11 \times (-0.5 - j0.866) \\ & + 100 \times (-2.44)(-0.5 + j0.866) = 0 - j1000 \end{aligned}$$

In phase $b$ the active (average) power is zero and the reactive power is 1000 var. The minus sign indicates that the reactive power is due to a current which *lags* the potential.

The vector volt-ampere input to phase $c$ is

$$\underline{E}_c I_c = 100 \times 3.33 + 100 \times 9.11 \times (-0.5 + j0.866) + 100 \times (-2.44)(-0.5 - j0.866)$$
$$= 0 + j1000$$

In phase $c$ the active (average) power is zero and the reactive power is 1000 var., but is due to a leading current, as shown by the plus sign.

The total vector volt-ampere input is

$$\underline{E}_a I_a + \underline{E}_b I_b + \underline{E}_c I_c = (1000 + j0) + (0 - j1000) + (0 + j1000)$$
$$= 1000 + j0$$

Or, by equation (52),

$$\text{Total } (\underline{E}I) = 3 \times 100 \times 3.33$$
$$= 1000 + j0 \text{ watts}$$

The power pulsation in phase $a$ is [equation (53), page 79]

$$\begin{aligned} S_a &= 0 + 100 \times 9.11 + 0 \\ &\quad 0 + 100 \times (-2.44) + 0 \\ &\quad 0 + 100 \times 3.33 + 0 \\ &= 1000 \text{ watts (maximum)} \end{aligned}$$

The power pulsation in phase $b$ is

$$S_b = 100 \times 9.11 + 100 \times (-2.44)(-0.5 - j0.866) + 100 \times 3.33(-0.5 + j0.866)$$
$$= 867 + j500 \text{ watts (maximum)}$$

This shows that the pulsating power in phase $b$ is equal to, but *leads* by 30 degrees, the pulsating power in phase $a$.

The pulsating power in phase $c$ is

$$S_c = 100 \times 9.11 + 100 \times (-2.44)(-0.5 + j0.866) + 100 \times 3.33(-0.5 - j0.866)$$
$$= 867 - j500 \text{ watts (maximum)}$$

The pulsating power in phase $c$ is equal to, but *lags* by 30 degrees, the pulsating power in phase $a$.

The total pulsating power is

$$E_aI_a + E_bI_b + E_cI_c = (1000 + j0) + (867 + j500) + (867 - j500)$$
$$= 2733 \text{ watts } (\textit{maximum})$$

Or, by equation (54),

$$\text{Total } S = 3(100 \times 9.11)$$
$$= 2733 \text{ watts } (\textit{maximum})$$

This is nearly three times the average power supplied to the circuit.

**21. Fall in Potential Due to Self-impedance.**—In writing the Kirchhoff equations for potential, it is frequently convenient to express the potential between two points as the product of current and impedance. Let the three currents $I_a$, $I_b$, and $I_c$ flow in the three impedances $Z_a$, $Z_b$, and $Z_c$. These may be the impedances of the transmission-line conductors or they may be the impedances of a $Y$-connected load. The problem of the line conductors is discussed at length in Chapter IV. Also let the symmetrical components of the current $I_a$ be $I_0$, $I_1$, and $I_2$, and the corresponding symmetrical components of $Z_a$ be $Z_0$, $Z_1$, and $Z_2$. The impedance drop in $Z_a$ is

$$\begin{aligned} I_aZ_a = {} & I_0Z_0 + I_1Z_2 + I_2Z_1 \\ & + I_0Z_1 + I_1Z_0 + I_2Z_2 \\ & + I_0Z_2 + I_1Z_1 + I_2Z_0 \end{aligned}$$

The Sequence Rule again operates just as when we derived the expression for the power pulsation. It shows that the first line consists of zero-sequence terms; the second, of positive-sequence terms; and the third, of negative-sequence terms. Thus we may write

$$\left.\begin{aligned} V_0 &= I_0Z_0 + I_1Z_2 + I_2Z_1 \\ V_1 &= I_0Z_1 + I_1Z_0 + I_2Z_2 \\ V_2 &= I_0Z_2 + I_1Z_1 + I_2Z_0 \end{aligned}\right\} \quad (55)$$

where $V_0$, $V_1$, and $V_2$ are the zero-, positive-, and negative-sequence components of the potential drop in the direction of the current through the impedance $Z_a$. For example, note that the positive-sequence current $I_1$ and the negative-sequence

impedance $Z_2$ produce a zero-sequence component in the potential drop and that the positive-sequence components in current and impedance produce a negative-sequence component in the potential drop. In general, each of these components of potential drop in equations (55) is due in part to each of the symmetrical components of the current. If, however, the impedances $Z_a$, $Z_b$, and $Z_c$ are exactly alike, *i.e.*, if they are balanced, they will have neither positive- nor negative-sequence components, and the zero-, positive-, and negative-sequence components of potential drop will then be proportional, respectively, to the zero-, positive-, and negative-sequence components of current.

Consider the case where $Z_a$, $Z_b$, and $Z_c$ are the impedances of a $Y$-connected load without a neutral connection, and assume that the potentials impressed across the terminals of this load are known. The analysis of these line potentials (page 53) at once determines the positive- and negative-sequence components of the potential to neutral *without regard to the particular values of the load impedances $Z_a$, $Z_b$, and $Z_c$*. Since we are assuming that there is no neutral connection, the zero-sequence current $I_0$ is zero. If the positive- and negative-sequence components of $V_a$ are $V_1$ and $V_2$, the general equations reduce to:

$$\begin{aligned} V_0 &= I_1Z_2 + I_2Z_1 \\ V_1 &= I_1Z_0 + I_2Z_2 \\ V_2 &= I_2Z_0 + I_1Z_1 \end{aligned}$$

From the second and third of these equations we can determine the positive- and negative-sequence currents, and then, by substituting these values in the first equation, the zero-sequence component, $V_0$, of the phase potential can be obtained. Observe that, if the load impedances are balanced, the zero-sequence component of the phase potential is zero. Thus, if a balanced impedance load is connected in parallel with the unbalanced one, the potential between the neutral points of the two loads is the zero-sequence component of the phase potential of the unbalanced load. For this reason the zero-sequence component of the phase potential is referred to as the displacement of the neutral of an unbalanced load. The true neutral is that of the balanced impedances.

Notice that, when balanced potentials ($V_2 = 0$) are impressed on an unbalanced $Y$-connected load having no neutral connection, the ratio of the negative- to the positive-sequence current is $-Z_1/Z_0$. Consider the following numerical illustration: Let $Z_a$, $Z_b$, and $Z_c$ be $10\underline{/0^\circ}$, $10\underline{/60^\circ}$, and $10\underline{/-60^\circ}$, respectively, and let each of the line potentials be 173.2 volts.

The symmetrical components of $Z_a$ are

$$\begin{aligned} Z_0 &= 6.67 + j0 \\ Z_1 &= -3.33 + j0 \\ Z_2 &= 6.67 + j0 \end{aligned}$$

Also we may take

$$\begin{aligned} V_1 &= 100 + j0 \\ V_2 &= 0 \end{aligned}$$

Thus

$$\begin{aligned} 100 + j0 &= 6.67I_1 + 6.67I_2 \\ 0 &= 6.67I_2 - 3.33I_1 \end{aligned}$$

from which it follows that

$$\begin{aligned} I_1 &= 10 + j0 \\ I_2 &= 5 + j0 \\ V_0 &= 50 + j0 \end{aligned}$$

The resulting current in phase $a$ is

$$\begin{aligned} I_a &= (10 + j0) + (5 + j0) \\ &= 15 + j0 \end{aligned}$$

The resulting current in phase $b$ is

$$\begin{aligned} I_b &= (10 + j0)(-0.5 - j0.866) + (5 + j0)(-0.5 + j0.866) \\ &= -7.5 - j4.33 \end{aligned}$$

The resulting current in phase $c$ is

$$\begin{aligned} I_c &= (10 + j0)(-0.5 + j0.866) + (5 + j0)(-0.5 - j0.866) \\ &= -7.5 + j4.33 \end{aligned}$$

**22. $Y$- and $\Delta$-connected Loads.**—In writing the Kirchhoff equations for current, it is frequently convenient to express the current as the product of potential and admittance. First we shall consider a $Y$-connected load, the admittances of whose branches are $y_a$, $y_b$, and $y_c$. Let the potentials across these

branches be $E_a$, $E_b$, and $E_c$. Resolve these admittances and potentials into their symmetrical components so that we may write

$$\begin{aligned} y_a &= y_0 + y_1 + y_2 \\ E_a &= E_0 + E_1 + E_2 \end{aligned}$$

Note that the positive- and negative-sequence components of $E_a$ are determined by the line potentials across the terminals of the load without regard to the character of the load itself. The zero-sequence component, $E_0$, however, is determined by the character of the load if there is no neutral connection, or by the existing potentials between the line and neutral conductors if there is a neutral connection.

The line current, $I_a$, is

$$\begin{aligned} I_a = E_a y_a = {} & E_0 y_0 + E_1 y_2 + E_2 y_1 \\ & + E_0 y_1 + E_1 y_0 + E_2 y_2 \\ & + E_0 y_2 + E_1 y_1 + E_2 y_0 \end{aligned}$$

The first line consists of zero-sequence components; the second line, of positive-sequence components; the third line, of negative-sequence components. That is

$$\left.\begin{aligned} I_0 &= E_0 y_0 + E_1 y_2 + E_2 y_1 \\ I_1 &= E_0 y_1 + E_1 y_0 + E_2 y_2 \\ I_2 &= E_0 y_2 + E_1 y_1 + E_2 y_0 \end{aligned}\right\} \tag{56}$$

If there is a neutral connection, the three symmetrical components of the applied potential are known, and the components of the current are directly determined. Where there is no neutral connection, there is no zero-sequence component of the current and

$$E_0 y_0 + E_1 y_2 + E_2 y_1 = 0$$

and

$$E_0 = -\frac{E_1 y_2 + E_2 y_1}{y_0}$$

If this value of $E_0$ is substituted in the equations for the positive- and negative-sequence currents,

$$\left.\begin{aligned} I_1 &= \frac{E_1(y_0^2 - y_1 y_2) + E_2(y_0 y_2 - y_1^2)}{y_0} \\ I_2 &= \frac{E_1(y_0 y_1 - y_2^2) + E_2(y_0^2 - y_1 y_2)}{y_0} \end{aligned}\right\} \tag{57}$$

When there is no neutral connection the $Y$-connected load can be replaced by its equivalent $\Delta$-connected load. The positive- and negative-sequence line currents are somewhat simpler in form when expressed in terms of the potentials to neutral and the admittances of the $\Delta$ branches. Let $y_{ab}$, $y_{bc}$, and $y_{ca}$ be the admittances of the equivalent $\Delta$-connected load or of an actual load. The simplest method of determining the positive- and negative-sequence components of the line current $I_a$ is as follows:

Let $y_0$, $y_1$, and $y_2$ represent the symmetrical components of the admittance $y_{bc}$, and let $E_{bc1}$ and $E_{bc2}$ be the positive- and negative-sequence components of the line potential $E_{bc}$. Then

$$\left.\begin{aligned} I_{bc1} &= E_{bc1}y_0 + E_{bc2}y_2 \\ I_{bc2} &= E_{bc2}y_0 + E_{bc1}y_1 \end{aligned}\right\} \tag{58}$$

The positive- and negative-sequence components of the line current $I_a$ are in quadrature with the corresponding components of the current $I_{bc}$ (page 64). A similar relation holds for the potential to neutral $E_a$ and the line potential $E_{bc}$ (page 55). Thus

$$I_{a1} = j\sqrt{3}I_{bc1} \tag{59}$$

and

$$I_{a2} = -j\sqrt{3}I_{bc2} \tag{60}$$

Similarly,

$$E_{a1} = j\frac{1}{\sqrt{3}}E_{bc1} \tag{61}$$

$$E_{a2} = -j\frac{1}{\sqrt{3}}E_{bc2} \tag{62}$$

If these values are substituted in equations (58), we have:

$$\left.\begin{aligned} I_{a1} &= 3E_{a1}y_0 - 3E_{a2}y_2 \\ I_{a2} &= 3E_{a2}y_0 - 3E_{a1}y_1 \end{aligned}\right\} \tag{63}$$

Note that, if there is a single load of admittance $y$ (the symmetrical components of this admittance are each $\frac{1}{3}y$) connected across the terminals $b$ and $c$ only, the components of the current in line $a$ are

$$\left.\begin{aligned} I_{a1} &= (E_{a1} - E_{a2})y \\ I_{a2} &= (E_{a2} - E_{a1})y \end{aligned}\right\} \tag{64}$$

The sum of the positive- and negative-sequence component currents is zero. That is, the actual current in line $a$ is zero.

**Illustration.**—The line potentials $V_{ab}$, $V_{bc}$, and $V_{ca}$ are, respectively, 155.5, 220, and 155.5 volts. Three lamp loads, whose admittances $y_{ab}$, $y_{bc}$, and $y_{ca}$ are 0.4, 0.1, and 0.4 mho, are connected to the circuit. The geometrical construction given in Fig. 14 shows that the positive- and negative-sequence components of the potential to neutral, $E_a$, are $100.5 + j0$ and $-27.2 + j0$. The zero-, positive-, and negative-sequence components of the load admittance $y_{bc}$ are $0.3 + j0$, $-0.1 + j0$, and $-0.1 + j0$. It follows that the positive- and negative-sequence components of the line current $I_a$ are

$$\begin{aligned} I_{a1} &= 3[100.5 \times 0.3 - (-27.2)(-0.1)] \\ &= 82.3 + j0 \end{aligned}$$

and

$$\begin{aligned} I_{a2} &= 3[(-27.2)0.3 - 100.5(-0.1)] \\ &= 5.7 + j0 \end{aligned}$$

The usual computation shows that the line current flowing in at terminal $a$ is 88 amperes.

**23. Emf of Mutual Induction.**—Let the mutual reactance couplings between the lines of the circuit be arranged so that, if $I$ represents the current in any conductor flowing *toward* the *load*, the mutually induced fall in potential in either of the other conductors measured in the same direction is $jxI$. This mutual reactance $x$ may be either positive or negative. The fall in potential in line $a$ is

$$E_a = jx_{ab}I_b + jx_{ac}I_c$$

It is much better to express this potential in terms of the components of the current in line $a$ and the components of the reactance between two of the lines only. Brief consideration indicates that the components of reactance are probably those of $x_{bc}$. For this reason let

$$jx_{bc} = Z_0 + Z_1 + Z_2$$

Then

$$\begin{aligned} jx_{ab} &= Z_0 + Z_1\underline{/120°} + Z_2\underline{/-120°} \\ jx_{ac} &= Z_0 + Z_1\underline{/-120°} + Z_2\underline{/120°} \end{aligned}$$

Similarly, if $I_a = I_0 + I_1 + I_2$,

$$I_b = I_0 + I_1\underline{/-120°} + I_2\underline{/120°}$$
$$I_c = I_0 + I_1\underline{/120°} + I_2\underline{/-120°}$$

The zero-sequence component of $E_a$ is the sum of the zero-sequence components of $jx_{ab}I_b$ and $jx_{ac}I_c$. Thus

$$E_{a0} = Z_0I_0 + Z_1I_2\underline{/240°} + Z_2I_1\underline{/-240°} + Z_0I_0 + Z_1I_2\underline{/-240°} + Z_2I_1\underline{/240°}$$

From which it at once follows that

$$E_{a0} = 2Z_0I_0 - Z_1I_2 - Z_2I_1 \tag{65}$$

Similarly, it may be shown that

$$E_{a1} = -Z_0I_1 - Z_1I_0 + 2Z_2I_2 \tag{66}$$

and

$$E_{a2} = -Z_0I_2 + 2Z_1I_1 - Z_2I_0 \tag{67}$$

**Illustration.**—Between the lines $b$ and $c$ there is a mutual-reactance coupling of 0.9 ohm. The other mutual reactances are zero. The self-reactance in line $a$ is zero and in each of the lines $b$ and $c$ is 1.05 ohms. There is negligible resistance in each of the three lines. The zero-, positive-, and negative-sequence components of the self-reactance in line $a$ are $j0.7$, $-j0.35$, and $-j0.35$. The corresponding components of the mutual reactance between lines $b$ and $c$ are $j0.3$, $j0.3$, and $j0.3$. Without neutral connection, there are only positive- and negative-sequence components, $I_1$ and $I_2$, of the current in line $a$. For these values of current, the zero-, positive-, and negative-sequence components of the fall in potential in line $a$ are

$$\begin{aligned} E_{a0} &= -j0.35I_2 - j0.35I_1 - j0.3I_2 - j0.3I_1 \\ &= -j0.65I_1 - j0.65I_2 \\ E_{a1} &= +j0.7I_1 - j0.35I_2 - j0.3I_1 + j0.6I_2 \\ &= +j0.4I_1 + j0.25I_2 \\ E_{a2} &= j0.7I_2 - j0.35I_1 - j0.3I_2 + j0.6I_1 \\ &= j0.25I_1 + j0.4I_2 \end{aligned}$$

Observe that the total drop in line $a$ is

$$E_a = E_{a0} + E_{a1} + E_{a2} = 0$$

**24. General Static Circuit Having Self- and Mutual Impedance.**—We have seen in equation (55) how we may express the symmetrical components of the fall in potential through self-impedances in terms of the symmetrical components of these impedances and the symmetrical components of the currents in them. We have also seen in equations (65), (66), and (67) how we may express the symmetrical components of the fall in potential due to mutual reactance in terms of the symmetrical components of these reactances and the symmetrical components of the currents in them. If these two sets of equations are combined, we shall have the complete expressions for the fall in potential in a circuit whose phases are magnetically coupled. Thus

$$\left.\begin{aligned} V_{a0} &= I_{a0}(Z_{aa0} + 2Z_{bc0}) + I_{a1}(Z_{aa2} - Z_{bc2}) + I_{a2}(Z_{aa1} - Z_{bc1}) \\ V_{a1} &= I_{a0}(Z_{aa1} - Z_{bc1}) + I_{a1}(Z_{aa0} - Z_{bc0}) + I_{a2}(Z_{aa2} + 2Z_{bc2}) \\ V_{a2} &= I_{a0}(Z_{aa2} - Z_{bc2}) + I_{a1}(Z_{aa1} + 2Z_{bc1}) + I_{a2}(Z_{aa0} - Z_{bc0}) \end{aligned}\right\} \qquad (68)$$

It is important that we should consider the composition of these equations in some detail. While $Z_{aa0}$, $Z_{aa1}$, and $Z_{aa2}$ are the zero-, positive-, and negative-sequence components of the self-impedance $Z_{aa}$, and while $Z_{bc0}$, $Z_{bc1}$, and $Z_{bc2}$ are the zero-, positive-, and negative-sequence components of the mutual impedance $Z_{bc}$, the combinations of these symmetrical components which form the coefficients of the symmetrical components of the current in equations (68) cannot properly be called the symmetrical components of any set of three impedances whatsoever. The reason for this lies in the Sequence Rule. The coefficient of $I_{a1}$ in the first equation of (68) and the coefficient of $I_{a2}$ in the second equation both should be negative-sequence components. Since these two coefficients are different, it is not possible for them both to be negative-sequence components of any set of three impedances whatsoever.

It is possible, however, to adjust a 3-phase circuit, *without* mutual impedance between its phases, so that the circuit is in *some* respects equivalent to another 3-phase circuit with mutual impedance between its phases. For example, consider a $Y$-connected circuit, the neutral point of which is free. The zero-sequence component $I_0$ of the currents is zero, and the general equations (68) reduce to

$$
\begin{aligned}
V_0 &= I_1(Z_{aa2} - Z_{bc2}) + I_2(Z_{aa1} - Z_{bc1}) \\
V_1 &= I_1(Z_{aa0} - Z_{bc0}) + I_2(Z_{aa2} + 2Z_{bc2}) \\
V_2 &= I_1(Z_{aa1} + 2Z_{bc1}) + I_2(Z_{aa0} - Z_{bc0})
\end{aligned}
$$

The equations that apply to another $Y$-connected circuit, *without* mutual impedance between its phases, are

$$
\begin{aligned}
V_0 &= I_1 Z_{a2} + I_2 Z_{a1} \\
V_1 &= I_1 Z_{a0} + I_2 Z_{a2} \\
V_2 &= I_1 Z_{a1} + I_2 Z_{a0}
\end{aligned}
$$

It will be observed that if

$$
\begin{aligned}
Z_{a0} &= Z_{aa0} - Z_{bc0} \\
Z_{a1} &= Z_{aa1} + 2Z_{bc1} \\
Z_{a2} &= Z_{aa2} + 2Z_{bc2}
\end{aligned}
$$

the two circuits are identical with respect to the positive- and negative-sequence components of potential and current, but are not identical with respect to the zero-sequence component of potential. That is, if the same 3-phase potentials are applied to these two circuits, the positive- and negative-sequence components of current in one circuit are equal to the positive- and negative-sequence components of current in the other circuit. The zero-sequence components of potential, that is, the displacements of the neutral points, are *not* equal in the two circuits.

If the preceding equations for $Z_{a0}$, $Z_{a1}$, $Z_{a2}$ are expanded and written in terms of the impedances themselves,

$$
\begin{aligned}
Z_a + Z_b + Z_c &= Z_{aa} + Z_{bb} + Z_{cc} - (Z_{bc} + Z_{ca} + Z_{ab}) \\
Z_a + Z_b\underline{/120} + Z_c\underline{/-120} &= Z_{aa} + Z_{bb}\underline{/120} + Z_{cc}\underline{/-120} \\
&\quad + 2(Z_{bc} + Z_{ca}\underline{/120} + Z_{ab}\underline{/-120}) \\
Z_a + Z_b\underline{/-120} + Z_c\underline{/120} &= Z_{aa} + Z_{bb}\underline{/-120} + Z_{cc}\underline{/120} \\
&\quad + 2(Z_{bc} + Z_{ca}\underline{/-120} + Z_{ab}\underline{/120})
\end{aligned}
$$

On adding these three relations among the impedances,

$$Z_a = Z_{aa} + Z_{bc} - Z_{ca} - Z_{ab}$$

On advancing the subscripts one letter,

$$Z_b = Z_{bb} + Z_{ca} - Z_{ab} - Z_{bc}$$

Again on advancing the subscripts one letter,

$$Z_c = Z_{cc} + Z_{ab} - Z_{bc} - Z_{ca}$$

If $V_0'$ is the zero-sequence component of the potentials for the circuit *without* mutual impedance between its phases, and $V_0''$ is the zero sequence component of the potentials for the circuit *with* mutual impedance between its phases, it is readily shown that

$$V_0' - V_0'' = 3I_1Z_{bc2} + 3I_2Z_{bc1}$$

Therefore, it is possible to replace a 3-phase circuit *with* mutual impedance between its phases by an equivalent 3-phase circuit *without* mutual impedance between its phases, provided the equivalence is limited to *positive* and *negative sequence only.*

On examining equations (68) it will be observed that, if both $(Z_{aa2} - Z_{bc2})$ and $(Z_{aa1} + 2Z_{bc1})$ are zero and if $V_{a0}$ and $V_{a2}$ are also both zero, the equations reduce to the single relation

$$V_{a1} = I_{a1}(Z_{aa0} - Z_{bc0}) \tag{69}$$

The zero- and negative-sequence components of the current are both zero. That is, the application of positive-sequence potentials produces positive-sequence currents only. In this respect the circuit cannot be distinguished from a balanced one although it is not necessary that the circuit should be balanced in order to satisfy the foregoing impedance relations. *Thus the fact that a static circuit, the phases of which are magnetically coupled, takes balanced currents of a given phase order when balanced potentials of the same phase order* are applied to it does not prove that the *circuit itself is balanced.*

If negative-sequence potentials alone are impressed on this circuit, equations (68) reduce to

$$\begin{aligned} 0 &= I_{a0}(Z_{aa0} + 2Z_{bc0}) + I_{a2}(Z_{aa1} - Z_{bc1}) \\ 0 &= I_{a0}(Z_{aa1} - Z_{bc1}) + I_{a1}(Z_{aa0} - Z_{bc0}) + I_{a2}(Z_{aa2} + 2Z_{bc2}) \\ V_{a2} &= I_{a2}(Z_{aa0} - Z_{bc0}) \end{aligned} \tag{70}$$

The zero-, positive-, and negative-sequence currents are readily determined from these equations. If the neutral of the circuit is *free,* there can be no zero-sequence current. However,

there will be a zero-sequence component in the phase potentials to neutral which will be proportional to the negative-sequence current. In this case equations (70) become

$$
\begin{aligned}
V_{a0} &= I_{a2}(Z_{aa1} - Z_{bc1}) \\
0 &= I_{a1}(Z_{aa0} - Z_{bc0}) + I_{a2}(Z_{aa2} + 2Z_{bc2}) \\
V_{a2} &= I_{a2}(Z_{aa0} - Z_{bc0})
\end{aligned} \tag{71}
$$

The ratio, $(Z_{aa0} - Z_{bc0})$, of the negative-sequence potential and current equals the ratio of the positive-sequence potential and current [equation (69)], but the currents [equations (71)] are now unbalanced and have a positive-sequence component

$$I_{a1} = -I_{a2}\frac{(Z_{aa2} + 2Z_{bc2})}{Z_{aa0} - Z_{bc0}}.$$

Thus it is possible to adjust a 3-phase static circuit so that it will take balanced currents when potentials of one phase order are impressed on it but will take unbalanced currents if the phase order of the applied potentials is reversed.

Again it will be observed [equations (68), page 89] that, if both $(Z_{aa1} - Z_{bc1})$ and $(Z_{aa2} + 2Z_{bc2})$ are zero, only negative-sequence currents will flow if balanced potentials of negative-sequence phase order are impressed on the circuit. Also, in this case, the currents will become unbalanced if the phase order of the applied potentials is reversed.

It will be observed, however, that, if all four of these impedance relations, *viz.* $(Z_{aa2} - Z_{bc2}) = 0$, $(Z_{aa1} + 2Z_{bc1}) = 0$,

$$(Z_{aa1} - Z_{bc1}) = 0, \qquad \text{and} \qquad (Z_{aa2} + 2Z_{bc2}) = 0,$$

are simultaneously satisfied, the positive- and negative-sequence components of both the mutual and self-impedances are zero and the circuit is balanced. *Therefore, if the circuit takes balanced currents for both phase orders of the applied potentials, the circuit is balanced.*

**Illustration.**—Consider the 3-phase $Y$-connected circuit which is defined by the following constants.

$$
\begin{aligned}
Z_{aa} &= \frac{3\sqrt{3}}{2} + j\frac{3}{2}, & Z_{ab} &= 0 + j1 \\
Z_{bb} &= \sqrt{3} + j2, & Z_{bc} &= 0 + j2 \\
Z_{cc} &= 2\sqrt{3} + j2, & Z_{ca} &= 0 + j1
\end{aligned}
$$

From which it follows that

$$Z_{aa0} = \frac{3}{2}\sqrt{3} + j\frac{5.5}{3}, \qquad Z_{bc0} = 0 + j\tfrac{4}{3}$$
$$Z_{aa1} = 0 - j\tfrac{2}{3}, \qquad Z_{bc1} = 0 + j\tfrac{1}{3}$$
$$Z_{aa2} = 0 + j\tfrac{1}{3}, \qquad Z_{bc2} = 0 + j\tfrac{1}{3}$$

With positive-sequence potentials only applied to this circuit, the currents are balanced and

$$I_{a1} = \frac{V_{a1}}{1.5\sqrt{3} + j0.5}$$

With the neutral free and with negative-sequence potentials only applied to this circuit, the currents are unbalanced and

$$I_{a2} = \frac{V_{a2}}{1.5\sqrt{3} + j0.5}$$
$$I_{a1} = -\frac{V_{a2}}{(1.5\sqrt{3} + j0.5)} \frac{(0 - j1)}{(1.5\sqrt{3} + j0.5)}$$
$$V_{a0} = \frac{V_{a2}}{1.5\sqrt{3} + j0.5}(0 - j1)$$

An interesting condition arises when the circuit is balanced. In this case, since the positive- and negative-sequence components of both the self- and mutual impedances are zero, the three equations (68) become

$$\begin{aligned} V_{a0} &= I_{a0}(Z_{aa0} + 2Z_{bc0}) \\ V_{a1} &= I_{a1}(Z_{aa0} - Z_{bc0}) \\ V_{a2} &= I_{a2}(Z_{aa0} - Z_{bc0}) \end{aligned} \tag{72}$$

Since the circuit is balanced, balanced currents will flow in the phases if balanced potentials of either phase order are applied. The equations (72) agree with this well-known fact and show that when either positive- or negative-sequence potentials are applied, the impedance of each phase is $(Z_{aa0} - Z_{bc0})$. They also show that when zero-sequence potentials are applied, the impedance of each phase becomes $(Z_{aa0} + 2Z_{bc0})$. While these phase impedances have the characteristic of zero-sequence quantities, it would be confusing to refer to them simply as zero-sequence impedances without specifying which phase order of potential is implied. Thus, to avoid any ambiguity, it may be

said that $(Z_{aa0} - Z_{bc0})$ is the zero-sequence impedance to either positive- or negative-sequence potential and that $(Z_{aa0} + 2Z_{bc0})$ is the zero-sequence impedance to zero-sequence potential. The author prefers, however, to abbreviate these expressions and to say that for a balanced static circuit $(Z_{aa0} - Z_{bc0})$ is the impedance to either positive- or negative-sequence and that $(Z_{aa0} + 2Z_{bc0})$ is the impedance to zero-sequence. It has also been suggested that they be designated as the positive-, negative-, and zero-sequence impedances. These terms, however, are used to designate the symmetrical components of *unbalanced* impedances. To the author it seems unwise to use the same term for entirely different quantities. Consequently we shall carefully distinguish between the "zero-, positive-, and negative-sequence impedances," which are the symmetrical components of unbalanced impedances, and the "impedances to zero, positive, and negative sequence," which are the phase impedances of a balanced circuit to zero-, positive-, and negative-sequence current.

**25. Nomenclature.**—The nomenclature to be used in representing the symmetrical components of potential, current, impedance, and admittance has not become standardized as yet, but the author's experience during the last fifteen years leads him to believe that the general principles of the notation that Fortescue used in his original paper are the most satisfactory. The usual English-type letters $V$ or $E$, $I$, $Z$ or $z$, $Y$ or $y$ will be used to represent the potential, current, impedance, and admittance. A letter subscript ($a$, $b$, $c$, etc.) will be used to indicate to which phase or part of the circuit these quantities belong, and an additional number subscript (0, 1, 2, etc.) will be used to indicate their order or sequence. In order to distinguish, for example, between the "positive-sequence impedance" and the "impedance to positive sequence," which is in effect a zero-sequence quantity, it is suggested that an additional zero-sequence (0) subscript or superscript or some other distinguishing mark be used until one attains a sufficient understanding of the method, after which this distinguishing mark may be omitted. Thus $Z^0_{a0}$ would be the phase impedance of the $a$ portion of a balanced circuit to zero sequence, and $Z^0_{a1}$ and $Z^0_{a2}$ would be the phase-impedances of the $a$ portion of a *balanced* circuit to positive sequence and to negative sequence. The important point to recognize is that they are the constants of a *balanced* or *symmetri-*

*cal* circuit. That is to say, they are *zero-sequence* quantities and must be treated as such when applying the Sequence Rule in the process of multiplication. When the additional zero (0) or other mark is omitted, it must be understood that $Z_{a1}$, for example, is the *positive-sequence impedance* of the $a$ phase if that portion of the circuit is *unsymmetrical*, but that it is the *impedance to positive sequence* if the $a$ phase forms part of a *symmetrical* portion of the circuit. It is interesting to note at this time that the impedance of a symmetrically wound rotating machine is not the same to positive sequence as it is to negative sequence, whereas the impedance of a balanced static circuit is the same to positive as to negative sequence.

## CHAPTER III

# FOUR-PHASE STATIC CIRCUITS

We have seen that all the quantities encountered in 3-phase circuits, potentials, currents, self-impedances and admittances, and mutual reactances are subject to the same analysis. In general, each one of these quantities can be resolved into three components. Furthermore, when one of these quantities is multiplied by another, the operation is always performed in the same way. In the analysis of 4-phase circuits, however, we shall find that a similar procedure cannot be followed in every case. Whereas in 3-phase circuits the *three* potentials, currents, or impedances can always be resolved into *three* components, in 4-phase circuits it is often necessary to consider six quantities, as, for example, the six impedances of the general 4-phase mesh or the six mutual reactances between the four line conductors of the circuit. Again the 4-phase circuit has as special cases the 2-phase 4-wire circuit, and the 2-phase 3-wire circuit. We shall find that, while the general method of resolution developed in Chapter I should not be followed blindly, the fundamental principle of the analysis, *viz.*, the resolution into symmetrical components, should still be the guiding one.

**26. Resolution of Potentials.**—Let $V_a$, $V_b$, $V_c$, and $V_d$ represent the 4-phase potentials whose symmetrical components we wish to determine. Furthermore, let us consider the special case in which they represent the potentials measured from the lines, *a*, *b*, *c*, and *d* to the neutral *n* of the circuit. Since the characteristic angle, $\beta$, for a 4-phase system is 90 degrees, the symmetrical components of these potentials are determined by [see equations (2*a*), page 29]

$$\begin{aligned} V_a &= V_0 + V_1 + V_2 + V_3 \\ V_b &= V_0 + V_1\underline{/-90^\circ} + V_2\underline{/-180^\circ} + V_3\underline{/-270^\circ} \\ V_c &= V_0 + V_1\underline{/-180^\circ} + V_2\underline{/-360^\circ} + V_3\underline{/-540^\circ} \\ V_d &= V_0 + V_1\underline{/-270^\circ} + V_2\underline{/-540^\circ} + V_3\underline{/-810^\circ} \end{aligned}$$

These equations can be written in much simpler form if the symbol $j$ is substituted for 90 degrees. Thus

$$\left.\begin{aligned} V_a &= V_0 + V_1 + V_2 + V_3 \\ V_b &= V_0 - jV_1 - V_2 + jV_3 \\ V_c &= V_0 - V_1 + V_2 - V_3 \\ V_d &= V_0 + jV_1 - V_2 - jV_3 \end{aligned}\right\} \qquad (73)$$

Each of the phase potentials, $V_a$, $V_b$, $V_c$, and $V_d$, consists of four components: a zero-order or zero-sequence component; a first-order or positive-sequence component; a second-order component; and a third-order or negative-sequence component. These

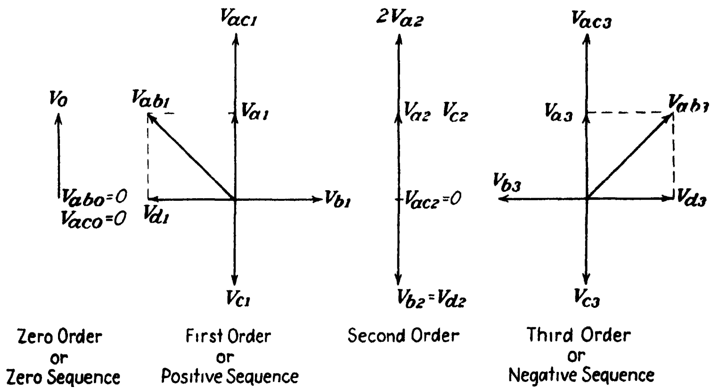



FIG. 31.

components are shown in the vector diagrams of Fig. 31. There are also shown in these vector diagrams the potentials between the adjacent terminals $a$ and $b$ and between the diametrical terminals $a$ and $c$. The vector relations shown in these diagrams are useful when computing the currents taken by the general six-branch mesh circuit. No significance should be attached to the fact that the four components of $V_a$ are equal vectors.

The magnitudes of these symmetrical components are determined by the method indicated in equations (4), (6), (7), and (9), page 31. Thus

$$\left.\begin{aligned} V_0 &= \tfrac{1}{4}(V_a + V_b + V_c + V_d) \\ V_{a1} &= \tfrac{1}{4}(V_a + jV_b - V_c - jV_d) \\ V_{a2} &= \tfrac{1}{4}(V_a - V_b + V_c - V_d) \\ V_{a3} &= \tfrac{1}{4}(V_a - jV_b - V_c + jV_d) \end{aligned}\right\} \qquad (74)$$

When the phase potentials, $V_a$, $V_b$, $V_c$, and $V_d$, are determined by their complex expressions, these equations lend themselves to a ready computation of the symmetrical components.

If each of the four line-to-neutral potentials and also *three* of the line-to-line potentials are determined by voltmeter measurements, the complete vector diagram of these potentials can be constructed as indicated in Fig. 32. Since the zero-sequence component of the line-to-neutral potentials is one-fourth their sum, it equals the vector $V_0$ drawn from $n$ to the midpoint of the line $gh$, which bisects the opposite sides of the polygon.

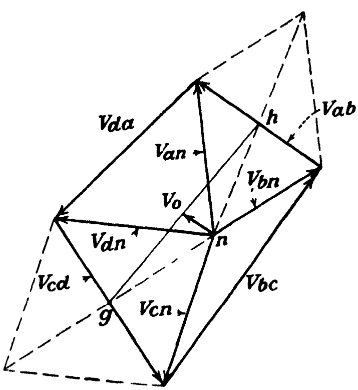


FIG. 32.

**27. Resolution of Line Potentials.**—In order to determine the polygon of line potentials, it is necessary to measure not only the four potentials between successive line conductors but also the potential between two of the diametrical conductors, $V_{ac}$, for example. Since the line potentials have no zero-sequence component, one of the line potentials can be eliminated from the general expressions [equations (74)] for the symmetrical components, that is, for example, $V_{da} = -V_{ab} - V_{bc} - V_{cd}$. By following a process similar to that used in equation (31), page 53, we obtain for the symmetrical components of the line potential $V_{ab}$

$$\left.\begin{aligned} V_{ab1} &= \tfrac{1}{4}[(1+j)V_{ab} + j2V_{bc} - (1-j)V_{cd}] \\ V_{ab2} &= \tfrac{1}{2}(V_{ab} + V_{cd}) \\ V_{ab3} &= \tfrac{1}{4}[(1-j)V_{ab} - j2V_{bc} - (1+j)V_{cd}] \end{aligned}\right\} \quad (75)$$

If the equations for the positive- and negative-sequence components are rearranged, they indicate a simple geometrical construction for determining the components of the line potentials.

$$\left.\begin{aligned} V_{ab1} &= \tfrac{1}{4}\{V_{ab} - V_{cd} + j[(V_{ab} + V_{bc}) + (V_{bc} + V_{cd})]\} \\ V_{ab3} &= \tfrac{1}{4}\{V_{ab} - V_{cd} - j[(V_{ab} + V_{bc}) + (V_{bc} + V_{cd})]\} \end{aligned}\right\} \quad (76)$$

Note that $(V_{ab} + V_{bc})$ and $(V_{bc} + V_{cd})$ are the diagonals $V_{ac}$ and $V_{bd}$ of the polygon of line potentials.

The geometrical construction is shown in Fig. 33.

If it is desirable to express the first-, second-, and third-order components of the potentials to neutral in terms of the line potentials, this may be done by noting their relative values as shown in the vector diagrams of Fig. 31. Thus, since

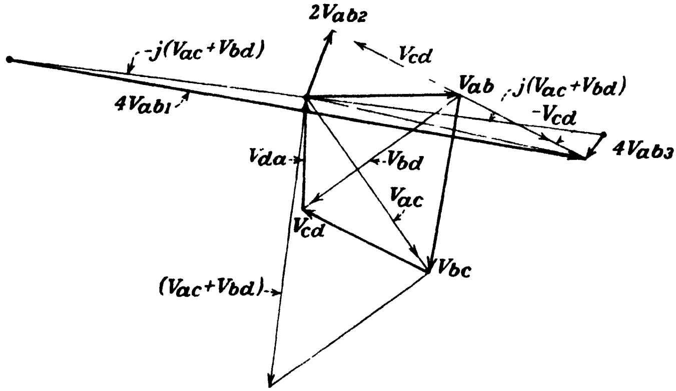



FIG. 33.

$V_{ab1} = (1 + j)V_{a1}, \quad V_{ab2} = 2V_{a2}, \quad \text{and} \quad V_{ab3} = (1 - j)V_{a3},$
it follows that

$$\left.\begin{aligned} V_{a1} &= \tfrac{1}{4}[V_{ab} + (1 + j)V_{bc} + jV_{cd}] \\ V_{a2} &= \tfrac{1}{4}(V_{ab} + V_{cd}) \\ V_{a3} &= \tfrac{1}{4}[V_{ab} + (1 - j)V_{bc} - jV_{cd}] \end{aligned}\right\} \tag{77}$$

Notice that all the components of the star potentials, except the zero-sequence component, are directly determined by the line potentials and in no way depend upon the character of any star load.

If the equations for the positive- and negative-sequence components are rearranged, they indicate a simple geometrical construction for determining the components of the star potentials.

$$\left.\begin{aligned} V_{a1} &= \tfrac{1}{4}[V_{ab} + V_{bc} + j(V_{bc} + V_{cd})] \\ V_{a3} &= \tfrac{1}{4}[V_{ab} + V_{bc} - j(V_{bc} + V_{cd})] \end{aligned}\right\} \tag{78}$$

The positive- and negative-sequence components $V_{a1}$ and $V_{a3}$ of the star potentials are determined by combining the two diagonals of the polygon of line potentials after one of the diagonals has been rotated 90 degrees.

The geometrical constructions indicated are shown in Fig. 34.

**28. Resolution of Currents.**—If the currents flowing toward the neutral $n$ of a 4-phase star load are represented by $I_a$, $I_b$, $I_c$, and $I_d$, the symmetrical components of $I_a$ can be determined by

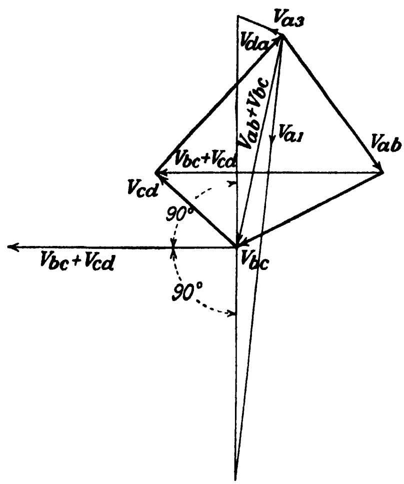



FIG. 34.

the process indicated in equations (74). If there is no neutral connection, the zero-sequence component is zero. Otherwise it is by definition one-fourth the neutral current. If the currents flowing between the adjacent terminals of a 4-phase mesh

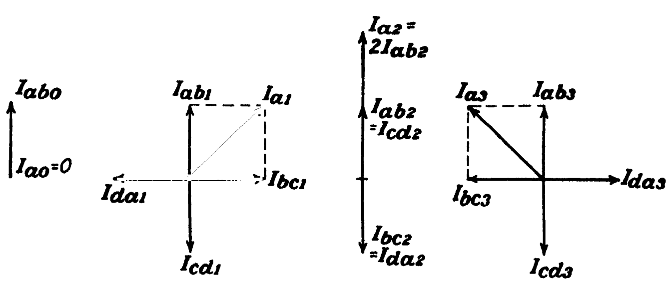



FIG. 35.

load are represented by $I_{ab}$, $I_{bc}$, $I_{cd}$, and $I_{da}$, the symmetrical components of $I_{ab}$ can be determined by a similar process [equations (74)]. These components are shown in the vector diagram of Fig. 35. The diagrams also show the components of the line current flowing *in* at the terminal *a*. No significance should be

attached to the fact that the four components of $I_{ab}$ are shown as equal vectors.

If the polygon of the line currents is given, *i.e.*, if their magnitudes and phase relations are known, their symmetrical components can be determined by the geometrical construction given in Fig. 33, in which the line currents $I_a$, $I_b$, $I_c$ and $I_d$ replace the line potentials $V_{ab}$, $V_{bc}$, $V_{cd}$, and $V_{da}$.

If there are two mesh circuits, one of which is connected across diametrical conductors *a-c* and the other is connected across conductors *b-d*, the currents in the four lines due to these circuits are $I_a$, $I_b$, $-I_a$, and $-I_b$. If these current values are substituted in the general equations [equations (74)], the symmetrical components reduce to

$$\left.\begin{aligned} I_0 &= \tfrac{1}{4}(I_a + I_b - I_a - I_b) \\ &= 0 \\ I_{a1} &= \tfrac{1}{4}(I_a + jI_b + I_a + jI_b) \\ &= \tfrac{1}{2}(I_a + jI_b) \\ I_{a2} &= \tfrac{1}{4}(I_a - I_b - I_a + I_b) \\ &= 0 \\ I_{a3} &= \tfrac{1}{4}(I_a - jI_b + I_a - jI_b) \\ &= \tfrac{1}{2}(I_a - jI_b) \end{aligned}\right\} \qquad (79)$$

For this type of load both the zero- and second-order components of the line currents are zero. The expressions for the positive- and negative-sequence components are as given in equation (79).

If both the 4-phase and 2-phase mesh loads are present simultaneously, the symmetrical components of the *resulting* line currents are the sums of the corresponding symmetrical components of the line currents due to the *individual* loads.

**29. Resolution of Impedances.**—If $Z_a$, $Z_b$, $Z_c$, and $Z_d$ are the four self-impedances of the line conductors or of a star-connected load, the symmetrical components of $Z_a$ are

$$\left.\begin{aligned} Z_0 &= \tfrac{1}{4}(Z_a + Z_b + Z_c + Z_d) \\ Z_1 &= \tfrac{1}{4}(Z_a + jZ_b - Z_c - jZ_d) \\ Z_2 &= \tfrac{1}{4}(Z_a - Z_b + Z_c - Z_d) \\ Z_3 &= \tfrac{1}{4}(Z_a - jZ_b - Z_c + jZ_d) \end{aligned}\right\} \qquad (80)$$

**30. Resolution of Admittances.**—If $y_a$, $y_b$, $y_c$, and $y_d$ are the admittances of a 4-phase star-connected load or of a 4-phase

mesh load connected between adjacent line conductors, the symmetrical components of $y_a$ are determined by equations of the same form as equation (80).

If $y_a$ and $y_b$ are the admittances of a 2-phase load that is connected between the diametrical conductors *a-c* and *b-d*, the symmetrical components of $y_a$ can be determined in the following manner. Since there are but two quantities to resolve, each should consist of but two components. This conclusion is readily confirmed. The vector diagrams of Fig. 31 show that only positive- and negative-sequence components of potential exist across the diametrical conductors. Likewise, equations (79) show that the currents taken by this 2-phase load consist of positive- and negative-sequence components only. Since current equals the product of potential and admittance, the Sequence Rule (page 41) shows that the admittance can consist of *zero-* and *second*-order components *only* [equations (85), page 104]. It is only with components of admittance of these orders that both the potential and current have none but positive- or negative-sequence components.

The zero and second order components of $y_a$ are

$$\left.\begin{aligned} y_0 &= \tfrac{1}{2}(y_a + y_b) \\ y_2 &= \tfrac{1}{2}(y_a - y_b) \end{aligned}\right\} \qquad (81)$$

**31. Vector Volt-amperes.**—Consider a star load for which the symmetrical components of the potential and current in phase *a* are

$$\begin{aligned} E_a &= E_0 + E_1 + E_2 + E_3 \\ I_a &= I_0 + I_1 + I_2 + I_3 \end{aligned}$$

The vector volt-ampere input to phase *a* is

$$\begin{aligned} \underset{\smile}{E}_a I_a = {} & \underset{\smile}{E}_0 I_0 + \underset{\smile}{E}_1 I_1 + \underset{\smile}{E}_2 I_2 + \underset{\smile}{E}_3 I_3 \\ & + \underset{\smile}{E}_0 I_1 + \underset{\smile}{E}_1 I_2 + \underset{\smile}{E}_2 I_3 + \underset{\smile}{E}_3 I_0 \\ & + \underset{\smile}{E}_0 I_2 + \underset{\smile}{E}_1 I_3 + \underset{\smile}{E}_2 I_0 + \underset{\smile}{E}_3 I_1 \\ & + \underset{\smile}{E}_0 I_3 + \underset{\smile}{E}_1 I_0 + \underset{\smile}{E}_2 I_1 + \underset{\smile}{E}_3 I_2 \end{aligned}$$

The application of the Sequence Rule and the Rule for Conjugates shows that the first line in the equation for vector volt-ampere input consists of zero-sequence components; the second line consists of positive-sequence components; the third line

consists of second-order components; the fourth line consists of negative-sequence components. The total vector volt-ampere input is

$$\text{Total } (\underset{\vee}{E}I) = 4(\underset{\vee}{E}_0 I_0 + \underset{\vee}{E}_1 I_1 + \underset{\vee}{E}_2 I_2 + \underset{\vee}{E}_3 I_3) \qquad (82)$$

If the star load has no neutral connection or if the load is mesh connected, the zero-sequence component of the line currents is zero and the total vector volt-ampere input is

$$\text{Total } (\underset{\vee}{E}I) = 4(\underset{\vee}{E}_1 I_1 + \underset{\vee}{E}_2 I_2 + \underset{\vee}{E}_3 I_3)$$

In this expression the potentials are the components of the star potential $E_a$ and the currents are the components of the line current $I_a$.

**32. Pulsating Power.**—Consider the same star load for which the vector volt-ampere input has just been computed. The pulsating power of phase $a$ is

$$\begin{aligned} S_a = {} & E_0I_0 + E_1I_3 + E_2I_2 + E_3I_1 \\ & + E_0I_1 + E_1I_0 + E_2I_3 + E_3I_2 \\ & + E_0I_2 + E_1I_1 + E_2I_0 + E_3I_3 \\ & + E_0I_3 + E_1I_2 + E_2I_1 + E_3I_0 \end{aligned}$$

The application of the Sequence Rule shows that the first line in this equation consists of zero-sequence components; the second line consists of positive-sequence components; the third line consists of second-order components; the fourth line consists of negative-sequence components. The total pulsating power is

$$\text{Total } S = 4(E_0I_0 + E_1I_3 + E_2I_2 + E_3I_1) \qquad (83)$$

If the star load has no neutral connection or if the load is mesh connected, the zero-sequence component of the line currents is zero and the total pulsating power is

$$\text{Total } S = 4(E_1I_3 + E_2I_2 + E_3I_1)$$

In this expression the potentials are the components of the star potential $E_a$, and the currents are the components of the line current $I_a$.

**33. Product *IZ*.**—If $Z_a$, $Z_b$, $Z_c$, and $Z_d$ are the impedances of a star-connected load or are the self-impedances of the four line conductors, and if $I_a$, $I_b$, $I_c$, and $I_d$ are the currents in them, the symmetrical components of the potential drop $I_aZ_a$ are

$$\left.\begin{aligned} V_0 &= I_0Z_0 + I_1Z_3 + I_2Z_2 + I_3Z_1 \\ V_1 &= I_0Z_1 + I_1Z_0 + I_2Z_3 + I_3Z_2 \\ V_2 &= I_0Z_2 + I_1Z_1 + I_2Z_0 + I_3Z_3 \\ V_3 &= I_0Z_3 + I_1Z_2 + I_2Z_1 + I_3Z_0 \end{aligned}\right\} \qquad (84)$$

The currents and impedances in equations (84) are the symmetrical components of the current and impedance in phase $a$. Note that the terms are grouped in accordance with the Sequence Rule.

If the neutral of a star-connected load is not connected to the neutral conductor of the circuit, the zero-sequence current, $I_0$, is zero. In this case if $Z_0$, $Z_1$, $Z_2$, and $Z_3$ are the symmetrical components of the impedance of branch $a$ of this star-connected load and $V_1$, $V_2$, and $V_3$ are the symmetrical components of the potential from the terminal $a$ of this load toward its neutral, the last three of equations (84) can be solved for the first-, second-, and third-order components of the line current in phase $a$. If the values of these component currents are now substituted in the *first* of equations (84), the zero-sequence component $V_0$ of the star potential will be determined. Note that $V_1$, $V_2$, and $V_3$ are determined by the potentials applied at the terminals of the load [equations (77)].

**34. Product *Ey*.**—First consider the four admittances $y_{ab}$, $y_{bc}$, $y_{cd}$, and $y_{da}$ connected between adjacent terminals as indicated by the subscripts, and let the potentials between these terminals be $V_{ab}$, $V_{bc}$, $V_{cd}$, and $V_{da}$. If $y_0$, $y_1$, $y_2$, and $y_3$ and $V_1$, $V_2$, and $V_3$ are the symmetrical components of $y_{ab}$ and $V_{ab}$, the symmetrical components of $I_{ab}$ are

$$\left.\begin{aligned} I_0 &= V_1y_3 + V_2y_2 + V_3y_1 \\ I_1 &= V_1y_0 + V_2y_3 + V_3y_2 \\ I_2 &= V_1y_1 + V_2y_0 + V_3y_3 \\ I_3 &= V_1y_2 + V_2y_1 + V_3y_0 \end{aligned}\right\} \qquad (85)$$

Note that the potentials have no zero-sequence components. The relations between the mesh current $I_{ab}$ and the line current $I_a$ are shown in Fig. 35. From these relations

$$\left.\begin{aligned} I_{a1} &= (1 - j)(V_1y_0 + V_2y_3 + V_3y_2) \\ I_{a2} &= 2(V_1y_1 + V_2y_0 + V_3y_3) \\ I_{a3} &= (1 + j)(V_1y_2 + V_2y_1 + V_3y_0) \end{aligned}\right\} \qquad (86)$$

It may be desirable to express the line current in terms of components of the star potentials. The relations between the mesh potentials and the star potentials are shown in Fig. 31. If these relations are substituted in equation (86)

$$\left.\begin{aligned} I_{a1} &= 2[V_{a1}y_0 + V_{a2}(1 - j)y_3 - V_{a3}jy_2] \\ I_{a2} &= 2[V_{a1}(1 + j)y_1 + V_{a2}2y_0 + V_{a3}(1 - j)y_3] \\ I_{a3} &= 2[V_{a1}jy_2 + V_{a2}(1 + j)y_1 + V_{a3}y_0] \end{aligned}\right\} \quad (87)$$

Let us now assume that, in addition to this mesh load, a 2-phase load whose admittances are $y_{ac}$ and $y_{bd}$ is connected across the diametrical terminals. The symmetrical components of the admittances of this 2-phase load are [equations (81)]

$$\left.\begin{aligned} y_0 &= \tfrac{1}{2}(y_{ac} + y_{bd}) \\ y_2 &= \tfrac{1}{2}(y_{ac} - y_{bd}) \end{aligned}\right\} \quad (88)$$

The positive and negative-sequence components of the current, $I_a$, taken by this 2-phase load are

$$\begin{aligned} I_{a1} &= 2(V_{a1}y_0 + V_{a3}y_2) \\ I_{a3} &= 2(V_{a1}y_2 + V_{a3}y_0) \end{aligned}$$

where $y_0$ and $y_2$ are defined in equation (88). Note that the arrangement of the terms agrees with the Sequence Rule.

**35. Mutual Reactance.**—There are six possible mutual reactances, *viz.*, $x_{ab}$, $x_{bc}$, $x_{cd}$, $x_{da}$, $x_{ac}$, and $x_{bd}$. These can be divided into two groups. The first four form the first group and are treated as a 4-phase system. The last two form the second group and are treated as a 2-phase system. In this respect the mutual reactances are similar to the general six-branch mesh load that has already been discussed. Let the mutual reactance couplings between the lines of the circuit be arranged so that, if $I$ represents the current in any conductor flowing *toward the load*, the mutually induced fall in potential in any of the other conductors measured in the *same* direction is $jxI$. This mutual reactance may be either positive or negative. First, let us consider the mutual reactances between adjacent conductors, *viz.*, $x_{ab}$, $x_{bc}$, $x_{cd}$, and $x_{da}$. Let the symmetrical components of $jx_{ab}$ be $Z_0$, $Z_1$, $Z_2$, and $Z_3$, and the corresponding components of the current $I_a$ be $I_0$, $I_1$, $I_2$, and $I_3$. The mutually induced fall in potential in phase $a$ due to the currents in $b$ and $d$ is

$$V_a = jx_{ab}I_b + jx_{ad}I_d \quad (89)$$

If $x_{ad}$ is expressed in terms of the symmetrical components of $x_{ab}$,

$$jx_{ad} = Z_0 + jZ_1 - Z_2 - jZ_3$$

Similarly, if $I_b$ and $I_d$ are expressed in terms of the symmetrical components of $I_a$,

$$I_b = I_0 - jI_1 - I_2 + jI_3$$
$$I_d = I_0 + jI_1 - I_2 - jI_3$$

If the foregoing values are substituted in equation (89) for the fall in potential in phase $a$

$$V_a = V_0 + V_1 + V_2 + V_3$$

where

$$\left.\begin{aligned} V_0 &= 2Z_0I_0 + (1 - j)Z_3I_1 + (1 + j)Z_1I_3 \\ V_1 &= (1 + j)Z_1I_0 - (1 - j)Z_3I_2 + j2Z_2I_3 \\ V_2 &= -(1 + j)Z_1I_1 - 2Z_0I_2 - (1 - j)Z_3I_3 \\ V_3 &= (1 - j)Z_3I_0 - j2Z_3I_1 - (1 + j)Z_1I_2 \end{aligned}\right\} \qquad (90)$$

It is interesting to note that, if there are *equal* reactance couplings between the adjacent phases, all the terms in equations (90) will vanish except

$$V_0 = 2Z_0I_0$$

and

$$V_2 = -2Z_0I_2$$

Thus positive- and negative-sequence currents will produce no mutual-reactance drop in potential in the circuit.

Let us now consider the mutual reactance between the diametrical phases $a$-$c$ and $b$-$d$. The mutually induced potentials in phases $a$, $b$, $c$, and $d$ are

$$V_a = jx_{ac}I_c \qquad V_c = jx_{ac}I_a$$
$$V_b = jx_{bd}I_d \qquad V_d = jx_{bd}I_b$$

The symmetrical components of the reactance $x_{ac}$ are

$$\begin{aligned} jx_0 &= j\tfrac{1}{2}(x_{ac} + x_{bd}) \\ jx_1 &= j\tfrac{1}{4}(x_{ac} + jx_{bd} - x_{ac} - jx_{bd}) \\ &= 0 \\ jx_2 &= j\tfrac{1}{2}(x_{ac} - x_{bd}) \\ jx_3 &= j\tfrac{1}{4}(x_{ac} - jx_{bd} - x_{ac} + jx_{bd}) \\ &= 0 \end{aligned}$$

The symmetrical components of the current $I_a$ have already been taken as $I_0$, $I_1$, $I_2$, and $I_3$ The current $I_c$ expressed in terms of these symmetrical components is

$$I_c = I_0 - I_1 + I_2 - I_3$$

It follows that the symmetrical components of the potential $V$ are

$$\left.\begin{aligned} V_0 &= jx_0I_0 + jx_2I_2 \\ V_1 &= jx_0I_1 + jx_2I_3 \\ V_2 &= jx_2I_0 + jx_0I_2 \\ V_3 &= jx_2I_1 + jx_0I_3 \end{aligned}\right\} \qquad (91)$$

The combination of equations (90) and (91) will give the symmetrical components of the mutual-reactance potential due to any combination of the six possible mutual reactances.

## TWO-PHASE CIRCUITS

**36. Potential and Current.**—Two-phase, 4-wire circuits are a special case of 4-phase circuits in which the currents in diametrical conductors are equal in magnitude and opposite in phase. This results from the fact that there is no electrical connection between adjacent conductors. The load on a 2-phase circuit consists of combinations of independent single-phase loads connected to diametrical conductors or of 2-phase motors in which the phases of the stator windings are not interconnected. We have already seen (page 102) that, when a 2-phase load of this type is considered as a special case of a 4-phase load, the line currents consist of positive- and negative-sequence components only. Both the zero- and second-order components are zero. Furthermore, it is most convenient in the analysis of 2-phase motors (see page 241) to resolve the applied potentials into positive- and negative-sequence components. Thus, if $V_a$ and $V_b$ represent either the 2-phase potentials $V_{ac}$ and $V_{bd}$ or the 2-phase currents $I_a$ and $I_b$, the symmetrical components of $V_a$ are

$$\left.\begin{aligned} V_1 &= \tfrac{1}{2}(V_a + jV_b) \\ V_3 &= \tfrac{1}{2}(V_a - jV_b) \end{aligned}\right\} \qquad (92)$$

**37. Self-impedance and Admittance.**—Having determined that the 2-phase potentials and currents should be resolved into first- and third-order components, the application of the Sequence

Rule fixes the resolution of the self-impedances and self-admittances. Since we must be able to write $E = IZ$ and $I = Ey$, it follows that both $Z$ and $y$ must be resolved into zero- and second-order components. Thus, if $Z_a$ and $Z_b$ represent either the line or the load impedances of the 2-phase circuit, the symmetrical components of $Z_a$ are

$$\left.\begin{aligned} Z_0 &= \tfrac{1}{2}(Z_a + Z_b) \\ Z_2 &= \tfrac{1}{2}(Z_a - Z_b) \end{aligned}\right\} \qquad (93)$$

**38. Products *IZ* and *Ey*.**—If the symmetrical components of $I_a$ are $I_1$ and $I_3$ and the symmetrical components of $Z_a$ are $Z_0$ and $Z_2$, the symmetrical components of $I_aZ_a$ are

$$\left.\begin{aligned} (I_aZ_a)_1 &= I_1Z_0 + I_3Z_2 \\ (I_aZ_a)_3 &= I_1Z_2 + I_3Z_0 \end{aligned}\right\} \qquad (94)$$

Note that the terms are arranged in accordance with the Sequence Rule.

**39. Vector Volt-ampere Input.**—The total vector volt-ampere input supplied to both phases is twice the zero-sequence component of the vector volt-ampere input of either phase. Thus, if the symmetrical components of $E_a$ are $E_1$ and $E_3$ and the symmetrical components of $I_a$ are $I_1$ and $I_3$, the total vector volt-ampere input is

$$\text{Total } (\underset{\smile}{E}I) = 2(\underset{\smile}{E}_1I_1 + \underset{\smile}{E}_3I_3) \qquad (95)$$

Note that the terms are arranged in accordance with the Sequence Rule and the Rule for Conjugates.

**40. Pulsating Power.**—The total pulsating power is twice the zero-sequence component of the pulsating power supplied to either one of the phases. Thus, if the symmetrical components of $E_a$ are $E_1$ and $E_3$ and the symmetrical components of $I_a$ are $I_1$ and $I_3$, the total pulsating power is

$$\text{Total } S = 2(E_1I_3 + E_3I_1) \qquad (96)$$

**41. Mutual Reactance.**—If the mutual reactance between the two phases is $x_{ab}$, the mutually induced potentials in the phases are

$$\begin{aligned} E_a &= jx_{ab}I_b \\ E_b &= jx_{ab}I_a \end{aligned}$$

If the currents are expressed in terms of the symmetrical components of $I_a$ these potentials are written

$$E_a = jx_{ab}(-jI_{a1} + jI_{a3})$$

or

$$E_a = x_{ab}I_{a1} - x_{ab}I_a$$

and

$$E_b = jx_{ab}(I_{a1} + I_{a3})$$

Note that the component $jx_{ab}I_{a3}$ in $E_b$ *lags* the corresponding component $-x_{ab}I_{a3}$ in $E_a$ by 90 degrees, and that the component $jx_{ab}I_{a1}$ in $E_b$ *leads* the corresponding component $x_{ab}I_{a1}$ in $E_a$ by 90 degrees. This shows that the first is a positive-sequence component and that the second is a negative-sequence component. Thus

$$\left.\begin{aligned} E_{a1} &= -x_{ab}I_{a3} \\ E_{a3} &= x_{ab}I_{a1} \end{aligned}\right\} \tag{97}$$

It is interesting to note that mutual reactance *between* the phases acts like resistance, either negative or positive, *in* the phases, and further that the positive-sequence component of the mutually induced potential is due wholly to negative-sequence current and that the negative-sequence component of the mutually induced potential is due wholly to the positive-sequence current.

**42. Two-phase, Three-wire.**—As before, let $a$ and $b$ be the two line or phase conductors and $n$ the common or neutral conductor. We have already considered the potential drop $IZ$ in the line conductors (page 108). Let us now consider the potential drop in the neutral conductor, which we will assume has a self-impedance $Z_n$. The return current in the neutral conductor is $I_a + I_b$ so that the potential drop in phase $a$ due to the effect of the neutral conductor is

$$E_a = Z_nI_a + Z_nI_b \tag{98}$$

The potential drop in phase $b$ is the same, but for the purpose of this analysis it should be written with the terms reversed,

$$E_b = Z_nI_b + Z_nI_a \tag{99}$$

The first term in each of these potential drops is similar to a self-impedance drop while the second term is similar to a mutual-

impedance drop. Thus in the first terms, the impedance $Z_n$ is like the zero-sequence component of a self-impedance. In the second terms the same impedance $Z_n$ should be treated as mutual reactance was treated in Art. 41.

The anslysis may also be made as follows. If the symmetrical components of $I_a$ are $I_1$ and $I_3$, the corresponding components of $I_b$ are $-jI_1$ and $jI_3$. Thus equations (98) and (99) may be written in terms of the symmetrical components of the current in $a$.

$$\begin{aligned} E_a &= Z_n(I_1 + I_3 - jI_1 + jI_3) \\ E_b &= Z_n(-jI_1 + jI_3 + I_1 + I_3) \end{aligned}$$

Observe that the first terms in $E_a$ and in $E_b$ are positive-sequence components; the second terms in $E_a$ and in $E_b$ are negative-sequence components; the third terms in $E_a$ and in $E_b$ are negative-sequence components; the last terms in $E_a$ and in $E_b$ are positive-sequence components.
Thus

$$\left.\begin{aligned} E_{a1} &= Z_nI_1 + jZ_nI_3 \\ E_{a3} &= Z_nI_3 - jZ_nI_1 \end{aligned}\right\} \qquad (100)$$

Let us now consider the effect of a single-phase load that is directly connected between the phases $a$ and $b$. Since, owing to this load alone, the current $I_b$ equals $-I_a$, the positive- and negative-sequence components of $I_a$ are

$$\begin{aligned} I_{a1} &= \tfrac{1}{2}(I_a - jI_a) \\ I_{a3} &= \tfrac{1}{2}(I_a + jI_a) \end{aligned}$$

Furthermore, since the current $I_a$ equals $Y_{ab}(E_a - E_b)$, where $Y_{ab}$ is the admittance of the load and since

$$(E_a - E_b) = E_{a1} + E_{a3} + jE_{a1} - jE_{a3},$$

these expressions for the symmetrical components of $I_a$ readily reduce to the following forms

$$\left.\begin{aligned} I_{a1} &= Y_{ab}(E_{a1} - jE_{a3}) \\ I_{a3} &= Y_{ab}(E_{a3} + jE_{a1}) \end{aligned}\right\} \qquad (101)$$

# CHAPTER IV

## TRANSMISSION LINES

In computing the resistance, inductance, and capacitance of transmission lines, the customary simplifying assumptions will be made. The author deems it essential that these assumptions should be specified with considerable care in order that the reader may obtain a true perspective of the method that is employed and be keenly aware of the fundamental principles upon which the computations are based. This will be done at the proper time.

**43. Resistance and Inductance.**—We shall limit our discussion entirely to aerial lines in which the distance between conductors is many times their diameter. Furthermore, we shall disregard the eddy currents that may be produced in any neighboring conductor or conducting material by the action of the current in any conductor. That is, for the aerial conductors which we shall consider, it will be assumed that the so-called "proximity effect" is negligible. Likewise, the effect of all currents flowing in the earth, except conduction currents, will be neglected. For this reason the expression for the potential drop in one of the conductors of a transmission line [equation (105), page 113] differs slightly from that which would result if Carson's formulas for the self- and mutual impedance of a grounded conductor [equations (127) and (128), page 126] were used in the derivation of the potential drop.

Under the foregoing conditions, the magnetic field within any conductor is due essentially to the current within the conductor itself. For a solid cylindrical conductor, the current density is determined by a Bessel function and the internal impedance of the conductor per centimeter length is given by

$$Z_i = \frac{nR}{2} \frac{J_0(nR)}{J_1(nR)} r_{dc}$$

where

$$n = 2\pi\sqrt{-j\frac{2f\mu}{\rho}}$$

$R$ is the radius, $r_{dc}$ is the resistance of the conductor to direct current, $\mu$ is the permeability, and $\rho$ is the resistivity of the conductor. For solid copper conductors of less than 0.5 in. in diameter and at frequencies of not more than 60 cycles, the internal impedance per centimeter is approximately

$$Z_i = r_{dc} + j\frac{\omega}{2}10^{-9} \text{ vector ohms per cm.} \tag{102}$$

For larger solid copper conductors, for steel conductors, for bimetallic conductors, and for cables, the reader should consult the published sources which give the values of this internal impedance. Thus, in general, the internal impedance per centimeter is

$$Z_i = r + j\omega l 10^{-9} \text{ vector ohms per cm.} \tag{103}$$

where $r$ and $\omega l 10^{-9}$ are the resistance and internal reactance of the conductor per centimeter at the given frequency.

It should be noted that, where $r$ is greater than $r_{dc}$, the internal inductance, $l$, per centimeter is less than 0.5 abhenry.

If energy is transmitted over a number of parallel cylindrical conductors, $a$, $b$, $c$, . . . $n$, the potential drop per centimeter in $a$ is

$$V_a = I_a Z_i + j2\omega\left(I_a \ln\frac{1}{R} + I_b \ln\frac{1}{ab} + I_c \ln\frac{1}{ac} \cdots + I_n \ln\frac{1}{an}\right)10^{-9} \text{ vector volts per cm.} \tag{104}$$

where $I_a$, $I_b$, etc., are the vectors representing the currents in the conductors, $Z_i$ is the internal vector impedance of the $a$ conductor per centimeter, $R$ is the radius, and $ab$, $ac$, . . . $an$ are the distances between the $a$ conductor and the other conductors.

In deriving this expression it is assumed that the conductors are infinitely long, that the current in each conductor is the same throughout its entire length, and that there are no magnetically induced currents in the earth. It should be noted that the vector sum of the currents $I_a$, $I_b$, . . . $I_n$ is zero. That is, no portion of the current that is involved in the transmission of the energy flows in the earth. The effect of the earth as a conductor is discussed in Art. 48. It is often considered more convenient to modify the form of this expression (104) and write

$$V_a = I_a r + j2\omega\left(I_a \ln\frac{1}{d} + I_b \ln\frac{1}{ab} + I_c \ln\frac{1}{ac} \cdots + I_n \ln\frac{1}{an}\right)10^{-9}$$
vector volts per cm. (105)

where $r$ is the resistance per centimeter of the $a$ conductor to the alternating current and $d = R\epsilon^{-\frac{l}{2}}$. If the current density is assumed to be uniform over the cross section of the conductor, $d$ is the G.M.D. of the cross section of the conductor with respect to itself. In this case $d$ for a solid conductor is $0.779R$.

The potential drop per mile is

$$V_a = I_a r + j0.00466f\left(I_a \log\frac{1}{d} + I_b \log\frac{1}{ab} \cdots + I_n \log\frac{1}{an}\right)$$
vector volts per mile (106)

where $r$ is now the resistance of the $a$ conductor *per mile*, $f$ is the frequency, and the common logarithm (log) is taken instead of the natural logarithm (ln).

**44. Three-phase, Three Wires Untransposed.**—Assume that the conductors are of the same size. The resistances and internal reactances are then the same. Thus

$$V_a = I_a r + j0.00466f\left(I_a \log\frac{1}{d} + I_b \log\frac{1}{ab} + I_c \log\frac{1}{ac}\right) \text{ vector volts per mile}$$

$$V_b = I_b r + j0.00466f\left(I_b \log\frac{1}{d} + I_c \log\frac{1}{bc} + I_a \log\frac{1}{ab}\right) \text{ vector volts per mile}$$

$$V_c = I_c r + j0.00466f\left(I_c \log\frac{1}{d} + I_a \log\frac{1}{ac} + I_b \log\frac{1}{bc}\right) \text{ vector volts per mile}$$

The form of these expressions shows that $\left(r + j0.00466f \log\frac{1}{d}\right)$ may be treated as a self-impedance and that $0.00466f \log 1/ab$, $0.00466f \log 1/bc$, and $0.00466f \log 1/ca$ may be treated as mutual reactances. Since the self-impedances are equal, they contain zero-sequence components only. That is

$$Z_{a0} = r + j0.00466f \log\frac{1}{d} \text{ vector ohms per mile} \qquad (107)$$

In order to determine the symmetrical components of the potential drop in any conductor, as $a$, it is best to use the symmetrical components of the mutual reactance between the other two conductors, as $b$ and $c$. The zero-sequence component of this mutual reactance is

$$X_{bc0} = \frac{0.00466f}{3}\left(\log \frac{1}{ab \cdot bc \cdot ca}\right)$$
$$= 0.00466f \log \frac{1}{\sqrt[3]{ab \cdot bc \cdot ca}} \text{ ohms per mile} \quad (108)$$

The positive-sequence component is,

$$X_{bc1} = \frac{0.00466f}{3}\left(\log \frac{1}{bc} + \underline{/120°} \log \frac{1}{ca} + \underline{/240°} \log \frac{1}{ab}\right)$$
$$= 0.00155f\left(\log \frac{\sqrt{ca \cdot ab}}{bc} + j0.866 \log \frac{ab}{ca}\right) \text{ vector ohms per mile} \quad (109)$$

The negative-sequence component is

$$X_{bc2} = 0.00155f\left(\log \frac{\sqrt{ca \cdot ab}}{bc} - j0.866 \log \frac{ab}{ca}\right) \text{ vector ohms per mile} \quad (110)$$

Notice that the positive- and negative-sequence components of this mutual reactance are always conjugate. The total line drop, $V_a$, in conductor $a$ equals the sum of its zero-, positive- and negative-sequence components, that is,

$$V_a = V_{a0} + V_{a1} + V_{a2}$$

Since in the case we are considering there can be no zero-sequence component in the current, the zero-sequence component of the potential is of little importance. It indicates the displacement of the neutral. The components of the line drop are (page 89)

$$\left.\begin{aligned} V_{a0} &= -jX_{bc2}I_{a1} - jX_{bc1}I_{a2} \\ V_{a1} &= I_{a1}(Z_{a0} - jX_{bc0}) + j2X_{bc2}I_{a2} \\ V_{a2} &= I_{a2}(Z_{a0} - jX_{bc0}) + j2X_{bc1}I_{a1} \end{aligned}\right\} \quad (111)$$

**Illustration.**—Consider a 60-cycle circuit consisting of three No. 0000 copper conductors in the same plane, spaced 1 ft. on centers. These conductors have a resistance of 0.259 ohm

per mile and a diameter of 0.460 in. For a conductor of this size the current density at 60 cycles is substantially uniform. Let $a$ be the central conductor.

$$Z_{a0} - jX_{bc0} = 0.259 + j0.280 \log \frac{\sqrt[3]{12 \times 12 \times 24}}{0.779 \times 0.23}$$
$$= 0.259 + j0.538 \text{ vector ohm per mile}$$
$$2X_{bc1} = 0.186 \left( \log \frac{\sqrt{12 \times 12}}{24} + j0.866 \log \frac{12}{12} \right)$$
$$2X_{bc1} = -0.0560 + j0 \text{ ohm per mile}$$
$$2X_{bc2} = -0.0560 + j0 \text{ ohm per mile}$$

Thus the positive- and negative-sequence components of the mutual reactance are capacitive in their action.

**45. Effect of Transposition.**—It is self-evident that transposition has no effect on the zero-sequence components of either self- or mutual reactance. If, in the second section, conductor $a$ occupies the same relative position that conductor $b$ occupied in the first section, the mutual reactance between the $b$ and $c$ conductors in the second section is the same as the mutual reactance between the $a$ and $c$ conductors in the first section. Similarly, the mutual reactance between the $b$ and $c$ conductors in the third section is the same as the mutual reactance between the $a$ and $b$ conductors in the first section.

Thus in the second section

$$X_{bc1} = 0.00155f\left( \log \frac{\sqrt{ab \cdot bc}}{ca} + j0.866 \log \frac{bc}{ab} \right) \text{ vector ohms per mile}$$

$$X_{bc2} = 0.00155f\left( \log \frac{\sqrt{ab \cdot bc}}{ca} - j0.866 \log \frac{bc}{ab} \right) \text{ vector ohms per mile}$$

The distances $ab$, $bc$, and $ca$ have the same values as in the first section. For example, $ab$ is the distance between the $a$ and $b$ conductors in the first section but in the second section it is the distance between the $c$ and $a$ conductors, since the $c$ and $a$ conductors now occupy the same relative positions as did the $a$ and $b$ conductors in the first section.

In the third section

$$X_{bc1} = 0.00155f\left(\log \frac{\sqrt{bc \cdot ca}}{ab} + j0.866 \log \frac{ca}{bc}\right) \text{ vector ohms per mile}$$

$$X_{bc2} = 0.00155f\left(\log \frac{\sqrt{bc \cdot ca}}{ab} - j0.866 \log \frac{ca}{bc}\right) \text{ vector ohms per mile}$$

Again the distances *ab*, *bc*, and *ca* have the same values as they did in the first section.

If the three sections of the transposed line are of equal length, it will be observed that the average values of the positive- and negative-sequence components of the mutual reactances are both zero.

Thus there is no zero-sequence component in the line potential drop and the positive- and negative-sequence components are

$$V_{a1} = I_{a1}\left(r + j0.00466f \log \frac{\sqrt[3]{ab \cdot bc \cdot ca}}{d}\right) \text{ vector volts per mile} \quad (112)$$

$$V_{a2} = I_{a2}\left(r + j0.00466f \log \frac{\sqrt[3]{ab \cdot bc \cdot ca}}{d}\right) \text{ vector volts per mile} \quad (113)$$

Thus the impedance per conductor of the transposed line either to positive- or to negative-sequence is

$$Z = r + j0.00466f \log \frac{\sqrt[3]{ab \cdot bc \cdot ca}}{d} \text{ vector ohms per mile} \quad (114)$$

where $r$ is the resistance per mile of conductor.

**46. Three-phase, Three Wires and Neutral Conductor.**—The neutral conductor is not grounded and there is no earth current. As before let $r$ be the resistance of each of the line conductors per mile and let $r_n$ be the resistance of the neutral conductor per mile. Also let $d = R\epsilon^{-\frac{l}{2}}$, where $R$ and $l$ are the radius and internal inductance (abhenry per centimeter) of each of the line conductors. Let $d_n$ be the corresponding constant for the neutral conductor. The distances between the line conductors are *ab*, *bc*, and *ca* as before, while the distances between the line and neutral conductors are *an*, *bn*, and *cn*. The vector values of the line currents are $I_a$, $I_b$, and $I_c$ and thus the current in the neutral

conductor flowing in the same direction is $-(I_a + I_b + I_c)$. The potential drop $V_a$ in the $a$ conductor is

$$V_a = I_a r + j0.00466f\left[I_a \log\frac{1}{d} + I_b \log\frac{1}{ab} + I_c \log\frac{1}{ac} - (I_a + I_b + I_c)\log\frac{1}{an}\right] \text{ vector volts per mile}$$

The potential drop $V_n$ in the neutral conductor in the same direction is

$$V_n = -(I_a + I_b + I_c)r_n + j0.00466f\left[-(I_a + I_b + I_c)\log\frac{1}{d_n} + I_a\log\frac{1}{an} + I_b\log\frac{1}{bn} + I_c\log\frac{1}{cn}\right] \text{ vector volts per mile}$$

The difference of these two potentials is the resultant potential drop for the $a$ phase and is

$$V_{an} = I_a\left[r + r_n + j0.00466f\log\frac{(an)^2}{dd_n}\right] + I_b\left[r_n + j0.00466f\log\frac{an\cdot bn}{ab\cdot d_n}\right] + I_c\left[r_n + j0.00466f\log\frac{an\cdot cn}{ca\cdot d_n}\right] \text{ vector volts per mile}$$

Similarly the potential drop for the $b$ phase is

$$V_{bn} = I_b\left[r + r_n + j0.00466f\log\frac{(bn)^2}{dd_n}\right] + I_c\left[r_n + j0.00466f\log\frac{bn\cdot cn}{bc\cdot d_n}\right] + I_a\left[r_n + j0.00466f\log\frac{bn\cdot an}{ab\cdot d_n}\right] \text{ vector volts per mile}$$

Likewise the potential drop in the $c$ phase is

$$V_{cn} = I_c\left[r + r_n + j0.00466f\log\frac{(cn)^2}{dd_n}\right] + I_a\left[r_n + j0.00466f\log\frac{cn\cdot an}{ca\cdot d_n}\right] + I_b\left[r_n + j0.00466f\log\frac{cn\cdot bn}{bc\cdot d_n}\right] \text{ vector volts per mile}$$

Again the form of these expressions shows that the coefficients of $I_a$ in $V_a$, of $I_b$ in $V_b$ and of $I_c$ in $V_c$ are self-impedances. The *zero-, positive-, and negative-sequence components of $Z_a$ are*

$$\left.\begin{aligned}
Z_{a0} &= \left[r + r_n + j0.00466f \log \frac{(an \cdot bn \cdot cn)^{\frac{2}{3}}}{dd_n}\right] \text{ vector ohms per mile} \\
Z_{a1} &= j0.00155f\left[\log \frac{(an)^2}{bn \cdot cn} + j\sqrt{3} \log \frac{bn}{cn}\right] \text{ vector ohms per mile} \\
Z_{a2} &= j0.00155f\left[\log \frac{(an)^2}{bn \cdot cn} - j\sqrt{3} \log \frac{bn}{cn}\right] \text{ vector ohms per mile}
\end{aligned}\right\} \tag{115}$$

Further, it will be observed that the other coefficients of the line currents are mutual impedances. The zero-, positive-, and negative-sequence components of the mutual impedance between conductors $b$ and $c$ are

$$\left.\begin{aligned}
Z_{bc0} &= \left[r_n + j0.00466f \log \frac{\sqrt[3]{(an \cdot bn \cdot cn)^2}}{\sqrt[3]{(ab \cdot bc \cdot ca)}\, d_n}\right] \text{ vector ohms per mile} \\
Z_{bc1} &= j0.00155f\left[\log \frac{\sqrt{bn \cdot cn \cdot ab \cdot ca}}{bc \cdot an} + j0.866 \log \frac{cn \cdot ab}{bn \cdot ca}\right] \\
&\qquad \text{vector ohms per mile} \\
Z_{bc2} &= j0.00155f\left[\log \frac{\sqrt{bn \cdot cn \cdot ab \cdot ca}}{bc \cdot an} - j0.866 \log \frac{cn \cdot ab}{bn \cdot ca}\right] \\
&\qquad \text{vector ohms per mile}
\end{aligned}\right\} \tag{116}$$

If the line is transposed and has sections of equal length, the average values of the positive- and negative-sequence components of both the self- and mutual impedances are zero. In this case the zero-, positive-, and negative-sequence components of the potential drop per phase are (see page 93)

$$\begin{aligned}
V_{an0} &= I_0(Z_{a0} + 2Z_{bc0}) \\
&= I_0\left[r + 3r_n + j0.00466f \log \frac{(an \cdot bn \cdot cn)^2}{dd_n^3 \sqrt[3]{(ab \cdot bc \cdot ca)^2}}\right] \text{ vector} \\
&\qquad \text{volts per mile}
\end{aligned}$$

$$V_{an1} = I_1(Z_{a0} - Z_{bc0})$$
$$= I_1\left(r + j0.00466f \log \frac{\sqrt[3]{ab \cdot bc \cdot ca}}{d}\right) \text{ vector volts per mile} \quad (117)$$

Also

$$V_{an2} = I_2\left(r + j0.00466f \log \frac{\sqrt[3]{ab \cdot bc \cdot ca}}{d}\right) \text{ vector volts per mile}$$

Notice that the neutral conductor has no effect on the positive- and negative-sequence components of the potential drop when the line is symmetrically transposed. That is, the impedances of the transposed line per conductor to positive and to negative sequence are not affected by the presence of the neutral conductor. The impedance per phase, $V_{an0}/I_{a0}$, to zero sequence is

$$r + 3r_n + j0.00466f \log \frac{(an \cdot bn \cdot cn)^2}{dd_n^3\sqrt[3]{(ab \cdot bc \cdot ca)^2}} \text{ vector ohms per mile} \quad (118)$$

Note that $V_{an0}$ is the zero-sequence component of the potential drop for the loop consisting of a line conductor and the neutral conductor and that $I_0$ is the zero-sequence component of the line current. The current in the neutral conductor is $3I_0$.

**47. Two-phase, Three-wire Circuit.**—The currents in the two line conductors, $a$ and $b$, are $I_a$ and $I_b$. The current in the neutral conductor $n$, is $-(I_a + I_b)$. The resistance of each line conductor is $r$ while that of the neutral conductor is $r_n$. In this problem it may be more convenient to give the values of these resistances in ohms per 1000 ft. and this will be done. As before $d$ and $d_n$ are the G.M.D.'s for the line conductors and for the neutral conductor.

The potential drop in the line conductor $a$ per 1000 ft. is

$$V_a = I_ar + j0.000882f\left[I_a \log \frac{1}{d} + I_b \log \frac{1}{ab} - (I_a + I_b) \log \frac{1}{an}\right]$$
vector volts per 1000 ft.

Likewise

$$V_b = I_br + j0.000882f\left[I_b \log \frac{1}{d} + I_a \log \frac{1}{ab} - (I_a + I_b) \log \frac{1}{bn}\right]$$
vector volts per 1000 ft.

The potential drop in the neutral conductor per 1000 ft. is

$$V_n = -(I_a + I_b)r_n + j0.000882f\left[-(I_a + I_b)\log\frac{1}{d_n} + I_a\log\frac{1}{an} + I_b\log\frac{1}{bn}\right] \text{ vector volts per 1000 ft.}$$

Since the total line drop for the $a$ phase is $V_a - V_n$, or, as we shall write it, $V_{an}$, we have

$$V_{an} = I_a\left[r + r_n + j0.000882f\log\frac{(an)^2}{dd_n}\right] + I_b\left[r_n + j0.000882f\log\frac{an \cdot bn}{ab \cdot d_n}\right] \text{ vector volts per 1000 ft.}$$

Similarly, the total line drop for the $b$ phase is

$$V_{bn} = I_b\left[r + r_n + j0.000882f\log\frac{(bn)^2}{dd_n}\right] + I_a\left[r_n + j0.000882f\log\frac{an \cdot bn}{ab \cdot d_n}\right] \text{ vector volts per 1000 ft.}$$

If the coefficients of $I_a$ and $I_b$ are represented by $z_a$, $z_b$, and $z_{ab}$, we may write simpler forms for these line drops

$$V_{an} = I_a z_a + I_b z_{ab}$$
$$V_{bn} = I_b z_b + I_a z_{ab}$$

By following the procedure developed in Chap. IV (page 108), the positive- and negative-sequence components of $V_{an}$ are found to be

$$V_{an1} = I_{a1}\frac{z_a + z_b}{2} + I_{a2}\left(\frac{z_a - z_b}{2} + jz_{ab}\right)$$
$$V_{an2} = I_{a2}\frac{z_a + z_b}{2} + I_{a1}\left(\frac{z_a - z_b}{2} - jz_{ab}\right)$$

If the values of $z_a$, $z_b$ and $z_{ab}$ are now resubstituted, the equations for the positive- and negative-sequence components of the line drop become

$$V_{an1} = I_{a1}\left(r + r_n + j0.000882f\log\frac{an \cdot bn}{d \cdot d_n}\right) + I_{a2}\left[-0.000882f\log\frac{an \cdot bn}{ab \cdot d_n} + j\left(r_n + 0.000882f\log\frac{an}{bn}\right)\right] \text{ vector volts per 1000 ft.} \quad (119a)$$

and

$$V_{an3} = I_{a3}\left(r + r_n + j0.000882f \log \frac{an \cdot bn}{d \cdot d_n}\right)$$
$$+ I_{a1}\left[0.000882f \log \frac{an \cdot bn}{ab \cdot d_n} - j\left(r_n - 0.000882f \log \frac{an}{bn}\right)\right] \text{ vector volts per 1000 ft.} \quad (119b)$$

**Illustration.**—Let the $a$ and $b$ conductors be No. 3 copper wires spaced 2 ft. apart and the neutral conductor, No. 1 copper wire placed midway between them. The length of each of the conductors is 1500 ft. The resistances of these conductors are $r = 0.308$ ohm and $r_n = 0.194$ ohm. The G.M.D.'s of these conductors are $d = 0.0896$ in. and $d_n = 0.113$ in. The frequency is 60 cycles. If these values are substituted in equations (119$a,b$), we have

$$V_{an1} = I_{a1}(0.502 + j0.330) + I_{a3}(-0.137 + j0.194) \text{ vector volts}$$
$$V_{an3} = I_{a3}(0.502 + j0.330) + I_{a1}(0.137 - j0.194) \text{ vector volts}$$

**48. Effect of the Earth as a Conducting Body.**—We have hitherto assumed that the conductors through which the currents flow were isolated, whereas they are actually suspended above the earth. We have now reached a point in our analysis where it becomes necessary to consider the effect of the earth as a conducting body. To be sure it is a very poorly conducting body and one that is unfortunately very far from uniform. Its resistivity varies locally from about 500 to about 1,000,000 ohms per centimeter cube.[1] An average value is in the neighborhood of 2500[1] ohms per centimeter cube or about one billion times that of copper. On account of this extremely high resistivity, the eddy currents induced in the earth by the action of the currents in the transmission conductors are of minor importance. We shall presently estimate their effect in a typical case. Furthermore, the resistivity varies with the distance below the surface of the earth. Near the surface the resistivity may depend to a large extent upon the moisture content of the soil and thus may be controlled by the season or by the state of the weather. In such a problem as this all that we can hope to do is to assume some ideal condition that lends itself to mathematical analysis

[1] CLEM, J. E., Reactance of Transmission Lines with Ground Return, *Am. Inst. Elec. Eng.*, **50**, 901

and then so to temper our solution by experience that we may forecast results with reasonable accuracy.

The problem before us resolves itself into two component parts, that of finding the self-impedance of a circuit consisting of a straight wire parallel to and above the surface of the earth with a ground return and that of finding the mutual impedance of two such circuits. Figure 36 shows a longitudinal elevation of the *a* circuit which is grounded at *g*. The self-impedance equals the ratio of the vector volts applied between the conductor and ground at the open end to the vector current flowing in the conductor and earth. Figure 37 shows two such circuits in

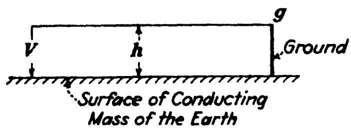


FIG. 36.

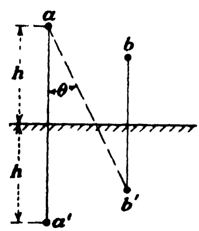


FIG. 37.

section. Both circuits are grounded at one end and are open at the other. The mutual impedance between *a* and *b* is the ratio of the vector volts appearing between the conductor and ground at the open end of *b* to the vector current that flows in *a*. Several different solutions, depending upon different assumptions,[1] have been given for these impedances. The one we shall use was developed by Carson. He assumed that the earth's conductivity was uniform and that its boundary surface was a plane to which the conductor was parallel; that the magnetic field within the conductor was due wholly to the current in the conductor itself, which is so long that end effects may be neglected. Carson's expression for the self-impedance, $Z_{aa}$, of a conductor-earth loop is in the form

$$Z_{aa} = Z_i + j2\omega \ln \frac{2h}{R} + 4\omega(P + jQ) \text{ c.g.s.}$$

[1] MAYR, Die Erde als Wechselstromleiter, *ETZ*, **46**, 1352, 1925.

RUDENBERG, REINHOLD, Die Ausbrietung der Erdströme in der Umgebung von Wechselstromleitungen, *Z. angew. Math. Mech.*, 1926, vol. 5, p 361.

or

$$Z_{aa} = r + j2\omega \ln \frac{2h}{d} + 4\omega(P + jQ) \tag{120}$$

The first term, $Z_i$, in the first of these two alternative expressions for $Z_{aa}$ is the internal impedance of the conductor itself. The second term is the reactance of the loop due to flux without the conductor on the assumption that the conductivity of the earth is infinite. In this term $R$ is the radius of the conductor and $h$ is its height above the surface of the *conducting* mass of the earth. If the upper layer of the earth is very dry and is therefore substantially nonconducting, $h$ may be greater than the height of the conductor above the actual surface of the earth. As usual $\omega$ equals $2\pi f$, where $f$ is the frequency. In the alternative expression, $r$ and $d$ have the significance previously attached to them [equation (105), page 113]. In both expressions the third term is a correction term which accounts for the fact that the conductivity of the earth is not infinite. The effect is additive for both the resistance and the reactance. $P$ and $Q$ are functions for which Carson[1] gives equations. Tabulated values of these functions computed from the Carson equations are given in Clem's paper noted above (page 121).

It is important to note that, while $4\omega P$ is the resistance of the earth path (c.g.s. units), $4\omega Q$ is *not* the reactance of the earth path. It is merely the difference between the actual reactance of the conductor-earth loop and the reactance that the loop would have were the conductivity of the earth infinite.

Before discussing these functions, we will write the expression for the mutual impedance between one conductor-earth loop and another similar loop

$$Z_{ab} = j2\omega \ln \frac{ab'}{ab} + 4\omega(P + jQ) \text{ c.g.s.} \tag{121}$$

where $ab$ is the distance between the conductors $a$ and $b$, and $ab'$ is the distance between the conductor $a$ and the image of the conductor $b$. The image of the conductor should be taken with respect to the surface of the *conducting mass of the earth,* which may in some localities be somewhat below the *actual surface* of

[1] CARSON, JOHN R., Wave Propagation in Overhead Wires with Ground Return, *Bell System Tech. J.*, October, 1926

the earth. In general $P$ and $Q$ depend upon a quantity $\rho$ and an angle $\theta$.

In the case of self-impedance $\rho$ equals $2\pi\sqrt{2f\lambda}$ times the distance between the conductor and its own image. In the case of mutual impedance, $\rho$ equals $2\pi\sqrt{2f\lambda}$ times the distance between the conductor $a$ and the image of the conductor $b$. If these distances are expressed in feet, this multiplying factor becomes $271\sqrt{f\lambda}$. In this and all that follows $\lambda$ is the conductivity of the earth in c.g.s. units. It varies between about $10^{-15}$ and $2 \times 10^{-12}$. For example, if $h = 100'$, $\lambda = 10^{-12}$ c.g.s. units and $f = 60$ cycles, $\rho = 0.42$. For power transmission lines this is a rather large value of $\rho$. In the majority of cases it rarely exceeds about 0.20. Carson gives equations for $P$ and $Q$ which may be used if $\rho$ is not greater than 0.25, *viz.*,

$$P = \frac{\pi}{8} - \frac{1}{3\sqrt{2}}\rho\cos\theta + \frac{\rho^2}{16}\left(0.673 + \ln\frac{2}{\rho}\right)\cos 2\theta + \frac{\rho^2}{16}\theta\sin 2\theta \tag{122}$$

$$Q = -0.0386 + \frac{1}{2}\ln\frac{2}{\rho} + \frac{1}{3\sqrt{2}}\rho\cos\theta \tag{123}$$

Carson also gives other expressions for $P$ and $Q$ when $\rho$ is greater than 0.25. The angle $\theta$ is zero when computing the self-impedance. When computing the mutual impedance, it is the angle between a perpendicular to the earth's surface and a line drawn from the conductor $a$ to the image of the conductor $b$ (see Fig. 37). If the conductors are supported from a single line of towers, the angle $\theta$ is so small that it affects the values of $P$ and $Q$ by not more than about 1 per cent, and thus need not be considered. In case the conductors under consideration are supported from parallel lines of towers, or in problems involving interference with parallel communication circuits, the values of $\rho$ and of $\theta$ may be so great as to warrant using values of $P$ and $Q$ computed from Carson's equations.

Thus, for the usual power lines, the effect of the earth depends almost entirely upon the value of $\rho$ which, as we have seen, is proportional to the height of the conductors above the earth, the square root of the frequency, and the square root of the conductivity of the earth. Of these factors, the frequency is alone susceptible of exact determination. Observe that, for the usual

values of $\rho$, the resistance of the earth, $4\omega P$, increases with the frequency and the *inductance* term, $4Q$, decreases with the frequency. These are the usual phenomena noted in the conduction of alternating currents. As previously mentioned, the height of the conductor should be its distance above the *conducting* mass of the earth and not above the surface of the earth. The effect of the sag in the transmission conductor may be accounted for by choosing the height, $h$, as the minimum clearance above the conducting mass of the earth plus one-third of the sag. This is probably as reasonable an approximation as can readily be made. It is suggested by the fact that the curve of the line conductor is approximately parabolic. The conductivity of the earth is by far the most indefinite of the quantities with which we have to deal. Fortunately, however, a great change in the conductivity affects the resistance and reactance of a conductor-earth loop by a relatively small amount. For example, consider a No. 0000 copper wire 50 ft. above the earth's conducting mass. The self-impedance per centimeter (c.g.s. units) at 60 cycles for this conductor-earth loop, on the assumption that the conductivity, $\lambda$, is $10^{-14}$ c.g.s. units, is

$$Z_{aa} = 1670 + j754 \ln \frac{100}{0.015} + 1508(P + jQ)$$

Since

$$\rho = 271\sqrt{60} \times 10^{-14} \times 100 = 0.021,$$
$$P = 0.38790$$

and

$$Q = 2.2466.$$

Thus

$$Z_{aa} = 2260 + j10{,}000$$

If the earth's conductivity is 100 times as great, *viz.* $10^{-12}$ c.g.s. units, $\rho$ becomes 10 times as great, *viz.* 0.21. For this value of $\rho$

$$P = 0.3514$$
$$Q = 1.1378$$
$$Z_{aa} = 2200 + j8350$$

That is, an increase in the earth's conductivity of 9900 per cent reduces the resistance of the conductor-earth loop by less than 2.7 per cent and reduces the reactance by about 16.5 per cent.

It is indeed fortunate that such extreme variations in the earth's conductivity have little effect upon the important impedance constants.

For the problems in which we are most interested the accuracy of the following simplified expressions for $P$ and $Q$ is sufficient.

$$P = 0.39 - \frac{\rho}{6} \tag{124}$$

$$Q = \frac{1}{2} \ln \frac{2}{\rho} \tag{125}$$

Within the practical range $\rho = 0.01$ to $\rho = 0.25$, the greatest error in each of the expressions (124) and (125) is about 1 per cent. When $\rho$ has a mean value of about 0.12, $P = 0.37$. If it is assumed to have this constant value, we may write

$$Z_{aa} = r + 4\omega \times 0.37 + j2\omega\left(\ln \frac{2h}{d} + \ln \frac{2}{\rho}\right) \text{ c.g.s.}$$

$$= r + 9.3f + j2\omega \ln \frac{4h}{\rho d} \text{ c.g.s.}$$

Letting $\frac{4h}{\rho} = D$, the value of $D$ is

$$D = \frac{1}{\pi\sqrt{2f\lambda}} \text{ cm.}$$

$$= \frac{0.00738}{\sqrt{f\lambda}} \text{ ft.} \tag{126}$$

The expression for the self-impedance becomes

$$Z_{aa} = r + 9.3f + j2\omega \ln \frac{D}{d} \text{ c.g.s.}$$

or

$$Z_{aa} = r + 0.0015f + j0.00466f \log \frac{D}{d} \text{ ohms per mile} \tag{127}$$

It is obvious that $r$ in the first of these expressions is the resistance of the conductor per centimeter in c.g.s. units whereas in the second expression it is the resistance of the conductor in ohms per mile. It is important to note that, within the limits of error of this simplified expression, the self-impedance of the conductor-earth loop, $Z_{aa}$, is independent of the height of the conductor

above the earth. It is thus the same for each of the conductors of a transmission line. Similarly the mutual impedance is

$$Z_{ab} = 0.0015f + j0.00466f \log \frac{D}{ab} \text{ ohms per mile} \qquad (128)$$

The mutual impedance between two conductor-earth loops depends upon the separation of the conductors. Thus the mutual impedances between the different conductor-earth loops of a transmission line are equal only when the conductors are equally spaced; that is, when $ab = bc = ca$. The distances, $D$, $d$, and $ab$ in equations (127) and (128) must be expressed in the same units. Notice that at 60 cycles the resistance of the earth return is 0.09 ohm per mile. From this comes the common statement that the resistance of the earth return is 0.1 ohm per mile at 60 cycles. We shall subsequently refer to these expressions, (127) and (128) as the "Carson impedances." Notice that $X_{aa}$ is the reactance of one of two conductors $D$ apart,[1] and thus it is sometimes said that the effect of the earth on the reactance is the same as that of another conductor $D$ distant from the transmission conductor. At 60 cycles and for an average value of the conductivity, $\lambda$, of $4 \times 10^{-13}$, $D = 1500$ ft. Thus in this case the reactance of the conductor-earth loop is one-half the reactance of a loop formed by two conductors of the same size as the transmission conductor placed 1500 ft. apart. If the reactance of the two-conductor loop is to be equal to the reactance of the conductor-earth loop, the separation of the two conductors must be $\sqrt{D \cdot d}$. If the conductors have a cross section of $10^6$ cir. mils and $D = 1500$ ft., this separation would be but 6.96 ft., which is much less than the height of the conductor above the earth. Further difficulties will arise if one attempts to replace both the $a$ and $b$ circuits by equivalent circuits which will correctly reproduce both the self- and mutual reactance. It seems very important to the author that the term $2\omega \ln D/d$, where $D = 0.00738/\sqrt{f\lambda}$ ft., should only be looked upon as a convenient way of expressing the reactance of the circuit formed by the conductor and the earth. The replacement of the actual circuit by an equivalent one is of doubtful value and a procedure

[1] The reactance of a circuit consisting of two parallel conductors whose self G.M.D's are $d$ and whose separation is $D$ is $4\omega \ln \frac{D}{d}$ per cm.

that may lead to considerable confusion if followed to a logical conclusion.

These two expressions [equations (127) and (128)] for the self- and mutual impedance of conductor-earth loops enable us to compute the approximate values of the impedance to zero sequence of any power circuit. The limitations of these expressions have been discussed in some detail. As we shall see presently, the impedances to positive and to negative sequence are practically independent of the effect of the earth and of ground wires and are computed by using equation (114), page 116.

**49. Three Wires Untransposed.**—Consider three conductors, *a*, *b*, and *c*, parallel to and above the surface of the earth. Let $Z_{aa}$, $Z_{bb}$, and $Z_{cc}$ be the Carson self-impedances per mile of the conductor-earth loops. Also let $Z_{ab}$, $Z_{bc}$, and $Z_{ca}$ be the Carson mutual impedances per mile between these conductor-earth loops. Furthermore, let $V_a$ be the potential drop per mile in the conductor-earth loop for conductor *a* and, similarly, let $V_b$ and $V_c$ be the potential drops for conductors *b* and *c*. Thus we may write

$$
\begin{aligned}
V_a &= I_a Z_{aa} + I_b Z_{ab} + I_c Z_{ac} \\
V_b &= I_b Z_{bb} + I_c Z_{bc} + I_a Z_{ab} \\
V_c &= I_c Z_{cc} + I_a Z_{ac} + I_b Z_{bc}
\end{aligned}
$$

These equations may be transformed into equations for the symmetrical components of $V_a$ as follows (see page 89)

$$
\begin{aligned}
V_{a0} &= I_{a0}(Z_{aa0} + 2Z_{bc0}) + I_{a1}(Z_{aa2} - Z_{bc2}) + I_{a2}(Z_{aa1} - Z_{bc1}) \\
V_{a1} &= I_{a1}(Z_{aa0} - Z_{bc0}) + I_{a2}(Z_{aa2} + 2Z_{bc2}) + I_{a0}(Z_{aa1} - Z_{bc1}) \\
V_{a2} &= I_{a2}(Z_{aa0} - Z_{bc0}) + I_{a0}(Z_{aa2} - Z_{bc2}) + I_{a1}(Z_{aa1} + 2Z_{bc1})
\end{aligned}
$$

**50. Effect of Transposition.**—We have already noted (page 126) that the self-impedances of the different conductor-earth loops are alike. Thus they have neither positive- nor negative-sequence components. On the other hand, the mutual-impedances between the conductor-earth loops of any single transposition section are, in general, different and thus have both positive- and negative-sequence components. If the line is symmetrically transposed, however, the average values of these mutual impedances over the entire line are equal. Or, conversely, it may be said that a line is symmetrically transposed when the *average* values of the mutual impedances between the different conductor-earth

loops are equal. Thus, if the line is properly transposed, the symmetrical components of $V_a$ are

$$\left.\begin{aligned} V_{a0} &= I_{a0}(Z_{aa0} + 2Z_{bc0}) \\ V_{a1} &= I_{a1}(Z_{aa0} - Z_{bc0}) \\ V_{a2} &= I_{a2}(Z_{aa0} - Z_{bc0}) \end{aligned}\right\} \tag{129}$$

Notice again that the circuit has the same impedance to both positive- and negative-sequence currents but that the impedance to zero-sequence current is quite different (see page 93).

**51. Induced Earth Currents.**—We have said that the effect of the eddy currents induced in the earth are negligible. Let us now estimate their effect in a typical case. We shall do this by comparing the value of the impedances to positive- and negative-sequence currents, $Z_{aa0} - Z_{bc0}$, when calculated by the Carson method and when calculated by the usual method which neglects the effect of the earth. There can be no conduction currents in the earth due to the positive- and negative-sequence component currents in the conductors since the sum of each of these components in the three conductors is zero. Only zero-sequence components can flow by conduction in the earth. However, since the earth is a conducting body, eddy currents will be induced in it by the action of the positive- and negative-sequence currents in the line conductors. There is a certain amount of energy loss due to these eddy currents which increases the apparent resistance of the conductors. Furthermore, owing to the effect of the magnetic field that is produced by these eddy currents, the reactance of the conductors is decreased. Let us now consider a typical 3-phase circuit consisting of three 1,000,000 cir.-mil cables lying in a horizontal plane 50 ft. above the surface of the earth, with a separation of 20 ft. between adjacent conductors. Assume that the conductivity of the earth has such a value that $\rho_{aa} = 0.2$. It then follows that $\rho_{ab} = 0.204$ with $\theta_{ab} = 11°$ and $\rho_{ac} = 0.214$ with $\theta_{ac} = 22°$.

The values of the required $P$ and $Q$ functions may be found from the tables by interpolation. The difference between the values of the impedance $(Z_{aa0} - Z_{bc0})$ calculated by both methods is

$$\text{Difference} = 2\omega(0.0006 - j0.0008)$$

This difference represents an increase of 0.13 per cent in the resistance of the conductor and a decrease of 0.012 per cent in the

reactance of the conductor as calculated by the usual method. We are therefore justified in calculating the impedances to positive- and negative-sequence currents of a properly transposed line without considering the effect of the earth.

**52. Impedance to Zero Sequence.**—If there is neither neutral conductor nor ground wire, all of the zero-sequence current is carried by the earth. The zero-sequence component of the line drop is

$$V_{a0} = I_{a0}(Z_{aa0} + 2Z_{bc0})$$

Since the self-impedance of each of the conductor-earth loops is the same, the self-impedance to zero sequence is

$$Z_{aa0} = r + 0.0015f + j0.00466f \log \frac{D}{d} \text{ vector ohms per mile} \quad (130)$$

The zero-sequence component of the mutual impedance of the conductor-earth loops is

$$Z_{bc0} = 0.0015f + j0.00466f \log \frac{D}{\sqrt[3]{ab \cdot bc \cdot ca}} \text{ vector ohms per mile} \quad (131)$$

Thus the zero-sequence component of the line drop may be written

$$V_{a0} = I_{a0}\left(r + 0.0045f + j0.00466f \log \frac{D^3}{d\sqrt[3]{(ab \cdot bc \cdot ca)^2}}\right) \text{ vector volts per mile} \quad (132)$$

As previously noted, this expression was derived by assuming the line was so long that the end effects might be neglected. In other words, it was assumed that the lines of current flow in the earth were parallel to the conductor.[1] This is reasonably true except where the current enters and leaves the earth. At these points the earth current converges upon the grounding conductors and there is a local potential drop of relatively considerable magnitude. This potential drop is the product of the current flowing into the earth at this point and the so-called "ground resistance." The ground resistance, $R_g$, may vary from about 1 ohm to 1000 or 2000 ohms depending upon the condition

[1] RUDENBERG, REINHOLD. See footnote, page 122.

of the soil and the size, shape and number of the grounding conductors and the depth to which they are buried.[1] Chemical treatment of the soil is sometimes resorted to in case the ground resistance is naturally high. At generating stations and at substations, more or less elaborate systems of grounding rods and buried conductors are used to insure a low ground resistance. At other points, such as tower footings, where a higher ground resistance is allowable, a single ground rod may be considered sufficient.

The effect of the ground resistance at the station and the ground resistance at the tower where the fault occurs may be introduced by adding to the impedance of the line to zero sequence, as determined by equation (132), three times the sum of these ground resistances. The current flowing through these ground resistances into the earth is three times the zero-sequence component of the line currents. Thus the impedance of the circuit to the zero sequence, including the effects of the terminal ground resistances, is

$$Z_0 = \frac{V_{a0}}{I_{a0}} = 3R_g + L\Big[r + 0.0045f + j0.00466f \log \frac{D^3}{d\sqrt[3]{(ab \cdot bc \cdot ca)^2}}\Big] \text{ vector ohms} \quad (133)$$

where $R_g$ is the sum of the terminal ground resistances at the station and at the fault, and $L$ is the distance in miles between them. $V_{a0}$ is the zero-sequence component in the phase potential (line-to-neutral) at the station. This assumes [equation (133)] that the line is open-circuited beyond the fault.

**53. Ground Wires.**—Many transmission lines are protected by ground wires, in which case the zero-sequence current is divided, part flowing through the earth and part through the ground wires. This division of the current between the ground wires and the earth is affected to a considerable extent by the magnitudes of the ground resistances.

In Fig. 38 the station-ground resistance, $R_g$, is assumed to be much smaller than the ground resistance, $r_g$, at the tower footings, $d$, $e$, $f$, etc. If a fault occurs just beyond $d$, it is evident

[1] DWIGHT, HERBERT B., Calculation of Resistances to Ground, *Am. Inst. Elec. Eng.*, **55**, 1319, 1936.

that the current in the ground wire between $d$ and $e$ is greater than that between $e$ and $f$. If the fault occurs at a point where the ground resistance is relatively high, the ground wire near that point will carry a greater proportion of the fault current than if the ground resistance were lower. Furthermore, it is evident that if, at the station, the ground wire terminates at a tower footing of resistance, $r_g$, the ground-wire current in the section between $h$ and $k$ will be less than if the ground wire is directly connected to the station ground at $k$, as shown by the dotted line. It thus appears that the determination of the current in the ground wire under actual conditions is a difficult problem.[1] There are

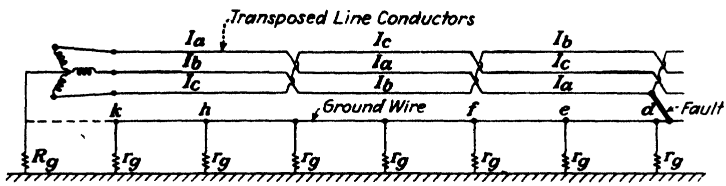


FIG. 38.

additional difficulties, since it is probable that the ground resistances, $r_g$, are not equal; also, if the line is one of considerable length, the conductivity of the earth may not be the same at different points along the line. For these reasons and because the author considers a more exact solution beyond the scope of this volume, we shall assume that the current in the ground wire is the same throughout its length, from the station ground to the fault. The error introduced by this assumption decreases as the distance from the station ground to the fault increases. If the ground-wire current is uniform, no current will flow into the earth at any of the tower footings except at $d$, where the fault occurs (see Fig. 38). Thus it follows that the potential between the ground wire and the earth at each of the tower footings, $e, f, h$, etc., must be zero; otherwise current will flow into the earth at these points equal in value to the potential between the ground wire and the earth divided by the ground resistance, $r_g$.

With this assumption in regard to the uniformity of the current in the ground wire, let us now consider a section of the ground wire between adjacent transposition points, that is to say, between

[1] See discussion of Clem's *Am. Inst. Elec. Eng.* paper.

the points $d$ and $f$ in Fig. 38. The expression for the fall in potential around this ground-wire-earth loop in terms of the Carson coefficients of self- and mutual impedance is

$$L(I_pZ_{pp} + I_aZ_{ap} + I_bZ_{bp} + I_cZ_{cp}) = 0$$

where $L$ is the length of the transposition section in miles. If potentials existed between the ground wire and earth at the points $d$ and $f$ (Fig. 38), the difference between these potentials would replace the zero at the right of the equality sign. The current in this section of the ground wire would then depend upon the currents in the adjacent sections. If, however, the ground-wire current is to be uniform as we have assumed, its value in terms of the line currents and the coefficients of self- and mutual impedance may at once be written

$$I_p = -\frac{1}{Z_{pp}}(I_aZ_{ap} + I_bZ_{bp} + I_cZ_{cp})$$

In the next transposition section nearer the station, it will be observed that the expression for the ground-wire current is

$$I_p = -\frac{1}{Z_{pp}}(I_aZ_{cp} + I_bZ_{ap} + I_cZ_{bp})$$

The mutual impedances have the same values as in the preceding expression.

In the next transposition section the expression for the ground-wire current is

$$I_p = -\frac{1}{Z_{pp}}(I_aZ_{bp} + I_bZ_{cp} + I_cZ_{ap})$$

These expressions for the ground-wire currents are somewhat different in the different sections. This indicates that current must flow from the ground wire to earth at each of the transposition points. If, as we have assumed, there is no difference in potential between the ground wire and earth at the transposition points, we are forced to assume that the ground resistance at these points is zero. If these ground resistances were very great, little current would flow to earth at the tower footings and the ground-wire current would be substantially uniform. Actually, the relatively high value of these ground resistances tends to prevent the flow of current to earth and thus tends to maintain

a uniform value of current in the ground wire. For this reason, we shall assume that the ground-wire current is the average of these three values. That is, since $I_a + I_b + I_c = 3I_0$,

$$I_p = -\frac{(Z_{ap} + Z_{bp} + Z_{cp})}{Z_{pp}} \frac{(I_a + I_b + I_c)}{3}$$

$$= -\frac{3Z_{ap0}}{Z_{pp}} I_0 \qquad (134)$$

Observe that

$$Z_{ap0} = 0.0015f + j0.00466f \log \frac{D}{\sqrt[3]{ap \cdot bp \cdot cp}}$$

vector ohms per mile (135)

where $D$ is defined in equation (126), page 126.

The current in the ground wire is directly proportional to the zero-sequence component of the currents in the transmission conductors. If the values of the self- and mutual impedances are substituted, the expression for the ground-wire current becomes [see equation (127), page 126]

$$I_p = -\frac{0.0015f + j0.00466f \log \dfrac{D}{\sqrt[3]{ap \cdot bp \cdot cp}}}{r_p + 0.0015f + j0.00466f \log \dfrac{D}{d_p}} 3I_0 \qquad (136)$$

Note that $\sqrt[3]{ap \cdot bp \cdot cp}$ is the geometric mean distance (G.M.D.) between the ground wire $p$ and the $(a, b, c)$ circuit. The potential drop per mile in the $a$ conductor-earth loop for the first transposition section is

$$V_{ag} = I_a Z_{aa} + I_b Z_{ab} + I_c Z_{ac} + I_p Z_{ap}$$

In the second transposition section where the $a$ conductor occupies the position of the $b$ conductor in the first section, the potential drop per mile in the $a$ conductor-earth loop is

$$V_{ag} = I_a Z_{bb} + I_b Z_{bc} + I_c Z_{ab} + I_p Z_{bp}$$

In the third section the potential drop per mile is

$$V_{ag} = I_a Z_{cc} + I_b Z_{ac} + I_c Z_{bc} + I_p Z_{cp}$$

Note that $Z_{aa} = Z_{bb} = Z_{cc}$ (see page 126). If the transposition sections are all of the same length the average value of the potential drop per mile in the $a$ conductor-earth loop is

$$V_{ag} = I_a\frac{(Z_{aa} + Z_{bb} + Z_{cc})}{3} + I_b\frac{(Z_{ab} + Z_{bc} + Z_{ca})}{3} + I_c\frac{(Z_{ab} + Z_{bc} + Z_{ca})}{3} + I_p\frac{(Z_{ap} + Z_{bp} + Z_{cp})}{3}$$

This may be written

$$V_{ag} = I_aZ_{aa0} + I_bZ_{bc0} + I_cZ_{bc0} + I_pZ_{ap0}$$

The average value of the potential drop per mile in the $b$ conductor-earth loop is

$$V_{bg} = I_bZ_{aa0} + I_cZ_{bc0} + I_aZ_{bc0} + I_pZ_{ap0}$$

The average value of the potential drop per mile in the $c$ conductor-earth loop is

$$V_{cg} = I_cZ_{aa0} + I_aZ_{bc0} + I_bZ_{bc0} + I_pZ_{ap0}$$

Notice that the current in the ground wire induces equal potentials, $I_pZ_{ap0}$, in the three conductor-earth loops. That is, the ground-wire current induces only a zero-sequence component of potential drop. The positive- and negative-sequence components of the average value of the line drop, $V_{ag}$, per mile are

$$\left.\begin{aligned} V_{ag1} &= I_{a1}(Z_{aa0} - Z_{bc0}) \\ V_{ag2} &= I_{a2}(Z_{aa0} - Z_{bc0}) \end{aligned}\right\} \qquad (137)$$

Thus for a line that is properly transposed, the positive- and negative-sequence components of the line drop are not affected by the presence of the ground wires. Furthermore, as we have already seen, the impedance of the line to the positive- or negative-sequence components of current can be calculated with sufficient accuracy without regard to the effect of the earth (see page 129).

The zero-sequence component of the average value of the potential drop in the conductor-earth loops per mile is

$$V_{ag0} = I_{a0}(Z_{aa0} + 2Z_{bc0}) + I_pZ_{ap0} \qquad (138)$$

If the value of $I_p$ in terms of $I_0$ is substituted we have

$$V_{ag0} = I_{a0}\left(Z_{aa0} + 2Z_{bc0} - \frac{3Z_{ap0}^2}{Z_{pp}}\right)$$

That is, the impedance of the line to zero sequence is

$$Z_0 = \frac{V_{ag0}}{I_{a0}} = Z_{aa0} + 2Z_{bc0} - \frac{3Z_{ap0}^2}{Z_{pp}} \text{ vector ohms per mile} \quad (139)$$

In this expression for the impedance to zero sequence the value of $Z_{aa0}$ is given in equation (130), page 130; the value of $Z_{bc0}$ is given in equation (131), page 130; the value of $Z_{ap0}$ is given in equation (135), page 134; and the value of $Z_{pp}$ is given in equation (127), page 126.

**54. Effect of the Ground Resistances.**—In deriving the preceding equation (139) for the impedance to zero sequence, it was assumed that the ground resistances at the station and at the fault were zero. The effect of these resistances may be introduced approximately by including the potential drop through them when writing the potential equations for the transmission conductor-earth loop and for the ground-wire-earth loop. Refer to Fig. 39 and equations (134) and (138). The following derivation assumes that the line is open-circuited beyond the fault.

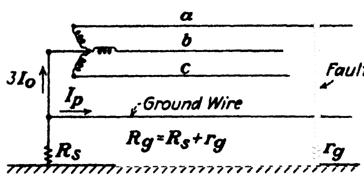


FIG. 39.

$$V_p = 0 = I_pZ_{pp} + 3I_0Z_{ap0} + (I_p + 3I_0)R_g \quad (140)$$
$$V_{a0} = I_0(Z_{aa0} + 2Z_{bc0}) + I_pZ_{ap0} + (I_p + 3I_0)R_g \quad (141)$$

where $R_g$ is the *sum* of the ground resistances at the station and at the fault and $V_{a0}$ is the zero-sequence component in the phase potential (line-to-neutral) at the station. The Carson impedances in this case are taken for the length of line between the station and the fault. It should be observed that in writing these equations it is assumed that the current throughout the ground wire is uniform, from which it follows that the ground resistance at each tower footing—except where the fault occurs—is infinite so that no current can flow to earth at these points.

Since, as has been noted previously (page 132), the ground wire carries a greater proportion of the fault current in sections near the fault, the current flowing to earth at the fault is less than the value assumed and thus the last term in each of these equations is too great. That is to say, the resistance $R_g$ should be somewhat less than the sum of the ground resistances at the station and the tower where the fault occurs.

The current in the ground wire is

$$I_p = -\frac{3(Z_{ap0} + R_g)}{Z_{pp} + R_g} I_0$$

The impedance of the circuit to zero sequence is

$$Z_0 = \frac{V_{a0}}{I_{a0}} = Z_{aa0} + 2Z_{bc0} + 3R_g - \frac{3(Z_{ap0} + R_g)^2}{Z_{pp} + R_g} \text{ vector ohms} \tag{142}$$

where the Carson impedances are for the length of line between the station and the fault.

Thus the impedance to zero sequence of the line alone, exclusive of the terminal resistance to ground, is

$$Z_{aa0} + 2Z_{bc0} - \frac{3(Z_{ap0} + R_g)^2}{Z_{pp} + R_g} \text{ vector ohms} \tag{143}$$

The *actual* value of this impedance probably lies between this value [equation (143)] and that given in equation (139).

**55. Two Ground Wires.**—With 2 ground wires, $p$ and $q$, and with the same assumptions as have already been outlined, the following equations, applying to one transposition section, may be written for the two ground wires

$$I_pZ_{pp} + I_qZ_{pq} + I_aZ_{ap} + I_bZ_{bp} + I_cZ_{cp} = 0$$
$$I_qZ_{qq} + I_pZ_{pq} + I_aZ_{aq} + I_bZ_{bq} + I_cZ_{cq} = 0$$

If need be, these equations may be solved for the ground-wire currents in terms of the currents in the transmission line conductors. In writing these equations we have assumed that the current in each ground wire is uniform. A high ground resistance at the tower footings tends to produce this result. However, with two or more ground wires, there is, another point to consider. At each tower the ground wires are connected together through the tower structure. Throughout a given transposition

section there is nothing to disturb the distribution of the total ground wire current among the several wires. In the next section, owing to the different arrangement of the transmission conductors, the mutually induced potentials in the ground wires will be different. The equations for the ground-wire currents become

$$\begin{aligned} I_pZ_{pq} + I_qZ_{pq} + I_aZ_{bp} + I_bZ_{cp} + I_cZ_{ap} &= 0 \\ I_qZ_{qq} + I_pZ_{pq} + I_aZ_{bq} + I_bZ_{cq} + I_cZ_{aq} &= 0 \end{aligned}$$

The solutions for the ground-wire currents in this section are different from the values obtained in the preceding section. The difference is due wholly to the action of the positive- and negative-sequence components of the line currents. In order to simplify the problem we shall choose as the ground-wire current its average value over three adjacent transposition sections of equal length. That is, although the ground-wire currents are free to redistribute themselves among the different ground wires at each transposition section, we shall in effect assume that no such redistribution takes place. Or it may be said that we shall neglect the effect of the positive- and negative-sequence components of the line currents in causing currents to flow in the ground wires. For some configurations the positive- and negative-sequence components of the line currents will actually have no effect on the currents in the ground wires, as for example, if the transmission conductors lie in a vertical plane with respect to which the ground wires are symmetrically located. Thus, if we assume that the line is properly transposed, the equations for the ground-wire currents become

$$\begin{aligned} I_pZ_{pp} + I_qZ_{pq} + 3I_{a0}Z_{ap0} &= 0 \\ I_qZ_{qq} + I_pZ_{pq} + 3I_{a0}Z_{aq0} &= 0 \end{aligned}$$

where $I_p$ and $I_q$ are the average values of the ground-wire currents. If the ground wires are the same size, their coefficients of self-impedance are the same.

$$Z_{pp} = Z_{qq} = r_p + 0.0015f + j0.00466f \log \frac{D}{d_p} \text{ vector ohms per mile}$$

Furthermore, if the ground wires are symmetrically arranged with respect to the transmission conductors, that is, if $Z_{ap0}$

equals $Z_{aq0}$, the currents in the ground wires are equal. That is

$$I_p = I_q = -\frac{Z_{ap0}}{Z_{pp} + Z_{pq}} 3I_{a0}$$

The expression for the zero-sequence component of the average value of the potential drop in the $a$ conductor-earth loop is now

$$V_{a0} = I_{a0}(Z_{aa0} + 2Z_{bc0}) + I_pZ_{ap0} + I_qZ_{aq0}$$

And since we are assuming that $I_p = I_q$ and that $Z_{ap0} = Z_{aq0}$

$$V_{a0} = I_{a0}\left(Z_{aa0} + 2Z_{bc0} - \frac{6Z^2_{ap0}}{Z_{pp} + Z_{pq}}\right) \qquad (144)$$

**56. Parallel Transmission Lines.**—Two 3-phase circuits are frequently carried on the same line of towers or, in some cases, on separate lines of towers that are located on the same right of way. In such cases the mutual reactance between the two circuits may be an important factor. Let us consider two parallel 3-phase circuits whose conductors are $a$, $b$, $c$, and $d$, $e$, $f$. We shall first determine the mutually induced potentials in $a$, $b$, $c$, due to the positive-sequence currents in $d$, $e$, $f$. The sections of the lines under consideration are not transposed.

For a single 3-phase circuit, we have already shown that the *eddy* currents induced in the earth produce a negligible effect on the resistance and inductance. We shall disregard these eddy currents—but not the zero-sequence conduction currents—when computing the mutually induced potentials. That is, we shall use the method indicated in equation (106), page 113, for determining the mutual-reactance potential due to positive- and negative-sequence currents. The mutual impedance potential in the first circuit, due to zero-sequence current in the second circuit, will be determined in terms of the Carson coefficients.

In general, the fall in potential per mile in conductor $a$ of the first circuit due to currents $I_d$, $I_e$, and $I_f$ in the second circuit is

$$V_a = j0.00466f\left(I_d \log \frac{1}{ad} + I_e \log \frac{1}{ae} + I_f \log \frac{1}{af}\right) \qquad (145)$$

If the currents in the other circuit are of positive sequence only, this expression may be written

$$V_a = j0.00466f\left(\log \frac{1}{ad} + \underline{/-120^\circ} \log \frac{1}{ae} + \underline{/-240^\circ} \log \frac{1}{af}\right)I_{d1}$$

Similarly, the fall of potential in conductor $b$ is

$$V_b = j0.00466f\left(\log\frac{1}{bd} + \underline{/-120°}\ \log\frac{1}{be} + \underline{/-240°}\ \log\frac{1}{bf}\right)I_{d1}$$

For conductor $c$,

$$V_c = j0.00466f\left(\log\frac{1}{cd} + \underline{/-120°}\ \log\frac{1}{ce} + \underline{/-240°}\ \log\frac{1}{cf}\right)I_{d1}$$

We shall now determine the zero-, positive-, and negative-sequence components of $V_a$. The units are vector volts per mile.

$$V_{a0} = j\frac{0.00466f}{3}\left(\log\frac{1}{ad\cdot bd\cdot cd} + \underline{/-120°}\ \log\frac{1}{ae\cdot be\cdot ce} + \underline{/-240°}\ \log\frac{1}{af\cdot bf\cdot cf}\right)I_{d1}$$

$$V_{a1} = j\frac{0.00466f}{3}\left(\log\frac{1}{ad\cdot be\cdot cf} + \underline{/-120°}\ \log\frac{1}{ae\cdot bf\cdot cd} + \underline{/-240°}\ \log\frac{1}{af\cdot bd\cdot ce}\right)I_{d1}$$

$$V_{a2} = j\frac{0.00466f}{3}\left(\log\frac{1}{ad\cdot bf\cdot ce} + \underline{/-120°}\ \log\frac{1}{ae\cdot bd\cdot cf} + \underline{/-240°}\ \log\frac{1}{af\cdot be\cdot cd}\right)I_{d1}$$

If the complex equivalents of the angles are substituted and the real and imaginary terms are collected, we have

$$\left.\begin{aligned}
V_{a0} &= j0.00466f\left(\log\frac{\sqrt[6]{ae\cdot be\cdot ce\cdot af\cdot bf\cdot cf}}{\sqrt[3]{ad\cdot bd\cdot cd}} + j0.866\log\sqrt[3]{\frac{ae\cdot be\cdot ce}{af\cdot bf\cdot cf}}\right)I_{d1}\text{ vector volts per mile}\\
V_{a1} &= j0.00466f\left(\log\frac{\sqrt[6]{ae\cdot bf\cdot cd\cdot af\cdot bd\cdot ce}}{\sqrt[3]{ad\cdot be\cdot cf}} + j0.866\log\sqrt[3]{\frac{ae\cdot bf\cdot cd}{af\cdot bd\cdot ce}}\right)I_{d1}\text{ vector volts per mile}\\
V_{a2} &= j0.00466f\left(\log\frac{\sqrt[6]{ae\cdot bd\cdot cf\cdot af\cdot be\cdot cd}}{\sqrt[3]{ad\cdot bf\cdot ce}} + j0.866\log\sqrt[3]{\frac{ae\cdot bd\cdot cf}{af\cdot be\cdot cd}}\right)I_{d1}\text{ vector volts per mile}
\end{aligned}\right\}\quad(146)$$

If the letters $a$ and $d$, $b$ and $e$, $c$ and $f$ are interchanged, there will result equations for the zero-, positive-, and negative-sequence components of the mutually induced potentials in line $d$ due to positive-sequence components of current in the $(a, b, c)$ circuit. A comparison of these two sets of equations shows that, in general, the ratio $V_{a0}/I_{d1}$ does not equal $V_{d0}/I_{a1}$, that the ratio $V_{a1}/I_{d1}$ is the negative of the conjugate of the ratio $V_{d1}/I_{a1}$; that the ratio $V_{a2}/I_{d1}$ equals the ratio $V_{d2}/I_{a1}$. The zero-sequence potential is usually of little importance when there is no zero-sequence current and it will not be considered further at this point.

The positive-sequence potential induced in the first circuit by unit positive-sequence current in the second circuit equals in magnitude *but is not in phase with* the positive-sequence potential induced in the second circuit by unit positive-sequence current in the first circuit. It is assumed that the unit positive-sequence currents in the two circuits are in phase. However, the negative-sequence potential induced in the first circuit by unit positive-sequence current in the second circuit is identical with the negative-sequence potential induced in the second circuit by unit positive-sequence current in the first circuit.

Let us now determine the positive- and negative-sequence components of potential induced in the first circuit due to negative-sequence current in the second circuit. Again neither circuit is transposed within the length being considered. The expressions for $V_a$, $V_b$, and $V_c$, due to negative-sequence current $I_{d2}$, will be the same as the expressions for positive-sequence current $I_{d1}$ except that the negative angles will be replaced by equal positive angles and $I_{d1}$ will be replaced by $I_{d2}$. If the same procedure is followed as before for determining the positive- and negative-sequence components of $V_a$, we shall have

$$\left.\begin{aligned} V_{a1} &= j0.00466f\left(\log \frac{\sqrt[6]{ae \cdot bd \cdot cf \cdot af \cdot be \cdot cd}}{\sqrt[3]{ad \cdot bf \cdot ce}} \right. \\ &\qquad \left. - j0.866 \log \sqrt[3]{\frac{ae \cdot bd \cdot cf}{af \cdot be \cdot cd}}\right) I_{d2} \text{ vector volts per mile} \\ V_{a2} &= j0.00466f\left(\log \frac{\sqrt[6]{ae \cdot bf \cdot cd \cdot af \cdot bd \cdot ce}}{\sqrt[3]{ad \cdot be \cdot cf}} \right. \\ &\qquad \left. - j0.866 \log \sqrt[3]{\frac{ae \cdot bf \cdot cd}{af \cdot bd \cdot ce}}\right) I_{d2} \text{ vector volts per mile} \end{aligned}\right\} \quad (147)$$

Note that the impedance ratio $V_{a1}/I_{d2}$ is the conjugate of $V_{a2}/I_{d1}$, and that $V_{a2}/I_{d2}$ is the negative of the conjugate of $V_{a1}/I_{d1}$.

If the letters *a* and *d*, *b* and *e*, *c* and *f* are interchanged, we shall obtain the expressions for the positive- and negative-sequence components of the potential $V_d$ due to the action of the current $I_{a2}$. Again a comparison of these two sets of equations shows that, in general, the ratio $V_{a1}/I_{d2}$ equals the ratio $V_{d1}/I_{a2}$ but that the ratio $V_{a2}/I_{d2}$ is the negative of the conjugate of the ratio $V_{d2}/I_{a2}$. These eight impedance ratios may be grouped as follows

$$\frac{V_{a1}}{I_{d1}} = \frac{V_{d2}}{I_{a2}} = -\text{conj.}\left(\frac{V_{a2}}{I_{d2}}\right) = -\text{conj.}\left(\frac{V_{d1}}{I_{a1}}\right)$$

$$\frac{V_{a2}}{I_{d1}} = \frac{V_{d2}}{I_{a1}} = -\text{conj.}\left(\frac{V_{a1}}{I_{d2}}\right) = -\text{conj.}\left(\frac{V_{d1}}{I_{a2}}\right)$$

These general relations show that it is necessary to compute only the positive- and negative-sequence components of the potential induced in the first circuit by positive-sequence current in the second circuit. If the two circuits are symmetrically arranged on opposite sides of a plane, as in Fig. 40, so that, with respect to this plane, the conductors of one circuit appear as the images of the conductors of the other circuit, the impedance ratio $V_{a1}/I_{d1}$ is wholly reactive, *i.e.*, of the form $jx$.

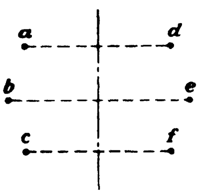


FIG. 40

**57. Effect of Transposition.**—We shall determine the effects produced by two types of transposition only. Let us first consider the case in which the first line is symmetrically transposed while the second is untransposed. Figure 41 illustrates this condition.

In this figure the dots indicate the positions of the conductors. The first column of letters indicates the positions of the *a*, *b*, *c* conductors in the first section; the second column, the positions of the same conductors in the second section; and the third column, the positions of these conductors in the third section. The sections are of equal length. The conductors, *d*, *e*, *f*, of the second circuit are not transposed. The distances between the conductors is indicated as that between them in the first section. For example, the distance between the *a* and *b* conductors in the third section is *ca*.

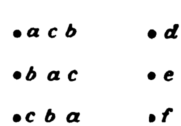


FIG. 41.

In the second transposition section the positive-sequence fall in potential in the first circuit, $V_{a1}$, due to positive-sequence current, $I_{d1}$, in the second circuit becomes

$$V_{a1} = j0.00466f\left(\log \frac{\sqrt[6]{be \cdot cf \cdot ad \cdot bf \cdot cd \cdot ae}}{\sqrt[3]{bd \cdot ce \cdot af}} + j0.866 \log \sqrt[3]{\frac{be \cdot cf \cdot ad}{bf \cdot cd \cdot ae}}\right) I_{d1} \text{ vector volts per mile}$$

If the three transposition sections are of equal length, we may write the expression for the average value of $V_{a1}$ (vector volts per mile) as follows

$$V_{a1} = j0.00466f\left(\log \frac{\sqrt[18]{\begin{matrix} ae \cdot bf \cdot cd \cdot af \cdot bd \cdot ce \\ be \cdot cf \cdot ad \cdot bf \cdot cd \cdot ae \\ ce \cdot af \cdot bd \cdot cf \cdot ad \cdot be \end{matrix}}}{\sqrt[9]{\begin{matrix} ad \cdot be \cdot cf \\ bd \cdot ce \cdot af \\ cd \cdot ae \cdot bf \end{matrix}}} + j0.866 \log \sqrt[9]{\frac{\begin{matrix} ae \cdot bf \cdot cd \\ be \cdot cf \cdot ad \\ ce \cdot af \cdot bd \end{matrix}}{\begin{matrix} af \cdot bd \cdot ce \\ bf \cdot cd \cdot ae \\ cf \cdot ad \cdot be \end{matrix}}}\right) I_{d1}$$

It will be observed that, since the distance ratios whose logarithms are taken are both unity, the average value of the positive-sequence potential drop in the first circuit, due to positive-sequence current in the second circuit, is zero. By a similar process it may be shown that the negative-sequence potential drop in the first circuit, due to positive-sequence current in the second circuit, is also zero. It will likewise be found that both the positive- and negative-sequence components of the potential in the first circuit due to negative-sequence current in the second circuit are zero. That is, with this type of transposition there is no mutual impedance between the two circuits. This is well known.

The second type of transposition we are to discuss is one in which both circuits are simultaneously and symmetrically transposed as indicated in Fig. 42.

The first column of letters in the first group indicates the positions of the conductors of the $(a,b,c)$ circuit in the first transposition

section. The first column of letters in the second group indicates the positions of the conductors of the $(d,e,f)$ circuit in the first transposition section. The second and third columns in each group indicate the positions of the same conductors in the second and in the third transposition sections. The distances between conductors in the second and third transposition sections are expressed as the distances between conductors in the first transposition section. For example, the distance between the $a$ and $f$ conductors in the third transposition section is $ce$. If these transpositions are made, it will be found that the positive-sequence component of the potential in the first circuit, due to positive-sequence current in the second circuit, is the same in each transposition section. The average value of the negative-sequence potential in the first circuit, due to positive-sequence current in the second circuit, is zero. Thus we may write

| | |
|---|---|
| •a c b | •d f e |
| •b a c | •e d f |
| •c b a | •f e d |

FIG. 42

$$\frac{V_{a1}}{I_{d1}} = j0.00466f\left(\log \frac{\sqrt[6]{ae \cdot bf \cdot cd \cdot af \cdot bd \cdot ce}}{\sqrt[3]{ad \cdot be \cdot cf}} + j0.866 \log \sqrt[3]{\frac{ae \cdot bf \cdot cd}{af \cdot bd \cdot ce}}\right) \text{ vector ohms per mile} \quad (148)$$

and

$$\frac{V_{a2}}{I_{d1}} = 0$$

Similarly it may be shown that, in the case of simultaneous and symmetrical transpositions, the positive- and negative-sequence components of potential in the first circuit, due to unit negative-sequence current in the second circuit, are

$$\frac{V_{a1}}{I_{d2}} = 0$$

and

$$\frac{V_{a2}}{I_{d2}} = j0.00466f\left(\log \frac{\sqrt[6]{ae \cdot bf \cdot cd \cdot af \cdot bd \cdot ce}}{\sqrt[3]{ad \cdot be \cdot cf}} - j0.866 \log \sqrt[3]{\frac{ae \cdot bf \cdot cd}{af \cdot bd \cdot ce}}\right) \text{ vector ohms per mile} \quad (149)$$

Furthermore, as we have previously shown, if the two circuits are symmetrically arranged on opposite sides of a plane as in Fig. 40, the real part of each of these impedance ratios [equations (148) and (149)] is zero, and we can write

$$\frac{V_{a1}}{I_{d1}} = \frac{V_{a2}}{I_{d2}} = j0.00466f \log \sqrt[3]{\frac{ae \cdot bf \cdot cd}{ad \cdot be \cdot cf}} \text{ vector ohms per mile} \quad (150)$$

There are other arrangements of the conductors that will produce a result similar to that indicated in equation (150). For example, if the conductors are located at the vertices of a regular hexagon, as in Fig. 43, not only is the real part of the mutual

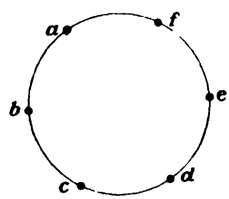


FIG. 43.

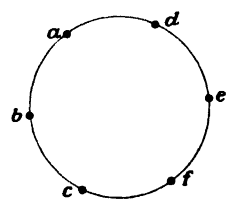


FIG. 44

impedance zero but the mutual reactance is *negative*. Whereas, if the $(d,e,f)$ circuit is arranged as in Fig. 44, the mutual reactance is positive.

The ratio of the mutual reactance between the $(a,b,c)$ and $(d,e,f)$ circuits for the arrangement of Fig. 43 to the mutual reactance for the arrangement of Fig. 44 is $-0.426/0.477$. Note the negative sign.

If the $(a,b,c)$ and $(d,e,f)$ circuits are properly transposed and are connected in parallel electrically (that is, $a$ is in parallel with $d$, $b$ is in parallel with $e$, and $c$ is in parallel with $f$), it is readily shown that their resulting impedance is, in general,

$$Z = \frac{Z_{aa}Z_{dd} - Z_{ad}^2}{Z_{aa} + Z_{dd} - 2Z_{ad}}$$

The impedance to positive, to negative, or to zero sequence is obtained by substituting the appropriate values of $Z_{aa}$, $Z_{dd}$, and $Z_{ad}$ in this expression. If the two circuits are alike, $Z_{aa}$ and $Z_{dd}$ are equal and the resulting impedance reduces to

$$Z = \tfrac{1}{2}(Z_{aa} + Z_{ad})$$

Since the mutual reactance to positive and to negative sequence is negative when the circuits are arranged as in Fig. 43, the resulting impedance of the parallel circuits is less for circuits arranged as in Fig. 43 than for circuits arranged as in Fig. 44.

**58. Effect of the Earth.**—When both the $(a,b,c)$ and the $(d,e,f)$ circuits are grounded, the zero-sequence component in the line drop becomes important. The mutually induced potential in the $a$ conductor-earth loop due to the currents in the $(d,e,f)$ circuit is

$$V_a = I_d Z_{ad} + I_e Z_{ae} + I_f Z_{af} \tag{151}$$

where $Z_{ad}$, etc., are the Carson coefficients of mutual impedance between the indicated conductor-earth loops. Similarly

$$V_b = I_d Z_{bd} + I_e Z_{be} + I_f Z_{bf}$$
$$V_c = I_d Z_{cd} + I_e Z_{ce} + I_f Z_{cf}$$

The zero-sequence component of $V_a$ is

$$V_{a0} = \frac{I_d}{3}(Z_{ad} + Z_{bd} + Z_{cd}) + \frac{I_e}{3}(Z_{ae} + Z_{be} + Z_{ce}) + \frac{I_f}{3}(Z_{af} + Z_{bf} + Z_{cf}) \tag{152}$$

We will first assume that the $(a,b,c)$ circuit is transposed while the $(d,e,f)$ circuit is not transposed (see Fig. 41). In the second transposition section the zero-sequence component of $V_a$ is

$$V_{a0} = \frac{I_d}{3}(Z_{bd} + Z_{cd} + Z_{ad}) + \frac{I_e}{3}(Z_{be} + Z_{ce} + Z_{ae}) + \frac{I_f}{3}(Z_{bf} + Z_{cf} + Z_{af})$$

It will be observed that this transposition of the conductors has not affected $V_{a0}$. We will further assume that the $(d,e,f)$ circuit is transposed at every third transposition point in the $(a,b,c)$ circuit. In the second transposition section of the $(d,e,f)$ circuit the zero-sequence component of $V_a$ is

$$V_{a0} = \frac{I_d}{3}(Z_{ae} + Z_{be} + Z_{ce}) + \frac{I_e}{3}(Z_{af} + Z_{bf} + Z_{cf}) + \frac{I_f}{3}(Z_{ad} + Z_{bd} + Z_{cd})$$

It will now be observed that the average value of the zero-sequence component of $V_a$ over three transposition sections of the $(d,e,f)$ circuit is

$$V_{a0} = \frac{1}{3}\left(\frac{I_d}{3} + \frac{I_e}{3} + \frac{I_f}{3}\right)\begin{pmatrix} Z_{ad} + Z_{bd} + Z_{cd} \\ +Z_{ae} + Z_{be} + Z_{ce} \\ +Z_{af} + Z_{bf} + Z_{cf} \end{pmatrix}$$

If the approximate values of the mutual impedances [equation (128), page 127] are substituted for $Z_{ad}$, $Z_{bd}$, etc., we have

$$V_{a0} = \frac{I_{d0}}{3}\left(0.0135f + j0.00466f \log \frac{D^9}{\begin{matrix} ad \cdot bd \cdot cd \\ ae \cdot be \cdot ce \\ af \cdot bf \cdot cf \end{matrix}}\right) \text{ vector volts per mile}$$

or we may write

$$V_{a0} = 3I_{d0}\left(0.0015f + j0.00466f \log \frac{D}{\sqrt[9]{\begin{matrix} ad \cdot bd \cdot cd \\ ae \cdot be \cdot ce \\ af \cdot bf \cdot cf \end{matrix}}}\right) \text{ vector volts per mile} \quad (153)$$

The ninth root of the product of the nine distances is the G.M.D. between the $(a,b,c)$ and the $(d,e,f)$ circuits. As an abbreviation, the G.M.D. between these circuits may be within ${}_{abc}\text{G.M.D.}_{def}$.

It should be noted that the logarithmic form of the foregoing mutual reactances is approximate (see Art. 48). When the $(a,b,c)$ and $(d,e,f)$ circuits are carried on separate towers, it may be desirable, in the solution of some problems, to use the more exact values of these reactances. Should the more exact form be used the computation of the impedance to zero sequence will be much more laborious than is indicated by equation (153).

If the $(a,b,c)$ and $(d,e,f)$ circuits are simultaneously and symmetrically transposed as in Fig. 42, the zero-sequence component of $V_a$ in the second transposition section becomes

$$V_{a0} = \frac{I_d}{3}(Z_{be} + Z_{ce} + Z_{ae}) + \frac{I_e}{3}(Z_{bf} + Z_{cf} + Z_{af}) + \frac{I_f}{3}(Z_{bd} + Z_{cd} + Z_{ad})$$

It will be observed that the average value of $V_{a0}$ over these three transposition sections is the same as previously deduced [equation (153)]. That is, for either of these two types of transposition we may write

$$\frac{V_{a0}}{I_{d0}} = 3\left(0.0015f + j0.00466f \log \frac{D}{_{abc}\text{G.M.D.}_{def}}\right) \text{ vector ohms per mile} \quad (154)$$

The mutual impedance to zero sequence between the two circuits is

$$Z_{(abc)(def)} = 0.0015f + j0.00466 \log \frac{D}{_{abc}\text{G.M.D.}_{def}}$$

Thus

$$V_{a0} = 3I_{d0}Z_{(abc)(def)} \quad (155)$$

**59. Effect of Ground Wires.**—Let us consider the case in which the $(a,b,c)$ circuit and the $(d,e,f)$ circuit are carried on the same

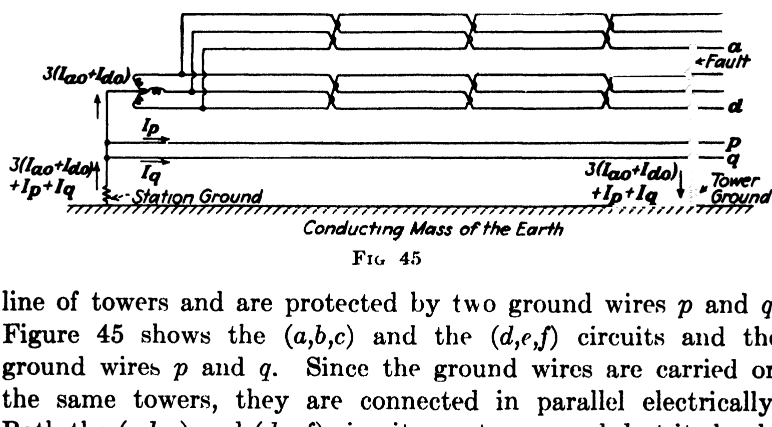



FIG. 45

line of towers and are protected by two ground wires $p$ and $q$. Figure 45 shows the $(a,b,c)$ and the $(d,e,f)$ circuits and the ground wires $p$ and $q$. Since the ground wires are carried on the same towers, they are connected in parallel electrically. Both the $(a,b,c)$ and $(d,e,f)$ circuits are transposed, but it should be remembered that transposition plays *no* part in determining the effect of the *zero-sequence* current in the transmission conductors. We shall assume that both ground wires and the neutrals of both transmission circuits are connected to the same station ground. The sum of the station ground resistance and the ground resistance at the tower footing where the fault occurs is $R_g$. We shall also assume that there is a simultaneous fault on both the $(a,b,c)$ and $(d,e,f)$ circuits at the same tower

as indicated in Fig. 45. We shall now write the equations for the potential drop around each of the ground-wire-earth loops. Refer to equation (140). It is assumed that the line is open-circuited beyond the fault.

$$\begin{aligned} V_p &= 0 = I_pZ_{pp} + I_qZ_{pq} + 3I_{a0}Z_{ap0} + 3I_{d0}Z_{dp0} \\ &\qquad + (I_p + I_q + 3I_{a0} + 3I_{d0})R_g \\ V_q &= 0 = I_qZ_{qq} + I_pZ_{pq} + 3I_{a0}Z_{aq0} + 3I_{d0}Z_{dq0} \\ &\qquad + (I_p + I_q + 3I_{a0} + 3I_{d0})R_g \end{aligned} \quad (155)$$

On collecting terms these equations become:

$$\left.\begin{aligned} &I_p(Z_{pp} + R_g) + I_q(Z_{pq} + R_g) + 3I_{a0}(Z_{ap0} + R_g) \\ &\qquad + 3I_{d0}(Z_{dp0} + R_g) = 0 \\ &I_q(Z_{qq} + R_g) + I_p(Z_{pq} + R_g) + 3I_{a0}(Z_{aq0} + R_g) \\ &\qquad + 3I_{d0}(Z_{dq0} + R_g) = 0 \end{aligned}\right\} \quad (156)$$

In these expressions the Carson impedances are for the length of line *between the station and the fault.* In order to simplify equations (156) as much as possible, we shall omit writing the $R_g$ and understand that this ground resistance is included in each Carson impedance. The Carson impedances will then represent the impedances, both self- and mutual, of the conductor-earth loops between the station and the fault. Thus equations (156) become

$$\left.\begin{aligned} I_pZ_{pp} + I_qZ_{pq} + 3I_{a0}Z_{ap0} + 3I_{d0}Z_{dp0} &= 0 \\ I_qZ_{qq} + I_pZ_{pq} + 3I_{a0}Z_{aq0} + 3I_{d0}Z_{dq0} &= 0 \end{aligned}\right\} \quad (157)$$

If the ground wires are of the same type and size, $Z_{pp} = Z_{qq}$.

It is seen that the currents in the ground wires each consist of two components, one due to the zero-sequence current in the $(a,b,c)$ circuit and the other due to the zero-sequence current in the $(d,e,f)$ circuit. If these equations are solved for the ground-wire currents, it will be found that the components of $I_p$ and $I_q$ that are proportional to $I_{a0}$ are

$$\left.\begin{aligned} \text{Component of } I_p &= -3\frac{Z_{ap0}Z_{pp} - Z_{aq0}Z_{qp}}{Z_{pp}^2 - Z_{qp}^2}I_{a0} \\ \text{Component of } I_q &= -3\frac{Z_{aq0}Z_{pp} - Z_{ap0}Z_{pq}}{Z_{pp}^2 - Z_{pq}^2}I_{a0} \end{aligned}\right\} \quad (158)$$

Observe that if there are three or more ground wires there will be a corresponding number of equations of the type of equations

(157) for determining the currents in the individual wires. The amount of computation involved in the solution of these simultaneous equations increases rapidly with their number. If there are many ground wires the computation can be simplified by assuming that the currents in the ground wires are equal.[1] Even in the case of two ground wires it simplifies considerably the form of the expressions for the ground-wire currents. Thus if the average value of the two ground-wire currents is determined from equations (157) we have

$$\frac{1}{2}(I_p + I_q) = -\frac{3}{2}\frac{Z_{ap0} + Z_{aq0}}{Z_{pp} + Z_{pq}}I_{a0} - \frac{3}{2}\frac{Z_{dp0} + Z_{dq0}}{Z_{pp} + Z_{pq}}I_{d0} \quad (159)$$

Observe that

$$\frac{1}{2}(Z_{ap0} + Z_{aq0}) = 0.0015f + \frac{R_g}{L} + j0.00466f \log \frac{D}{\sqrt[6]{\overline{ap} \cdot \overline{bp} \cdot \overline{cp} \cdot \overline{aq} \cdot \overline{bq} \cdot \overline{cq}}}$$

vector ohms per mile (160)

where $L$ is the number of miles between the station and the fault and $\sqrt[6]{\overline{ap} \cdot \overline{bp} \cdot \overline{cp} \cdot \overline{aq} \cdot \overline{bq} \cdot \overline{cq}}$ is the G.M.D. between the two ground wires and the $(a,b,c)$ circuit. A similar expression can be written for $(Z_{dp0} + Z_{dq0})$ by replacing $a$, $b$, and $c$ by $d$, $e$, and $f$. Observe also that:

$$(Z_{pp} + Z_{pq}) = r_p + 2\left(0.0015f + \frac{R_g}{L} + j0.00466f \log \frac{D}{\sqrt{d_p \cdot \overline{pq}}}\right)$$

vector ohms per mile (161)

A similar formula may be derived for the average value of the ground-wire currents when there are three or more ground wires. The average zero-sequence potential in the $(a,b,c)$ circuit if the line is open-circuited beyond the fault is,

$$V_{a0} = I_{a0}(Z_{aa0} + 2Z_{bc0}) + 3I_{d0}Z_{(abc)(def)} + I_pZ_{ap0} + I_qZ_{aq0} + (3I_{a0} + 3I_{d0} + I_p + I_q)R_g \quad (162)$$

The first term is the potential due directly to the zero-sequence current in the $(a,b,c)$ circuit [equation (129)]. The second term is the potential due directly to the zero-sequence current in the $(d,e,f)$ circuit [equation (154)]. The next two terms are the two

[1] CLEM. See footnote, page 121.

potentials due to the currents in the ground wires $p$ and $q$. The sum of the components of these two potentials due to that part of the ground-wire current which is proportional to the zero-sequence current in the $(a,b,c)$ circuit is, referring to equations (158)

$$\frac{-3[(Z_{ap0}^2 + Z_{aq0}^2)Z_{pp} - 2Z_{ap0}Z_{aq0}Z_{pq}]I_{a0}}{Z_{pp}^2 - Z_{qp}^2}$$

The sum of the corresponding components of the two potentials which are proportional to the zero-sequence current in the $(d,e,f)$ circuit is

$$\frac{-3[(Z_{ap0}Z_{dp0} + Z_{aq0}Z_{dq0})Z_{pp} - (Z_{ap0}Z_{dp0} + Z_{aq0}Z_{dp0})Z_{pq})]I_{d0}}{Z_{pp}^2 - Z_{pq}^2}$$

If the average values of the ground-wire currents, as determined by equation (159), are used instead of their more exact values, the corresponding and somewhat simpler expressions for these two components of potential are

$$-\frac{3}{2}\frac{(Z_{ap0} + Z_{aq0})^2}{Z_{pp} + Z_{qp}}I_{a0} \tag{163}$$

and

$$-\frac{3}{2}\frac{(Z_{ap0} + Z_{aq0})(Z_{dp0} + Z_{dq0})}{Z_{pp} + Z_{pq}}I_{d0} \tag{164}$$

Note that, if the ground wires are symmetrically arranged with respect to the $(a,b,c)$ and $(d,e,f)$ circuits,

$$Z_{ap0} = Z_{dp0} \qquad \text{and} \qquad Z_{aq0} = Z_{dq0}.$$

and equations (163) and (164) may be combined as

$$-\frac{3}{2}\frac{(Z_{ap0} + Z_{aq0})^2}{Z_{pp} + Z_{qp}}(I_{a0} + I_{d0})$$

The last term in equation (162) is the potential drop through the ground resistances at the station and at the fault. If the line is *not* open-circuited beyond the fault, $R_g$ may be taken as the ground resistance at the station. The ground resistance at the fault could then be considered as a resistance load at the point of fault. In short-circuit computations all resistances are frequently disregarded.

The solution for the zero-sequence component of potential in the $(d,e,f)$ circuit can be obtained in a similar manner.

If the ground wires are symmetrically arranged with respect to the $(a,b,c)$ and $(d,e,f)$ circuits and if the average values of the ground-wire currents are used, the expression for the zero-sequence potential in the $(a,b,c)$ circuit, exclusive of the effect of the ground resistances at the station and at the fault, is

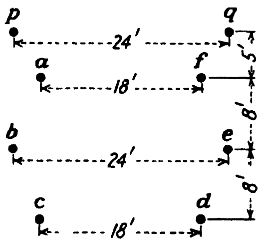



FIG. 46.—66 kv, 50 miles, double-circuit, A.C.S.R, 397,500 cir. mils, 30 strands al, 7 strands steel. Similarly and symmetrically transposed. Two ground wires, same size and material as line wires. Frequency = 60 cycles.

$ae = bf = 22.47'$
$ad = cf = 24.10'$
$cd = 18.0'$
$be = 24.0'$
$ab = bc = 8.54'$
$ac = 16.0'$
Self G.M.D, $d = 0.0278'$
$r = 0.259$ ohms per mile at 60 cycles.

$$V_{a0} = I_{a0}\left[Z_{aa0} + 2Z_{bc0} - \frac{3}{2}\frac{(Z_{ap0} + Z_{aq0})^2}{Z_{pp} + Z_{qp}}\right] + I_{d0}\left[3Z_{(abc)(def)} - \frac{3}{2}\frac{(Z_{ap0} + Z_{aq0})^2}{Z_{pp} + Z_{pq}}\right] \quad (165)$$

Since it is assumed that the two circuits are symmetrically arranged, a similar expression can be written for the zero-sequence potential, $V_{d0}$, in the $(d,e,f)$ circuit. Thus if $V_{a0}$ is written

$$V_{a0} = AI_{a0} + BI_{d0}$$

then

$$V_{d0} = DI_{d0} + BI_{a0}$$

where $A$ and $D$ are self-impedances and $B$ is the mutual impedance between the two circuits. If the two lines are connected in parallel, $V_{a0} = V_{d0}$, and therefore the combined impedance of both lines, exclusive of the ground resistances, to zero sequence is

$$\frac{V_{a0}}{I_{a0} + I_{d0}} = \frac{1}{2}\left[Z_{aa0} + 2Z_{bc0} + 3Z_{(abc)(def)} - 3\frac{(Z_{ap0} + Z_{aq0})^2}{Z_{pp} + Z_{qp}}\right] \text{ vector ohms per mile} \quad (166)$$

**Illustration.**—Consider a double-circuit line arranged as indicated in Fig. 46. Each circuit is symmetrically transposed. We shall first compute the self- and mutual impedance constants per mile.

*Positive and Negative Sequence.*—By equation (114), page 116,

$$Z_{aa1} = Z_{dd1} = Z_{aa2} = Z_{dd2} = 0.259 + j0.280 \log 379$$
$$= 0.259 + j0.721 \text{ vector ohms per mile}$$

By equation (150), page 145,

$$Z_{ad1} = Z_{ad2} = j0.0932 \log 0.652$$
$$= -j0.0173 \text{ vector ohms per mile}$$

With the two circuits in parallel

$$Z_1 = Z_2 = \tfrac{1}{2}(Z_{aa1} + Z_{ad1}) = 0.130 + j0.352 \text{ vector ohms per mile}$$

*Zero Sequence.*—The self- and mutual impedances are defined in equation (165), in which assume $R_g = 0$. Assuming earth conductivity $\lambda = 10^{-13}$,

$$D = \frac{0.00738}{\sqrt{f\lambda}} = 3010 \text{ ft.}$$
$$r + 0.0015f = 0.349 \text{ ohm per mile}$$
$$\text{Ground wire self G.M.D. } d_p = 0.0278'$$

By equation (127), page 126,

$$Z_{aa0} = Z_{pp} = Z_{qq} = 0.349 + j1.41 \text{ vector ohms per mile}$$

By equation (131), page 130,

$$Z_{bc0} = 0.09 + j0.686 \text{ vector ohms per mile}$$

By equation (135), page 134,

$$Z_{ap0} = 0.09 + j0.674 \text{ vector ohms per mile}$$
$$Z_{aq0} = 0.09 + j0.577 \text{ vector ohms per mile}$$

By equation (128), page 127,

$$Z_{pq} = 0.09 + j0.587 \text{ vector ohms per mile}$$

By equations (153), (155), page 147,

$$Z_{(abc)(def)} = 0.09 + j0.598 \text{ vector ohms per mile}$$
$$\frac{3}{2}\frac{(Z_{ap0} + Z_{aq0})^2}{Z_{pp} + Z_{pq}} = 0.081 + j1.17 \text{ vector ohms per mile}$$

On substituting these values in equation (165), page 152,

$$V_{a0} = (0.448 + j1.61)I_{a0} + (0.189 + j0.622)I_{d0} \text{ vector volts per mile}$$

If the two circuits are in parallel, by equation (166), page 152,

$$\frac{V_{a0}}{I_{a0} + I_{d0}} = Z_0 = 0.319 + j1.11 \text{ vector ohms per mile}$$

A single fault occurs at a point $F$, one-third the length of the line from one end. We shall determine the equivalent circuits that can be used to represent this line for zero, positive, and negative sequence. The two circuits are connected in parallel,

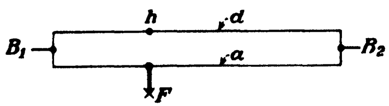



FIG. 47.

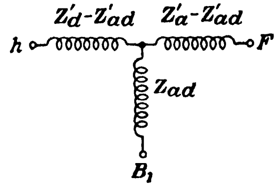



FIG. 48.

or bussed, at each end of the line. This is indicated in Fig. 47, which shows the $a$ and $d$ conductors bussed at $B_1$ and $B_2$, with a fault on the $(a,b,c)$ circuit at $F$. The point $h$ is on the $(d,e,f)$ circuit opposite the fault on the $(a,b,c)$ circuit, but there is no fault at $h$. Since the $(a,b,c)$ and $(d,e,f)$ circuits between $B_1$ and $F$ have both self- and mutual impedance, they have the characteristics of the primary and secondary of a transformer and can be represented by the $T$-equivalent circuit of Fig. 48.

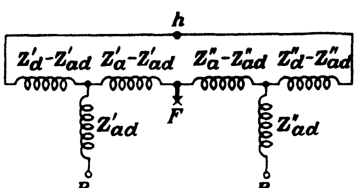



FIG. 49.

In this figure $Z_a'$ is the self-impedance of the $(a,b,c)$ circuit between $B_1$ and $F$, $Z_d'$ is the self-impedance of the $(d,e,f)$ circuit between $B_1$ and $h$, and $Z_{ad}'$ is the mutual impedance between these two circuits. Note that the distance between $B_1$ and $F$ is $16\frac{2}{3}$ miles. The values of these impedances per mile are the same in both positive- and negative-sequence equivalent circuits and are given on page 153. The values of these impedances per mile in the *zero-sequence* equivalent circuit are also given on page 153.

The equivalent circuit which represents the two line circuits between $F$ and $B_2$ is determined in a similar manner. The equivalent circuits for the two parts of the line are connected as indicated in Fig. 49.

The values of the impedance constants in both positive- and negative-sequence equivalent circuits for the two parts of the line are

$$
\begin{array}{lll}
Z_a' = 4.32 + j12.0 & Z_d' = 4.32 + j12.0 & Z_{ad}' = 0 - j0.288 \\
Z_a'' = 8.64 + j24.0 & Z_d'' = 8.64 + j24.0 & Z_{ad}'' = 0 - j0.576
\end{array}
$$

The values of the impedance constants in the zero-sequence equivalent circuit are

$$
\begin{array}{lll}
Z_a' = 7.47 + j26.8 & Z_d' = 7.47 + j26.8 & Z_{ad}' = 3.15 + j10.4 \\
Z_a'' = 14.9 + j53.5 & Z_d'' = 14.9 + j53.5 & Z_{ad}'' = 6.30 + j20.7
\end{array}
$$

**60. Capacitance.**—When calculating the capacitance of transmission lines we shall assume that the line conductors are long, straight, parallel cylinders which are so far apart that the density of the surface charge on each conductor is sensibly uniform. We shall also assume that the potential of each conductor is the same throughout its entire length, in which case there can be no longitudinal component of the electric field. Under these commonly made assumptions the electric field intensity in stat volts per centimeter at a radial distance of $x$ cm. from the axis of the cylinder is $2q/x$, where $q$ is the surface charge in stat coulombs per centimeter length of the conductor. If $ab$ and $ac$ are the radial distances from the axis of the conductor, $a$, to the points $b$ and $c$, the fall in potential from $b$ to $c$, due to this surface charge, is $2q \ln ac/ab$ stat volts. For our purposes it is more convenient to express this potential in terms of the time rate of change of the charge rather than in terms of the charge itself. Since we shall assume that the charge varies sinusoidally with the time, the general vector relation between the charge $Q$ and the time rate of change of the charge, that is to say, the charging current $I$, is $I = j\omega Q$. Thus, in vector notation, the fall in potential from $b$ to $c$ due to the charging current $I_a$ on the conductor $a$ is

$$
\begin{aligned}
V_{bc} &= -j\frac{2I_a\,9 \times 10^{11} \times 2.303}{5280 \times 12 \times 2.54 \times 2\pi f}\log\frac{ac}{ab}\text{ volts} \\
&= -j\frac{4.10 \times 10^6}{f}I_a \log\frac{ac}{ab}\text{ volts} \qquad (167)
\end{aligned}
$$

where $I_a$ represents the charging current in amperes per mile and the common logarithm rather than the natural logarithm is

taken. As usual, $f$ is the frequency. This fundamental relation between the potential and the charging current forms the basis of all the subsequent computations of capacitance.

At this point it is interesting to compare the problem of calculating the inductance of a transmission line with that of calculating its capacitance. The electromagnetically produced fall in potential along a transmission conductor is due in part to the current in the conductor itself and in part to the currents in the other conductors of the system. Likewise the electric potential of any one of the conductors must equal the potential that would be produced by the joint action of the charge on the conductor itself and the charges on the other conductors. Since inductance is the electromagnetically induced potential per unit rate of change of the current, the first relation [equation (106), page 113] is already expressed in the proper form for determining the coefficients of induction. On the other hand, since capacitance is charge per unit difference in potential, the second relation [equation (168), page 157], which fundamentally determines the potential in terms of the charge, must be inverted so as to express the charge in terms of the potential. This is accomplished by solving a number of simultaneous equations, there being one equation for each conductor in the system. It is for this reason that the determination of the capacitance is much more laborious than the determination of the inductance. With three conductors the labor is not unduly great, but, if the transmission circuit is protected by two or more ground wires or if the circuit is double and consists of not less than six conductors, the solution of the necessary equations is tedious. Fortunately, however, there is an approximate method of solution which, if properly applied, yields satisfactory results. The approximate method is similar to and is no more difficult to apply than the method used in determining the inductance. Briefly it consists in assuming that the charges have certain known values from which the *average* values of the line potentials may be computed. The capacitance is then taken as the ratio of the assumed charge to the *average* value of the potential. The obvious advantage of this method is that it is direct and does not involve the solution of simultaneous equations. By comparing, in a number of typical cases, the result obtained by the approximate method with that obtained by exact analysis,

we shall have a fair estimate of the accuracy of the approximate method. As with other approximate methods some discretion is necessary in applying it.

**61. Three-phase Line, Ungrounded and Untransposed.**—Consider three parallel conductors, $a$, $b$, and $c$, the distances between which are $ab$, $bc$, and $ca$. Each of the conductors has a radius $r$. If the charging currents per mile are $I_a$, $I_b$, and $I_c$, the potential between $a$ and $b$ in terms of these charging currents is

$$V_{ab} = -j\frac{4.10 \times 10^6}{f}\left(I_a \log\frac{ab}{r} + I_b \log\frac{r}{ab} + I_c \log\frac{bc}{ca}\right) \text{ vector volts} \quad (168)$$

Since the line is ungrounded and the effect of the ground is to be neglected, all the tubes of electric flux must originate and end on the line conductors. Thus the sum of the charging currents must be zero, and $I_c$ may be replaced by $-I_a - I_b$. If this substitution is made, the expression for the potential becomes

$$V_{ab} = -j\frac{4.10 \times 10^6}{f}\left(I_a \log\frac{ab \cdot ca}{r \cdot bc} - I_b \log\frac{ab}{r} \cdot \frac{bc}{ca}\right)$$

Similarly the potentials between $b$ and $c$ and between $c$ and $a$ are

$$V_{bc} = -j\frac{4.10 \times 10^6}{f}\left(I_b \log\frac{bc \cdot ab}{r \cdot ca} - I_c \log\frac{bc \cdot ca}{r \cdot ab}\right)$$

$$V_{ca} = -j\frac{4.10 \times 10^6}{f}\left(I_c \log\frac{ca \cdot bc}{r \cdot ab} - I_a \log\frac{ca \cdot ab}{r \cdot bc}\right)$$

It will be observed that these equations may be written in the following form:

$$V_{ab} = I_a Z_a - I_b Z_b$$
$$V_{bc} = I_b Z_b - I_c Z_c$$
$$V_{ca} = I_c Z_c - I_a Z_a$$

This shows that the charging currents per mile are the same as the currents supplied to three condensers which are connected in $Y$ across the circuit and whose reactances are

$$\left.\begin{aligned} x_a &= \frac{4.10 \times 10^6}{f} \log \frac{ab \cdot ca}{r \cdot bc} \\ x_b &= \frac{4.10 \times 10^6}{f} \log \frac{ab \cdot bc}{r \cdot ca} \\ x_c &= \frac{4.10 \times 10^6}{f} \log \frac{ca \cdot bc}{r \cdot ab} \end{aligned}\right\} \tag{169}$$

If $Z_{a0}$, $Z_{a1}$ and $Z_{a2}$ are the symmetrical components of the reactance $jx_a$, $I_1$, and $I_2$ the positive- and negative-sequence components of the charging current $I_a$, and $V_1$ and $V_2$ the corresponding components of the line-to-neutral potential $V_a$, we may write

$$\left.\begin{aligned} V_1 &= I_1 Z_{a0} + I_2 Z_{a2} \\ V_2 &= I_2 Z_{a0} + I_1 Z_{a1} \end{aligned}\right\} \tag{170}$$

Solving these equations for the positive and negative-sequence components of the charging current gives

$$I_1 = \frac{V_1 Z_{a0} - V_2 Z_{a2}}{Z_{a0}^2 - Z_{a1} Z_{a2}} \tag{171}$$

$$I_2 = \frac{V_2 Z_{a0} - V_1 Z_{a1}}{Z_{a0}^2 - Z_{a1} Z_{a2}} \tag{172}$$

Note that $Z_{a1}Z_{a2}$ is a zero-sequence quantity (Sequence Rule).

**62. Effect of Transposition.**—If in the second section the $a$ conductor occupies the $b$ position, the positive- and negative-sequence components of the charging current, $I_a$, become

$$I_1 = \frac{V_1 Z_{b0} - V_2 Z_{b2}}{Z_{b0}^2 - Z_{b1} Z_{b2}}$$

$$I_2 = \frac{V_2 Z_{b0} - V_1 Z_{b1}}{Z_{b0}^2 - Z_{b1} Z_{b2}}$$

In the third section we may write similar expressions except that the subscript $b$ should be replaced by the subscript $c$. Since the sum of the positive-sequence components of impedance $Z_{a1}$, $Z_{b1}$, and $Z_{c1}$ is zero and, similarly, since the sum of the negative-sequence components is also zero and since $Z_{a1}Z_{a2}$ equals $Z_{b1}Z_{b2}$, it is evident that the average values of the positive- and negative-sequence components of the charging current per mile are

$$\frac{V_1 Z_0}{Z_0^2 - Z_1 Z_2}$$
$$\frac{V_2 Z_0}{Z_0^2 - Z_1 Z_2}$$

Thus it may be said that the average value of the charging impedance per mile for either positive- or negative-sequence potential is

$$Z_0 - \frac{Z_1 Z_2}{Z_0} \qquad \text{or} \qquad Z_0\left(1 - \frac{Z_1 Z_2}{Z_0^2}\right) \text{ vector ohms per mile}$$

The zero-sequence component of the reactances [equations (169)] is

$$Z_0 = -j\frac{4.10 \times 10^6}{f} \log \frac{\sqrt[3]{ab \cdot bc \cdot ca}}{r} \text{ vector ohms per mile}$$

The product of the positive- and negative-sequence components of the reactances is

$$\begin{aligned} Z_1 Z_2 &= -j\tfrac{1}{9}[x_a(x_a - x_b) + x_b(x_b - x_c) + x_c(x_c - x_a)] \\ &= -j\frac{1}{9}\,\frac{4.10 \times 10^6}{f}\left[\log\frac{ab \cdot ca}{r \cdot bc}\log\left(\frac{ca}{bc}\right)^2 + \log\frac{ab \cdot bc}{r \cdot ca}\log\left(\frac{ab}{ca}\right)^2 \right. \\ &\qquad \left. + \log\frac{ca \cdot bc}{r \cdot ab}\log\left(\frac{bc}{ab}\right)^2\right] \end{aligned}$$

For an equilateral spacing this product, $Z_1 Z_2$, is zero and the charging impedance is $Z_0$. For a flat spacing in which $ab = bc = ca/2$ and $ab/r = R$,

$$Z_1 Z_2 = -j\frac{4.10 \times 10^6}{f}(\log \sqrt[3]{4})^2$$

The corrective term becomes

$$\frac{Z_1 Z_2}{Z_0^2} = \left(\frac{\log \sqrt[3]{4}}{\log \sqrt[3]{2R}}\right)^2$$

For conductors having a diameter of about 1 in. and spaced about 20 ft. apart, this corrective term is about 0.5 per cent. For conductors having a diameter of 1 in. but spaced 50 in. apart, the corrective term becomes 1 per cent. For all practical cases

it is thus sufficiently accurate to assume that the charging impedance either to positive sequence or to negative sequence is

$$-j\frac{4.10 \times 10^{-6}}{f} \log \frac{\sqrt[3]{ab \cdot bc \cdot ca}}{r} \text{ vector ohms per mile} \qquad (173)$$

Let us further consider equations (170), (171), and (172), which apply to any *one transposition* section of the line. Inspection of equation (171) shows that, if the line potentials are *balanced,* the positive-sequence charging current per mile is the same in each transposition section of the line. Furthermore, equation (172) shows that, with balanced line potentials, the charging current has a negative-sequence component which has the same magnitude in each transposition section. For the flat spacing cited, in which $ab = bc = ca/2$, the ratio of the negative-sequence to the positive-sequence charging current in each transposition section is $\frac{2}{3}\frac{\log 2}{\log \sqrt[3]{2}R}$ if the line potentials are balanced. If $R = 400$, the negative-sequence charging current is 0.074 of the positive-sequence charging current. There is a positive-sequence component in potential due to this negative-sequence component in the charging current whose value is, by equations (170),

$$-V_1 \frac{Z_{a1}Z_{a2}}{Z_{a0}^2 - Z_{a1}Z_{a2}}$$

That is, the negative-sequence component in the charging current produces the same positive-sequence component in potential in each of the transposition sections. For the particular case just mentioned this potential has a value of $-0.0055V_1$ and may properly be considered negligible. If the line potentials are unbalanced owing to some unsymmetry in the circuit, the relative value of the negative-sequence charging current may become much greater.

It is interesting to point out that, had we used the approximate method and assumed that the charging current per mile on the *a* conductor is the same in each of the transposition sections and that this is likewise true for the charging currents per mile on both the *b* and *c* conductors, we should have found that the ratio

of the *average* value of the positive-sequence potential to neutral due to this charging current is

$$-j\frac{4.10 \times 10^{-6}}{f}I_a \log \frac{\sqrt[3]{ab \cdot bc \cdot ca}}{r}$$

Thus, as we have shown, the error in the approximate method is generally less than 1 per cent.

**63. Effect of Earth.**—As is customary, we shall assume that the earth is a conducting body of infinite extent whose surface is a plane. For the purpose of computing capacitance the effect of the earth may be simulated by replacing it by the images of all the conductors of the system. This assumes that the earth charges reside on the surface of the earth. If these surface charges vary, as they do when the potentials are alternating, it is probably more accurate to assume that as the charges move from one point to another on the earth's surface they pass, not along the surface, but through the body of the earth, in accordance with the principles by which Carson determined the impedance of a conductor-earth loop.

Consider three parallel conductors, $a$, $b$, and $c$, the distances between which are $ab$, $bc$, and $ca$, and whose heights above the conducting surface of the earth are $ag$, $bg$, and $cg$. Each of the conductors has a radius $r$. The images of the conductors are $a'$, $b'$, and $c'$. The distance between conductor $a$ and its image $a'$ is $aa' = 2ag$. The distance between the conductor $a$ and the image of the conductor $b$ is $ab'$. This is also the distance between $b$ and the image of $a$. The fictitious charging current on the image of a conductor is the negative of the charging current on the conductor itself.

The fall in potential from the conductor $a$ to earth, due solely to the charging current of $I_a$ amperes per mile on $a$ and to $-I_a$ amperes per mile on the image of $a$, is

$$V_{ag} = -j\frac{4.10 \times 10^{6}}{f}\left(I_a \log \frac{ag}{r} - I_a \log \frac{a'g}{a'a}\right)$$

$$= -j\frac{4.10 \times 10^{6}}{f}I_a \log \frac{2ag}{r}$$

The fall in potential from the conductor $a$ to earth, due solely to the charging current of $I_b$ amperes per mile on $b$ and to $-I_b$ amperes per mile on the image of $b$, is

$$V_{ag} = -j\frac{4.10 \times 10^6}{f}\left(I_b \log \frac{bg}{ba} - I_b \log \frac{b'g}{b'a}\right)$$
$$= -j\frac{4.10 \times 10^6}{f} I_b \log \frac{b'a}{ba}.$$

We can now write the expression for the fall in potential from each line conductor to earth. Thus

$$V_{ag} = -j\frac{4.10 \times 10^6}{f}\left(I_a \log \frac{2ag}{r} + I_b \log \frac{ab'}{ab} + I_c \log \frac{ca'}{ca}\right)$$
$$V_{bg} = -j\frac{4.10 \times 10^6}{f}\left(I_a \log \frac{ab'}{ab} + I_b \log \frac{2bg}{r} + I_c \log \frac{bc'}{bc}\right)$$
$$V_{cg} = -j\frac{4.10 \times 10^6}{f}\left(I_a \log \frac{ca'}{ca} + I_b \log \frac{bc'}{bc} + I_c \log \frac{2cg}{r}\right)$$

It will simplify the nomenclature if we write these potential equations as

$$\left.\begin{aligned} V_{ag} &= -j(x_{aa}I_a + x_{ab}I_b + x_{ca}I_c) \\ V_{bg} &= -j(x_{ab}I_a + x_{bb}I_b + x_{bc}I_c) \\ V_{cg} &= -j(x_{ca}I_a + x_{bc}I_b + x_{cc}I_c) \end{aligned}\right\} \tag{174}$$

The solution of these three equations for the charging currents gives

$$I_a = j\frac{\begin{vmatrix} V_{ag} & x_{ab} & x_{ca} \\ V_{bg} & x_{bb} & x_{bc} \\ V_{cg} & x_{bc} & x_{cc} \end{vmatrix}}{|D|}$$
$$I_b = j\frac{\begin{vmatrix} x_{aa} & V_{ag} & x_{ca} \\ x_{ab} & V_{bg} & x_{bc} \\ x_{ca} & V_{cg} & x_{cc} \end{vmatrix}}{|D|}$$
$$I_c = j\frac{\begin{vmatrix} x_{aa} & x_{ab} & V_{ag} \\ x_{ab} & x_{bb} & V_{bg} \\ x_{ca} & x_{bc} & V_{cg} \end{vmatrix}}{|D|}$$

where

$$|D| = \begin{vmatrix} x_{aa} & x_{ab} & x_{ca} \\ x_{ab} & x_{bb} & x_{bc} \\ x_{ca} & x_{bc} & x_{cc} \end{vmatrix}$$

If the determinants are expanded, the equations for the charging currents will be

$$\left.\begin{aligned} I_a &= Y_{aa}V_{ag} + Y_{ab}V_{bg} + Y_{ca}V_{cg} \\ I_b &= Y_{ab}V_{ag} + Y_{bb}V_{bg} + Y_{bc}V_{cg} \\ I_c &= Y_{ca}V_{ag} + Y_{bc}V_{bg} + Y_{cc}V_{cg} \end{aligned}\right\} \tag{175}$$

where, for example, the admittance $Y_{aa}$ is

$$j\frac{(x_{bb}x_{cc} - x_{bc}^2)}{|D|}$$

and the admittance $Y_{ab}$ is

$$j\frac{(x_{bc}x_{ca} - x_{cc}x_{ab})}{|D|}$$

The other admittance, $Y_{bb}$, $Y_{cc}$, $Y_{bc}$, $Y_{ca}$, will have similar forms.

If the admittances in equations (175) are resolved into their symmetrical components, the equations for the symmetrical components of the charging currents become

$$I_{a0} = (Y_{aa0} + 2Y_{bc0})V_{ag0} + (Y_{aa2} - Y_{bc2})V_{ag1} + (Y_{aa1} - Y_{bc1})V_{ag2} \tag{176}$$

$$I_{a1} = (Y_{aa1} - Y_{bc1})V_{ag0} + (Y_{aa0} - Y_{bc0})V_{ag1} + (Y_{aa2} + 2Y_{bc2})V_{ag2} \tag{177}$$

$$I_{a2} = (Y_{aa2} - Y_{bc2})V_{ag0} + (Y_{aa1} + 2Y_{bc1})V_{ag1} + (Y_{aa0} - Y_{bc0})V_{ag2} \tag{178}$$

If in the second section of the line the $a$ conductor occupies the position that was occupied by the $b$ conductor in the first section, the subscripts on both the reactances and admittances should be advanced one letter. Thus in the *second* section the equation for the zero-sequence component of the charging current on the $a$ conductor becomes

$$I_{a0} = (Y_{bb0} + 2Y_{ca0})V_{ag0} + (Y_{bb2} - Y_{ca2})V_{ag1} + (Y_{bb1} - Y_{ca1})V_{ag2} \tag{179}$$

where the subscripts indicate the values of the admittances in the *first* section.

The equations for the positive- and negative-sequence components of the charging current in the *second* section can likewise be written by advancing by one letter the admittance subscripts in equations (177) and (178). By a similar process of again

advancing the admittance subscripts one letter, the equations for the symmetrical components of the charging currents in the *third* section are obtained. Since the average values of the positive- and negative-sequence admittance coefficients in the three sections are both zero, the average values of the symmetrical components of the charging currents are, if the transposition sections have equal lengths

$$I_{a0} = (Y_{aa0} + 2Y_{bc0})V_{ag0} \text{ vector amperes per mile} \quad (180)$$
$$I_{a1} = (Y_{aa0} - Y_{bc0})V_{ag1} \text{ vector amperes per mile} \quad (181)$$
$$I_{a2} = (Y_{aa0} - Y_{bc0})V_{ag2} \text{ vector amperes per mile} \quad (182)$$

Thus, as we should expect, in a properly transposed line the symmetrical components of the charging currents are proportional to the corresponding components of the line-to-ground potentials. It is interesting to note that, when the line-to-ground potentials are balanced, the average values of both the zero- and negative-sequence charging currents are zero [see equations (180) and (182)]. Also equation (177) shows that the positive-sequence component of the charging current is the same in each section. There are, however, both zero-sequence and negative-sequence components of charging current in each section as is shown by equations (176) and (178).

$$I_{a0} = (Y_{aa2} - Y_{bc2})V_{ag1}$$
$$I_{a2} = (Y_{aa1} + 2Y_{bc1})V_{ag1}$$

Each of these components of the charging current contribute slightly toward the positive-sequence line-to-ground potential. The amount of this contribution can be determined by resolving equations (174) into their symmetrical components. This contribution accounts for the difference that exists between the exact and approximate methods of computing the charging impedance to positive sequence. The difference is very small and in the two typical cases we shall presently discuss amounts to less than 0.5 per cent. We shall now determine the values of the zero-sequence admittances.

$$Y_{aa0} = j\frac{1}{3}\left(\frac{4.10 \times 10^6}{f}\right)^2\left(\log\frac{2ag}{r}\log\frac{2bg}{r} + \log\frac{2bg}{r}\log\frac{2cg}{r}\right.$$
$$\left. + \log\frac{2cg}{r}\log\frac{2ag}{r} - \log^2\frac{ab'}{ab} - \log^2\frac{bc'}{bc} - \log^2\frac{ca'}{ca}\right) \div |D| \quad (183)$$

$$Y_{bc0} = j\frac{1}{3}\left(\frac{4.10 \times 10^6}{f}\right)^2\left(\log\frac{ab'}{ab}\log\frac{bc'}{bc} + \log\frac{bc'}{bc}\log\frac{ca'}{ca} + \log\frac{ca'}{ca}\log\frac{ab'}{ab} - \log\frac{2ag}{r}\log\frac{bc'}{bc} - \log\frac{2bg}{r}\log\frac{ca'}{ca} - \log\frac{2cg}{r}\log\frac{ab'}{ab}\right) \div |D| \quad (184)$$

where

$$D = \left(\frac{4.10 \times 10^6}{f}\right)^3\left(\log\frac{2ag}{r}\log\frac{2bg}{r}\log\frac{2cg}{r} + 2\log\frac{ab'}{ab}\log\frac{bc'}{bc}\log\frac{ca'}{ca} - \log\frac{2ag}{r}\log^2\frac{bc'}{bc} - \log\frac{2bg}{r}\log^2\frac{ca'}{ca} - \log\frac{2cg}{r}\log^2\frac{ab'}{ab}\right) \quad (185)$$

The prediction is fulfilled that it is relatively laborious to compute the exact values of the charging admittances.

**64. Charging Admittance by Approximate Method.**—We shall now determine these admittances by the approximate method and compare, in two typical cases, the values as calculated by the two methods. Assume that the positive-sequence charging current per mile is the same in each of the three transposition sections. This is true only when the line-to-ground potentials are balanced, as shown by equation (177). We can then write the equations for the line-to-ground potentials in each section due to this positive-sequence component of the charging current. Thus in the first section, since $I_b = \alpha^2 I_a$ and $I_c = \alpha I_a$, where $\alpha = \underline{/120°}$

$$V_a = -j\frac{4.10 \times 10^6}{f}I_{a1}\left(\log\frac{2ag}{r} + \alpha^2\log\frac{ab'}{ab} + \alpha\log\frac{ca'}{ca}\right)$$

$$V_b = -j\frac{4.10 \times 10^6}{f}I_{a1}\left(\log\frac{ab'}{ab} + \alpha^2\log\frac{2bg}{r} + \alpha\log\frac{bc'}{bc}\right)$$

$$V_c = -j\frac{4.10 \times 10^6}{f}I_{a1}\left(\log\frac{ca'}{ca} + \alpha^2\log\frac{bc'}{bc} + \alpha\log\frac{2cg}{r}\right)$$

Since the positive-sequence component of $V_a$ is

$$\tfrac{1}{3}(V_a + \alpha V_b + \alpha^2 V_c),$$

$$\frac{V_{a1}}{I_{a1}} = -j\frac{4.10 \times 10^6}{f}\log\frac{\sqrt[3]{ab \cdot bc \cdot ca}}{r}\sqrt[3]{\frac{2ag \cdot 2bg \cdot 2cg}{ab' \cdot bc' \cdot ca'}} \text{ vector ohms per mile} \quad (186)$$

From the symmetry of this expression it is apparent that the positive-sequence component of line-to-ground potential, due to the positive-sequence component of the charging current, is the same in each of the three transposition sections. The error in equation (186) results from assuming that the positive-sequence component in the potential is due solely to the positive-sequence component in the charging current (see page 164). Compare this approximate value of the charging impedance with the corresponding value when the effect of the earth is neglected [equation (173), page 160]. The approximate value of the charging impedance to be used with the negative-sequence component of potential is the same as that used with the positive-sequence component of potential since the circuit has been made symmetrical by transposition. In determining the approximate value of the charging impedance to zero-sequence we shall assume that the zero-sequence component of the line-to-ground potential is due solely to the zero-sequence component of the charging current. The line-to-ground potential, $V_a$, due to the zero-sequence component of the charging current, is

$$V_a = -j\frac{4.10 \times 10^6}{f}I_0\left(\log\frac{2ag}{r} + \log\frac{ab'}{ab} + \log\frac{ca'}{ca}\right)$$

Similar expressions may be written for $V_b$ and $V_c$. Since the zero-sequence component of these potentials, $V_0$, is

$$\tfrac{1}{3}(V_a + V_b + V_c),$$

$$\frac{V_0}{I_0} = -j\frac{4.10 \times 10^6}{f}\log\frac{\sqrt[3]{2ag \cdot 2bg \cdot 2cg}}{r}\sqrt[3]{\left(\frac{ab' \cdot bc' \cdot ca'}{ab \cdot bc \cdot ca}\right)^2} \text{ vector ohms per mile} \quad (187)$$

**65. Comparison of Exact and Approximate Values of Charging Impedances.**—We shall now compare the approximate values of these charging impedances with their exact values in two typical cases.

*Case I.*—Horizontal spacing in which $ag = bg = cg = 50'$; $ab = bc = 20'$; $ac = 40'$; radius $r = 0.5''$.

$$\log\frac{2ag}{r} = \log\frac{100 \times 12}{0.5} = 3.3802$$

$$ab' = bc' = \sqrt{100^2 + 20^2} = 102.0 \qquad ca' = 107.7$$

$$\log \frac{ab'}{ab} = \log \frac{102.0}{20} = 0.7076$$

$$\log \frac{aa'}{ca} = \log 2.693 = 0.4303$$

$$|D| = \left(\frac{4.10 \times 10^6}{f}\right)^3 [3.3802^3 + 2 \times 0.7076^2 \times 0.4303 - 3.3802(2 \times 0.7076^2 + 0.4303^2)]$$

$$= \left(\frac{4.10 \times 10^6}{f}\right)^3 35.04$$

$$Y_{aa0} = j\frac{1}{3}\left(\frac{4.10 \times 10^6}{f}\right)^2 (3 \times 3.3802^2 - 2 \times 0.7076^2 - 0.4303^2) \div \left(\frac{4.10 \times 10^6}{f}\right)^3 35.04$$

$$Y_{aa0} = j\frac{f}{4.10 \times 10^6} 0.3147$$

$$Y_{bc0} = j\frac{1}{3}\left(\frac{4.10 \times 10^6}{f}\right)^2 [0.7076^2 + 2 \times 0.7076 \times 0.4303 - 3.3802(2 \times 0.7076 + 0.4303)] \div \left(\frac{4.10 \times 10^6}{f}\right)^3 35.04$$

$$= -j\frac{f}{4.10 \times 10^6} 0.0484$$

Thus the exact value of the charging admittance for positive and for negative sequence is

$$Y_{aa0} - Y_{bc0} = j\frac{f}{4.10 \times 10^6} 0.3631 \text{ vector mhos per mile}$$

The exact value of the charging impedance for positive and for negative sequence is

$$-j\frac{4.10 \times 10^6}{f} 2.755 \text{ vector ohms per mile}$$

The approximate value of this charging impedance [equation (186)] is

$$-j\frac{4.10 \times 10^6}{f} 2.765 \text{ vector ohms per mile}$$

Had the effect of the earth been neglected, the approximate value of this charging impedance [equation (173)] would have been $-j\frac{4.10 \times 10^6}{f}2.780$ vector ohms per mile.

*Case II.*—Vertical spacing in which $ag = 80'$; $bg = 60'$; $cg = 40'$; $ab = bc = 20'$; $ac = 40'$; radius $r = 0.5''$.

$$\log\frac{2ag}{r} = 3.5843, \qquad \log\frac{2bg}{r} = 3.4594, \qquad \log\frac{2cg}{r} = 3.2833$$

$$\log\frac{ab'}{ab} = 0.8451, \qquad \log\frac{bc'}{bc} = 0.6990, \qquad \log\frac{ca'}{ca} = 0.4771$$

$$D = \left(\frac{4.10 \times 10^6}{f}\right)^3 36.38$$

$$Y_{aa0} = j\frac{f}{4.10 \times 10^6}0.3123$$

$$Y_{bc0} = -j\frac{f}{4.10 \times 10^6}0.0513$$

Thus the exact value of the charging admittance for positive and for negative sequence is

$$j\frac{f}{4.10 \times 10^6}0.3636 \text{ vector mhos per mile}$$

The exact value of the charging impedance for positive and for negative sequence in this case is

$$-j\frac{4.10 \times 10^6}{f}2.750 \text{ vector ohms per mile}$$

The approximate value of this charging impedance [equation (186)] is

$$-j\frac{4.10 \times 10^6}{f}2.769 \text{ vector ohms per mile}$$

If the effect of the earth is neglected, the approximate value of the charging impedance [equation (173)] is

$$-j\frac{4.10 \times 10^6}{f}2.782 \text{ vector ohms per mile}$$

When one considers that the *conducting* surface of the earth is not plane, that on account of the sag the line conductors are not straight, and that the effect of the capacitance between the line

conductors and the supporting towers is neglected, it is evident that there is no justification for computing the value of the charging impedance by the relatively laborious exact method. Some may question the need for taking into account the effect of the earth and for this reason would use equation (173) when computing the charging impedance to positive and to negative sequence. Others will prefer to consider the effect of the earth and compute the charging impedance to positive and to negative sequence by equation (186).

*Zero Sequence.*—We shall now compute the charging impedance to zero sequence in each of these cases by both the exact and the approximate method.

*Case I.*—The exact value of the charging admittance for zero sequence is

$$Y_{aa0} + 2Y_{bc0} = j\frac{f}{4.10 \times 10^6}(0.315 - 0.097) \text{ vector mhos per mile}$$

Thus the exact value of the charging impedance is

$$-j\frac{4.10 \times 10^6}{f}4.59 \text{ vector ohms per mile}$$

The approximate value of this charging impedance is by equation (187)

$$-j\frac{4.10 \times 10^6}{f}4.61 \text{ vector ohms per mile}$$

*Case II.*—The exact value of the charging admittance for zero sequence is

$$Y_{aa0} + 2Y_{bc0} = j\frac{f}{4.10 \times 10^6}(0.312 - 0.102) \text{ vector mhos per mile}$$

Thus the exact value of the charging impedance is

$$-j\frac{4.10 \times 10^6}{f}4.77 \text{ vector ohms per mile}$$

The approximate value of this charging impedance is by equation (187)

$$-j\frac{4.10 \times 10^6}{f}4.79 \text{ vector ohms per mile}$$

It will be observed that the error in the charging impedance for zero sequence as calculated by the approximate method is less than 0.5 per cent in both the typical cases. This is ample justification for calculating the charging impedance by the approximate method.

**66. Ungrounded Line.**—An interesting condition arises when the transmission circuit is *not grounded*. If the line is completely transposed but ungrounded, the average value of the zero sequence charging current must be zero and thus by equation (180) the zero-sequence component of the line-to-ground potential is also zero. If the line potentials are balanced, their negative-sequence component is zero and by equation (176) the zero-sequence component of the charging current in the *first* transposition section is

$$I_{a0} = V_{ag1}(Y_{aa2} - Y_{bc2})$$

In the *second* section the zero-sequence component of the charging current becomes $V_{ag1}(Y_{bb2} - Y_{ca2})$ and in the *third* section, $V_{ag1}(Y_{cc2} - Y_{ab2})$. Thus there is in each section a zero-sequence component of charging current which has the same magnitude as that in the next section but which differs from it in time phase by 120 degrees. The sum of these zero-sequence components in three successive transposition sections is zero. That is, while there is a zero-sequence component of charging current in each section of the line, the total zero-sequence component of charging current is zero. Similarly, there is in each section a negative-sequence component of charging current. The component in the *first* section is $V_{ag1}(Y_{aa1} + 2Y_{bc1})$, that in the *second* section is $V_{ag1}(Y_{bb1} + 2Y_{ca1})$, and that in the *third* section is $V_{ag1}(Y_{cc1} + 2Y_{ab1})$. Thus the negative-sequence components of the charging current in three successive sections of the *a* conductor form a symmetrical system.

*Case I.*—The zero-sequence charging current per mile is 2.2 per cent of the positive-sequence charging current per mile while the negative-sequence component is 6.8 per cent of the positive-sequence component.

*Case II.*—The zero-sequence charging current per mile is 3.5 per cent of the positive-sequence charging current per mile, while the negative-sequence component is again 6.8 per cent of the positive-sequence component.

**67. Ground Wires.**—When the transmission line is protected by a single ground wire, $p$, the equations for the line-to-ground potentials become

$$V_{ag} = -j\frac{4.10 \times 10^6}{f}\left(I_a \log \frac{2ag}{r} + I_b \log \frac{ab'}{ab} + I_c \log \frac{ca'}{ca} + I_p \log \frac{ap'}{ap}\right)$$

$$V_{bg} = -j\frac{4.10 \times 10^6}{f}\left(I_a \log \frac{ab'}{ab} + I_b \log \frac{2bg}{r} + I_c \log \frac{bc'}{bc} + I_p \log \frac{bp'}{bp}\right)$$

$$V_{cg} = -j\frac{4.10 \times 10^6}{f}\left(I_a \log \frac{ca'}{ca} + I_b \log \frac{bc'}{bc} + I_c \log \frac{2cg}{r} + I_p \log \frac{cp'}{cp}\right)$$

$$0 = -j\frac{4.10 \times 10^6}{f}\left(I_a \log \frac{ap'}{ap} + I_b \log \frac{bp'}{bp} + I_c \log \frac{cp'}{cp} + I_p \log \frac{2pg}{r_p}\right)$$

It is a simple matter to substitute the value of $I_p$ as determined by the last equation of this group in each one of the others. The resulting equations for the charging currents will be like equations (174), except for the modified values of the coefficients. The analysis may then be completed by the exact method indicated on page 162, but this will not be done here. We shall give, however, the computed values of the charging admittances when a single ground wire is added to each of the two typical conductor arrangements so that the effect of the ground wire may be noted.

*Case I.*—In addition there is a ground wire of 0.5 in. diameter placed 15 ft. above the central conductor.

The exact value of the charging admittance per mile to positive and to negative sequence is $j\frac{f}{4.10 \times 10^6}0.366$ mho. Comparing this value with the value of the charging admittance when there is no ground wire shows that the effect of the ground wire is to increase the admittance by 0.83 per cent.

The exact value of the charging admittance per mile to zero-sequence is $j\frac{f}{4.10 \times 10^6}0.235$ mho. Comparing this value with the value of the charging admittance when there is no ground wire shows that the effect of the ground wire is to increase the admittance by 7.8 per cent.

*Case II.*—In addition there is a ground wire 0.5 in. in diameter placed 15 ft. above the topmost conductor. The exact value of the charging admittance per mile to positive and to negative sequence is $j\frac{f}{4.10 \times 10^6}0.367$ mho. Comparing this value with the value of the charging admittance when there is no ground wire shows that the effect of the ground wire is to increase the admittance by 0.82 per cent.

The exact value of the charging admittance per mile to zero sequence is $j\frac{f}{4.10 \times 10^6}0.227$ mho. Comparing this value with the value of the charging admittance when there is no ground wire shows that the effect of the ground wire is to increase the charging admittance by 8.0 per cent.

The general conclusions which may be drawn from the results just given are that, for circuits of approximately the arrangements cited, the ground wire has an insignificant effect on the charging admittance to positive and to negative sequence but has an appreciable effect on the charging admittance to zero sequence. We have previously shown that the charging admittance to positive and to negative sequence can be computed with sufficient accuracy by disregarding entirely the effect of the earth. We are thus justified in concluding that in cases such as we have cited it is proper to compute the charging admittance to positive and to negative sequence without considering the effect of either the earth or the ground wires. The general expression for the charging admittance per mile to positive and to negative sequence is [see equation (173)]

$$j\frac{f}{4.10 \times 10^6} \log \frac{\sqrt[3]{ab \cdot bc \cdot ca}}{r}$$

Thus we do not find it necessary to use the approximate method of analysis and so obtain a general expression for the value of the charging impedance to positive and to negative sequence

which takes into account the effect of the earth and the effect of ground wires. This is fortunate inasmuch as this expression for the charging impedance is somewhat cumbersome. Since the ground wire produces a noticeable effect on the charging admittance to zero sequence, we shall determine the value of this admittance by the approximate method and compare it with the exact value to see if there is a satisfactory agreement between the two values in the two cases we have chosen. If the approximate method of analysis is followed in determining the charging impedance to zero sequence

$$V_{ag0} = -j\frac{4.10 \times 10^6}{3f} \left\{ I_{a0}\left[\log \frac{2ag \cdot 2bg \cdot 2cg}{r^3} + \log\left(\frac{ab' \cdot bc' \cdot ca'}{ab \cdot bc \cdot ca}\right)^2\right] + I_p\left(\log \frac{ap' \cdot bp' \cdot cp'}{ap \cdot bp \cdot cp}\right)\right\}$$

and

$$I_p \log \frac{2pg}{r_p} = -I_{a0} \log \frac{ap' \cdot bp' \cdot cp'}{ap \cdot bp \cdot cp}$$

Thus the approximate value of the charging impedance to zero sequence is

$$\frac{V_{ag0}}{I_{a0}} = -j\frac{4.10 \times 10^6}{f}\left[\log \frac{\sqrt[3]{2ag \cdot 2bg \cdot 2cg}}{r} \sqrt[3]{\left(\frac{ab' \cdot bc' \cdot ca'}{ab \cdot bc \cdot ca}\right)^2} - \frac{1}{3}\,\frac{\log^2\left(\frac{ap' \cdot bp' \cdot cp'}{ap \cdot bp \cdot cp}\right)}{\log \frac{2pg}{r_p}}\right] \text{ ohms per mile} \qquad (188)$$

If we compare equation (188) with the expression obtained when there is no ground wire [equation (187)], it will be observed that the second term on the right-hand side of equation (188) corrects for the effect of the ground wire.

*Case I.*—The charging admittance to zero sequence as computed by the approximate method is 1.7 per cent greater than the value computed by the exact method.

*Case II.*—The charging admittance to zero sequence as computed by the approximate method is about 0.2 per cent smaller than the value computed by the exact method.

We may thus conclude that the approximate method which results in equation (188) is sufficiently accurate for computing the charging admittance to zero sequence for arrangements of conductors similar to those in the two cases we have considered.

**68. Parallel Transmission Lines without Ground Wires.**—Let us consider two parallel 3-phase circuits whose conductors are *a, b, c* and *d, e, f*. The fall in potential from the line conductor *a* to earth is

$$V_{ag} = -j\frac{4.10 \times 10^6}{f}\left(I_a \log \frac{2ag}{r} + I_b \log \frac{ab'}{ab} + I_c \log \frac{ac'}{ac} + I_d \log \frac{ad'}{ad} + I_e \log \frac{ae'}{ae} + I_f \log \frac{af'}{af}\right) \text{ volts} \quad (189)$$

The currents $I_a$, $I_b$, etc., are the charging currents in amperes per mile. Expressions similar to equations (189) can be written for each of the other five line-to-earth potentials. In order to obtain exact expressions for the charging admittances, it would be necessary to solve six simultaneous equations to obtain the expressions for the charging currents directly in terms of the potentials to earth. Since the literal solution of these equations for *all* the currents would contain more than 8000 logarithmic terms, it will not be attempted here. However, the author, with the assistance of L. F. Woodruff, did solve these equations and obtain exact values of the charging admittances for a typical arrangement of conductors. This was done for the purpose of comparing the exact and the approximate solutions. We shall consider only the case in which the circuits are simultaneously transposed in the same manner. The approximate solution consists naturally of two component parts: one the effect of the charging currents in one circuit upon that circuit; the other, the effect of the charging currents in one circuit upon the other circuit. The first of these has already been calculated and is given as a self-impedance in equations (186) and (187). The second or mutual effect between the two circuits can be calculated in exactly the same way that the inductive mutual reactance between parallel lines was calculated. On comparing equation (145), which relates to the electromagnetically induced potentials, with equation (189), which relates to the electrostatically induced potentials, it will be observed that the logarithmic coefficient

of $I_d$ in equation (145) is $\log \frac{1}{ad}$ and that the coefficient of $I_d$ in equation (189) is $\log \frac{ad'}{ad}\cdot$

Thus it is evident, from the form of equations (148) and (149), that, when the circuits are simultaneously transposed, the ratio of the average positive-sequence potential induced on the $(a,b,c)$ circuit *to* the average positive-sequence charging current on the $(d,e,f)$ circuit which induces this potential is,

$$\frac{V_{ag1}}{I_{d1}} = -j\frac{4.10 \times 10^6}{3f}\left(\log \sqrt{\frac{ae \cdot bf \cdot cd \cdot af \cdot bd \cdot ce}{ae' \cdot bf' \cdot cd' \cdot af' \cdot bd' \cdot ce'}\,\frac{ad' \cdot be' \cdot cf'}{ad \cdot be \cdot cf}}\right.$$

$$\left. + j0.866 \log \frac{ae \cdot bf \cdot cd \cdot af' \cdot bd' \cdot ce'}{ae' \cdot bf' \cdot cd' \cdot af \cdot bd \cdot ce}\right) \text{vector ohms per mile} \qquad (190)$$

The expression for ratio $V_{ag2}/I_{d2}$ is the same except that the sign before $j0.866$ is minus instead of plus.

For many arrangements of conductors (Figs. 43 and 44) the second term in these expressions is zero and the mutual charging impedance is the same for positive sequence as it is for negative sequence. In this case

$$\frac{V_{ag1}}{I_{d1}} = \frac{V_{ag2}}{I_{d2}} = -j\frac{4.10 \times 10^6}{f} \log \sqrt[3]{\frac{ae \cdot bf \cdot cd \cdot ad' \cdot be' \cdot cf'}{ae' \cdot bf' \cdot cd' \cdot ad \cdot be \cdot cf}} \qquad (191)$$

Compare equation (191) with equation (150), page 145. The mutual effect between the two circuits for zero sequence can be derived by following a procedure similar to that used in calculating the mutual inductive reactance. On comparing equation (189) with equation (151), it will be observed that the coefficient of $I_d$ in (189) corresponds to $Z_{ad}$ in equation (151), page 146. Thus from inspection of equation (153), page 147, and equation (128), page 127, it is evident that

$$V_{ag0} = -j\frac{4.10 \times 10^6}{f}\frac{I_{d0}}{3} \log \frac{ad' \cdot bd' \cdot cd' \cdot ae' \cdot be' \cdot ce' \cdot af' \cdot bf' \cdot cf'}{ad \cdot bd \cdot cd \cdot ae \cdot be \cdot ce \cdot af \cdot bf \cdot cf}$$

This can be written

$$V_{ag0} = -j\frac{4.10 \times 10^6}{f} 3I_{d0} \log \frac{{}_{abc}\text{G.M.D.}_{d'e'f'}}{{}_{abc}\text{G.M.D.}_{def}} \qquad (192)$$

It is usually more convenient to have expressions for the charging admittances rather than for the impedances. The self- and mutual admittances can be obtained in the following manner: Let $Z_{aa}$ and $Z_{dd}$ be the self-impedance as determined by equations (173) or (186) for positive and negative sequence and by equation (187) for zero sequence. Also let $Z_{ad}$ be the mutual impedance as determined by equations (190) or (191) for positive and negative sequence and by equation (192) for zero sequence. Then in general for any sequence order,

$$V_a = Z_{aa}I_a + Z_{ad}I_d$$
$$V_d = Z_{dd}I_d + Z_{ad}I_a$$

from which it follows that

$$I_a = \frac{Z_{dd}}{Z_{aa}Z_{dd} - Z_{ad}^2}V_a - \frac{Z_{ad}}{Z_{aa}Z_{dd} - Z_{ad}^2}V_d$$

A similar equation may be written for $I_d$.
Thus:

$$I_a = Y_{aa}V_a - Y_{ad}V_d$$

where

$$Y_{aa} = \frac{Z_{dd}}{Z_{aa}Z_{dd} - Z_{ad}^2} \tag{193}$$

and

$$Y_{ad} = \frac{Z_{ad}}{Z_{aa}Z_{dd} - Z_{ad}^2} \tag{194}$$

By following a procedure similar to that employed when analyzing a transformer, these expressions for the charging currents can be modified so as to become

$$I_a = V_a(Y_{aa} - Y_{ad}) + (V_a - V_d)Y_{ad}$$
$$I_d = V_d(Y_{dd} - Y_{ad}) + (V_d - V_a)Y_{ad}$$

These equations show that the charging currents are correctly computed by assuming that the line-to-earth admittances are $(Y_{aa} - Y_{ad})$ and $(Y_{dd} - Y_{ad})$ and that the admittance between the two circuits is $Y_{ad}$. If all the conductors are of the same size and are similarly arranged with respect to the earth, $Y_{aa} = Y_{dd}$ and both $(Y_{aa} - Y_{ad})$ and $(Y_{dd} - Y_{ad})$ reduce to

$$\frac{1}{Z_{aa} + Z_{ad}} \tag{195}$$

The mutual admittance $Y_{ad}$ then becomes

$$\frac{Z_{ad}}{Z_{aa}^2 - Z_{ad}^2} \tag{196}$$

If the two circuits are connected in parallel electrically, $V_a = V_d$ and

$$I_a = (Y_{aa} - Y_{ad})V_a$$

The total charging admittance for both circuits in terms of the self- and mutual charging impedances is

$$\frac{Z_{aa} + Z_{dd} - 2Z_{ad}}{Z_{aa}Z_{dd} - Z_{ad}^2} \tag{197}$$

If the circuits are alike ($Z_{aa} = Z_{dd}$) the total charging admittance is

$$\frac{2}{Z_{aa} + Z_{ad}} \tag{198}$$

In the numerical example referred to, the conductors were 1.0 in. in diameter and in each circuit were arranged in two vertical planes, the planes being 20 ft. apart. The upper and lower conductors of each circuit were 20 ft. from the middle conductor, which was 60 ft. above the conducting surface of the earth. The conductors of each circuit were symmetrically transposed in the same way. The exact values of the self- and mutual charging admittances were obtained by solving the original six simultaneous equations. They are expressed as mhos per mile.

The *exact* value of the self-admittance to positive or negative sequence is $j\frac{f}{4.10 \times 10^6}0.3660$, whereas the *approximate* value is $j\frac{f}{4.10 \times 10^6}0.3632$.

The *exact* value of the self-admittance to zero sequence is $j\frac{f}{4.10 \times 10^6}0.2483$. The *approximate* value is $j\frac{f}{4.10 \times 10^6}0.2476$.

The differences between the exact and approximate values of the self-admittances are insignificant.

The *exact* value of the mutual admittance to positive or negative sequence is $j\frac{f}{4.10 \times 10^6}0.0244$, whereas the *approximate* value is $j\frac{f}{4.10 \times 10^6}0.0269$, or about 10 per cent greater.

The *exact* value of the mutual admittance to zero sequence is $j\frac{f}{4.10 \times 10^6}0.0931$. The *approximate* value is $j\frac{f}{4.10 \times 10^6}0.0980$, or about 5 per cent greater. However, the effect of the mutual admittance is less important than that of the self-admittance. Also note that the *differences in mhos* between the exact and approximate values of the several admittances are the same order of magnitude.

If the circuits are connected in parallel electrically, the approximate value of the total admittance to positive sequence is 1.5 per cent less than the exact value. The approximate value of the total admittance to zero sequence is 3.6 per cent less than the exact value. It is important to remember that in the calculation of the exact values it was assumed that the conductors were straight and parallel to each other and to a conducting ground plane. This is an ideal condition which does not exist in practice.

**69. Long Lines.**—Consider a long 3-phase transmission line that is symmetrically transposed (page 128). When this line is operating in the *steady* state at normal frequency and under *balanced* conditions of zero or positive or negative sequence, the following differential equations determine the relations between the *vectors* which represent the line-to-earth potential, $E$, and the line current, $I$,

$$\frac{dE}{dx} = zI \tag{199}$$

$$\frac{dI}{dx} = yE \tag{200}$$

where $z$ and $y$ are the inductive impedance and the charging admittance per mile and $x$ is the distance in miles from the *receiving* end of the line. The solution of these equations shows that the relations between the potentials and currents at the two ends of the line are

$$E_a = E_b \cosh \theta + I_b Z_s \sinh \theta \tag{201}$$

$$I_a = E_b \frac{\sinh \theta}{Z_s} + I_b \cosh \theta \tag{202}$$

where the subscript $a$ indicates the sending end of the line, $b$ indicates the receiving end, $\theta$ is the angle of the line and equals $L\sqrt{yz}$, $L$ is the length of the line in miles, and $Z_s$ is the surge

impedance and equals $\sqrt{z/y}$. The line constants $\theta$ and $Z_s$ have the same values for positive and for negative sequence. Their values for zero sequence, however, are different from their values for positive and negative sequence. Equations (201) and (202) are frequently written

$$E_a = AE_b + BI_b \tag{203}$$
$$I_a = CE_b + DI_b \tag{204}$$

where $A$, $B$, $C$, and $D$ are the so-called *general circuit constants* of the line.

$$A = D = \cosh\theta = 1 + \frac{YZ}{2} + \frac{Y^2Z^2}{24} + \frac{Y^3Z^3}{720} + \cdots \tag{205}$$

$$B = Z_s \sinh\theta = Z\left(1 + \frac{YZ}{6} + \frac{Y^2Z^2}{120} + \frac{Y^3Z^3}{5040} + \cdots\right) \tag{206}$$

$$C = \frac{\sinh\theta}{Z_s} = Y\left(1 + \frac{YZ}{6} + \frac{Y^2Z^2}{120} + \frac{Y^3Z^3}{5040} + \cdots\right) \tag{207}$$

where $Z = Lz$ and $Y = Ly$ are the inductive impedance and the charging admittance for the entire line. The equivalent circuit of a single transmission line may take either the equivalent $T$ or the equivalent $\pi$ form. The constants of the equivalent $\pi$ circuit are

$$Z_{eq} = Z\frac{\sinh\theta}{\theta} = Z\left(1 + \frac{YZ}{6} + \frac{Y^2Z^2}{120} + \frac{Y^3Z^3}{5040} + \cdots\right) \tag{208}$$

$$Y_{eq} = Y\frac{\tanh\frac{\theta}{2}}{\theta/2} = Y\left(\frac{1}{2} - \frac{YZ}{24} + \frac{Y^2Z^2}{240} + \cdots\right) \tag{209}$$

**Illustration.**—Consider *one* of the two parallel circuits of the long line described in Fig. 50. Assume that this circuit receives power from an infinite source having balanced potentials with the neutral point grounded. Assume also that a simultaneous short circuit occurs between the three line conductors at the receiving end ($b$) which causes two of the circuit breakers at the sending end ($a$) to open, as indicated in Fig. 51.

We shall first compute the impedances, the admittances and the general circuit constants for the different sequences.

*Impedance to Positive and Negative Sequence.*—By equation (114), page 116,

$$Z_1 = Z_2 = 0.148 + j0.826 \text{ vector ohm per mile per circuit}$$

*Impedance to Zero Sequence.*—Assuming earth conductivity $\lambda = 10^{-13}$

$$D = \frac{0.00738}{\sqrt{f\lambda}} = 3010 \text{ ft.}$$

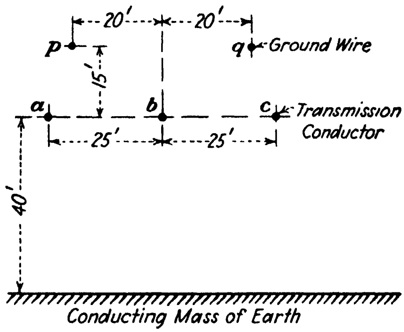



FIG. 50.—220 kv, 240 miles, 2 parallel circuits on separate towers. The arrangement of *one* circuit with its ground wires is indicated. The mutual impedance between the two circuits is neglected. A.C.S.R. 715,500 cir. mils, 54 strands al, 7 strands steel. Completely transposed every 30 miles. Average height above ground 40 ft. Frequency 60 cycles. Two galvanized steel ground wires, 0.5 inch diam.,

$ab = bc = 25'$, $ac = 50'$, $pq = 40'$, $pb = qb = 25'$
$ap = qc = 15.8'$, $pc = qa = 47.4'$
Self G.M.D. of line conductors $d = 0.0349$ ft.
Self G.M.D. of steel ground wires $d_p = 10^{-8}$ ft.
Resistance line conductors $r = 0.148$ ohm per mile at 60 cycles
Resistance steel ground wires $r_p = 5.0$ ohms per mile at 60 cycles.

By equation (130), page 130,

$$Z_{aa0} = 0.238 + j1.38 \text{ vector ohms per mile per circuit}$$

By equation (131), page 130,

$$Z_{bc0} = 0.090 + j0.554 \text{ vector ohms per mile per circuit}$$

By equation (127), page 126,

$$Z_{pp} = 5.09 + j3.21 \text{ vector ohms per mile per circuit}$$

By equation (135), page 134,

$$Z_{ap0} = 0.090 + j0.574 \text{ vector ohms per mile per circuit}$$

By equation (128), page 127,

$$Z_{pq} = 0.090 + j0.524 \text{ vector ohms per mile per circuit}$$

By equation (144), page 139,

$$Z_0 = 0.608 + j2.23 \text{ vector ohms per mile per circuit.}$$

*Charging Admittances to Positive and Negative Sequence.*—By equation (173), page 160 (radius of line conductor $r = 0.0432$ ft.),

$$Y_1 = Y_2 = j5.11 \times 10^{-6} \text{ vector mho per mile per circuit}$$

*Charging Admittance to Zero Sequence.*—With two ground wires the second term in equation (188), page 173, becomes

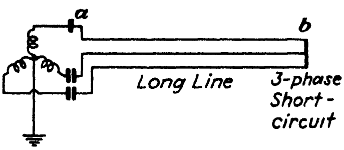



FIG. 51.

$$-\frac{2}{3}\,\frac{\log^2\left(\dfrac{ap'\cdot bp'\cdot cp'}{ap\cdot bp\cdot cp}\right)}{\log\dfrac{2pg\cdot pq'}{r_p\cdot pq}}$$

$$\sqrt[3]{2ag\cdot 2bg\cdot 2cg} = 80$$

$$\frac{ap'\cdot bp'\cdot cp'}{ap\cdot bp\cdot cp} = 51.7 \qquad \sqrt[3]{(ab'\cdot bc'\cdot ca')^2} = 7600.$$

$$\frac{2pg\cdot pq'}{r_p\cdot pq} = 15450 \qquad \sqrt[3]{ab\cdot bc\cdot ca} = 31.5$$

Radius of ground wire $r_p = 0.25$ in. $= 0.0208$ ft.
By equation (188), page 173, modified for two ground wires,

$$Y_0 = j3.97 \times 10^{-6} \text{ vector mho per mile per circuit.}$$

Angle of line (per circuit) for positive and negative sequence

$$\theta_1 = \theta_2 = 0.497\underline{/84.9^\circ}$$

$$\frac{\sinh\theta_1}{\theta_1} = \frac{\sinh\theta_2}{\theta_2} = 0.960\underline{/0.42^\circ}$$

$$\frac{\tanh\frac{\theta_1}{2}}{\frac{\theta_1}{2}} = \frac{\tanh\frac{\theta_2}{2}}{\frac{\theta_2}{2}} = 1.021\underline{/-0.2^\circ}$$

Angle of line (per circuit) for zero sequence

$$\theta_0 = 0.728\underline{/82.4^\circ}$$

$$\frac{\sinh \theta_0}{\theta_0} = 0.918\underline{/1.35^\circ}$$

$$\frac{\tanh \dfrac{\theta_0}{2}}{\dfrac{\theta_0}{2}} = 1.046\underline{/-0.7^\circ}$$

*General Circuit Constants*
For single circuit line.

$$A_1 = A_2 = D_1 = D_2 = \cosh \theta_1 = 0.881\underline{/1.4^\circ}$$
$$A_0 = D_0 = \cosh \theta_0 = 0.758\underline{/4.8^\circ}$$
$$B_1 = B_2 = Z_s \sinh \theta = Z_1 \frac{\sinh \theta_1}{\theta_1}$$
$$= 193.3\underline{/80.2^\circ} \text{ vector ohms per circuit}$$
$$B_0 = Z_0 \frac{\sinh \theta_0}{\theta_0}$$
$$= 510\underline{/76.15^\circ} \text{ vector ohms per circuit}$$
$$C_1 = C_2 = \frac{\sinh \theta_1}{Z_s} = Y_1 \frac{\sinh \theta_1}{\theta_1}$$
$$= 1.18 \times 10^{-3}\underline{/90.4^\circ} \text{ vector mho per circuit}$$
$$C_0 = Y_0 \frac{\sinh \theta_0}{\theta_0}$$
$$= 0.875 \times 10^{-3}\underline{/91.35^\circ} \text{ vector mho per circuit}$$

For double circuit line (2 lines in parallel).
The $A$ and $D$ constants are the same as for the single circuit line.

$$B_1 = B_2 = 96.7\underline{/80.2^\circ} \text{ vector ohms}$$
$$B_0 = 255\underline{/76.15^\circ} \text{ vector ohms}$$
$$C_1 = C_2 = 2.35 \times 10^{-3}\underline{/90.4^\circ} \text{ vector mho}$$
$$C_0 = 1.75 \times 10^{-3}\underline{/91.35^\circ} \text{ vector mho}$$

*For Positive- and Negative-sequence Equivalent π Circuit.*—By equation (208), page 180,

$$Z_{eq} = 193.3/80.2^\circ \text{ vector ohms per circuit}$$

For double circuit

$$Z_{eq} = 96.7\underline{/80.2^\circ} \text{ vector ohms}$$

By equation (209), page 180,

$$Y_{eq} = 1.25 \times 10^{-3}\underline{/89.2^\circ} \text{ vector mho per circuit}$$

For double circuit

$$Y_{eq} = 2.50 \times 10^{-3}\underline{/89.2^\circ} \text{ vector mho}$$

*For Zero-sequence Equivalent* $\pi$ *Circuit.*- By equation (208), page 180,

$$Z_{eq} = 510\underline{/76.2^\circ} \text{ vector ohms per circuit}$$

For double circuit

$$Z_{eq} = 255\underline{/76.2^\circ} \text{ vector ohms}$$

By equation (209), page 180,

$$Y_{eq} = 0.994 \times 10^{-3}\underline{/89.3^\circ} \text{ vector mho per circuit}$$

For double circuit

$$Y_{eq} = 1.99 \times 10^{-3}\underline{/89.3^\circ} \text{ vector mho}$$

The constants of the equivalent $\pi$ circuit are used in the illustration at the end of Chapter X.

We shall now proceed with the analysis of the conditions occasioned by the short-circuit at the end of the line.

Write the general circuit equations for positive, negative, and zero sequence for the phase which is still connected to the source. Positive sequence:

$$E_{a1} = A_1E_{b1} + B_1I_{b1}$$
$$I_{a1} = C_1E_{b1} + D_1I_{b1}$$

Negative sequence:

$$E_{a2} = A_2E_{b2} + B_2I_{b2}$$
$$I_{a2} = C_2E_{b2} + D_2I_{b2}$$

Zero sequence:

$$E_{a0} = A_0E_{b0} + B_0I_{b0}$$
$$I_{a0} = C_0E_{b0} + D_0I_{b0}$$

Owing to the short-circuit or fault at the receiving end of the circuit, the line potentials at the receiving end ($b$) are zero. Hence, $E_{b1} = 0$ and $E_{b2} = 0$. Since the line conductors are not

grounded at the fault, $E_{b0}$ *is not zero*, but the zero sequence component, $I_{b0}$, of the line currents at the fault *is zero*. At the sending end ($a$) of the line, two of the line currents are zero.
Hence

$$I_{a0} = I_{a1} = I_{a2} = \frac{I_a}{3}$$

Also

$$E_{a0} + E_{a1} + E_{a2} = V_a$$

If the relations among the symmetrical components of the potential and current at the two ends of the line are substituted in the general circuit equations
Positive sequence:

$$E_{a1} = 0 + B_1 I_{b1}$$
$$I_{a1} = 0 + D_1 I_{b1}$$

Negative sequence:

$$E_{a2} = 0 + B_2 I_{b2}$$
$$I_{a2} = 0 + D_2 I_{b2}$$

Zero sequence:

$$E_{a0} = A_0 E_{b0} + 0$$
$$I_{a0} = C_0 E_{b0} + 0$$

But

$$B_1 = B_2, \qquad D_1 = D_2$$

and

$$I_{a1} = I_{a2} = I_{a0}$$

Therefore

$$I_{b1} = I_{b2}, \qquad E_{a1} = E_{a2}, \qquad \text{and} \qquad E_{a0} = \frac{A_0}{C_0}\frac{D_1}{B_1}E_{a1}$$

Since

$$E_{a0} + E_{a1} + E_{a2} = V_2$$
$$\left(\frac{A_0 D_1}{C_0 B_1} + 1 + 1\right)E_{a1} = V_a$$

Therefore

$$E_{a1} = \frac{V_a}{2 + \dfrac{A_0 D_1}{C_0 B_1}} = E_{a2}$$

$$E_{a0} = \frac{V_a}{2 + \dfrac{A_0 D_1}{C_0 B_1}} \frac{A_0 D_1}{C_0 B_1}$$

On substituting the numerical values of the general circuit constants, $\frac{A_0D_1}{C_0B_1} = 3.95\underline{/-165.35°}$ and consequently

$$E_{a1} = E_{a2} = (-0.422 + j0.232)V_a$$
$$E_{a0} = (1.84 - j0.464)V_a$$

The potential across one of the open circuit breakers is

$$V_a\underline{/-120°} - (E_{a1}\underline{/-120°} + E_{a2}\underline{/120°} + E_{a0})$$

The potential across the other circuit breaker is

$$V_a\underline{/120°} - (E_{a1}\underline{/120°} + E_{a2}\underline{/-120°} + E_{a0})$$

On substituting the values of $E_{a0}$, $E_{a1}$, and $E_{a2}$ in terms of $V_a$ in these expressions, the numerical values of the potentials across the circuit breakers are $2.77\,|V_a|$ and $3.18\,|V_a|$.

With a line so short that the charging currents could be neglected, the potentials across the open circuit breakers both would be $1.73\,|V_a|$.

# CHAPTER V

## TRANSFORMERS

**70. Two-winding Transformers.**—Single-phase power transformers usually have either two or three independent windings on the same magnetic core. We shall first consider the case in which there are but two windings, a primary and a secondary.

At the rated frequency, since all capacitive effects in the windings may be neglected, the terminal potential of the primary winding (*a*) consists of but two components, one due to resistance and the other due to the rate of change of the total flux linkages. That is

$$v_a = i_a r_a + \frac{d\lambda_a}{dt} 10^{-8} \tag{210}$$

where $\lambda_a$ represents the total flux linkages with the primary winding. Similarly, the terminal potential of the secondary winding (*b*) is

$$v_b = i_b r_b + \frac{d\lambda_b}{dt} 10^{-8} \tag{211}$$

where $\lambda_b$ represents the total flux linkages with the secondary winding. The flux linkages $\lambda_a$ and $\lambda_b$ are produced by the combined action of the currents in both primary and secondary windings. Furthermore, these flux linkages are due almost entirely to a flux that is localized within the magnetic core. Herein lies the difficulty in the solution of these equations since, on account of the magnetic characteristics of the core, this flux is not proportional to the currents which produce it. This difficulty may be postponed by the following procedure: The portions of the total flux linkages, $\lambda_a$ and $\lambda_b$, which are produced by the flux that is confined wholly to the magnetic core are $A\phi$ and $B\phi$, where $A$ and $B$ are the numbers of turns in the primary and secondary windings and $\phi$ is the core flux. Note that this core flux links all the turns of both windings, whereas the total flux producing $\lambda_a$, for example, does not link

all the primary turns. The remainder $(\lambda_a - A\phi)$ of the primary flux linkages, being due to flux that is either wholly or partly in nonmagnetic material, is substantially *proportional* to the currents producing these flux linkages. Consequently the potential $\frac{d}{dt}(\lambda_a - A\phi)$ due to their time variation may be written $L_a\frac{di_a}{dt} + M_{ab}\frac{di_b}{dt}$, where $L_a$ and $M_{ab}$ are *constant* coefficients of self- and mutual induction. The potential $A\frac{d\phi}{dt}10^{-8}$, due to the core flux is $e_a'$. If the terminal potentials, $v_a$ and $v_b$, are sinusoidal and if the harmonics in the currents, due to the magnetic properties of the core, are ignored, the original equations (210) and (211) can be written in their vector forms. This neglect of the harmonics in the currents is the second step toward reducing the equations (210) and (211) to a workable form. It is amply justified in the problems we shall discuss, not only on the ground of expediency but because *near full load* the largest harmonic, the third, is not more than about 2 per cent of the fundamental current. The vector equations are

$$V_a = I_a r_a + jI_aX_{aa} + jI_bX_{ab} + E_a' \qquad (212)$$

$$V_b = I_b r_b + jI_bX_{bb} + jI_aX_{ab} + E_b' \qquad (213)$$

The potentials $E_a'$ and $E_b'$ are produced by the *resultant* core flux. They are proportional to the primary and secondary turns, or $E_a'/E_b' = A/B$. The ratio of turns, $A/B$, is the ratio of transformation of the transformer. The *resultant* core flux is produced by the *combined* ampere-turns $(AI_a + BI_b)$ of the primary and secondary windings. The value of the combined ampere-turns is usually assumed to be negligibly small as compared with either the primary or the secondary ampere-turns, or $AI_a$ is assumed to equal $-BI_b$. That is, the primary and secondary ampere-turns are assumed to be equal in magnitude and opposite in time-phase. This is the last step toward bringing equations (210) and (211) to a workable form. It is generally considered that this last assumption is fully justified in all the cases we shall discuss.

It is important to observe that, in writing these equations, we have adopted the convention that the primary and secondary currents are both positive when flowing right-handed, or clock-

wise, about the magnetic core with respect to an assigned direction in which the core flux is assumed to be positive. The positive direction of the potentials due to the core flux is the same as that of the currents. If it is assumed that $AI_a = -BI_b$, equations (212) and (213) may be written:

$$V_a = I_a\left(r_a + j(X_{aa} - \frac{A}{B}X_{ab})\right) + E_a' \qquad (214)$$

$$V_b = I_b\left(r_b + j(X_{bb} - \frac{B}{A}X_{ab})\right) + E_b' \qquad (215)$$

The quantity $\left(X_{aa} - \frac{A}{B}X_{ab}\right)$ is the *leakage reactance* of the primary winding with respect to the secondary winding. Similarly, $\left(X_{bb} - \frac{B}{A}X_{ab}\right)$ is the *leakage reactance* of the secondary winding with respect to the primary winding. The assumptions that have been made in the derivation of these equations should be carefully considered. The notation may be much simplified as follows

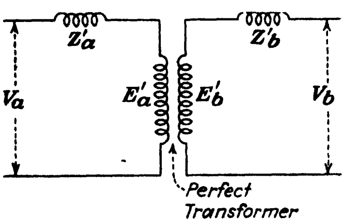


FIG. 52.

$$V_a = I_aZ_a' + E_a' \qquad (216)$$

$$V_b = I_bZ_b' + E_b' \qquad (217)$$

where $Z_a'$ and $Z_b'$ are the *leakage impedances* of the primary and secondary windings. These equations apply to the equivalent circuit of Fig. 52. For our purpose it is often more convenient to modify this equivalent circuit so that the leakage impedances appear on but one side. This can be accomplished readily since we have assumed $AI_a = -BI_b$. Equations (216) and (217) may be written

$$V_a = I_a\left(Z_a' + \frac{A^2}{B^2}Z_b'\right) + E_a$$

or

$$V_a = I_aZ_a + E_a \qquad (218)$$

and

$$V_b = E_b \qquad (219)$$

where $Z_a$ is the so-called equivalent or transient impedance of the transformer. That is, the leakage impedance of one winding

with respect to the other can be transferred either wholly or in part to the other winding, if the part that is transferred is first multiplied by the square of the ratio of transformation. Thus equations (218) and (219) can also be written

$$V_a = E_a \tag{220}$$
$$V_b = I_bZ_b + E_b \tag{221}$$

The values of $E_a$ and $E_b$ are *not* the same in equations (218) and (219) as in equations (220) and (221), but the ratio $E_a/E_b$ in

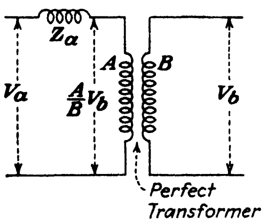


FIG. 53.

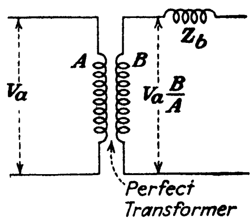


FIG. 54.

equations (218) and (219) *is* the same as in equations (220) and (221), and is

$$\frac{E_a}{E_b} = \frac{A}{B}$$

The equivalent circuits are in Figs. 53 and 54, in which

$$\frac{Z_a}{Z_b} = \frac{A^2}{B^2}.$$

It is important to note that the *actual* transformer is equivalent to an impedance in series with a *perfect* transformer, which has neither resistance nor leakage reactance, which has the same ratio of transformation as the actual transformer, but which requires no exciting current. Note, however, that the potentials $E_a$ and $E_b$ induced in this perfect transformer are no longer equal to those induced in the actual transformer by the core flux. This equivalence of the actual transformer and the perfect transformer with series impedance is a useful conception. The equivalent impedance $Z_a$ on the primary side is determined by measuring the potential, current, and power on the primary side while the secondary winding $b$ is short-circuited. The

equivalent impedance $Z_b$ on the secondary side is measured in a similar manner.

**71. Autotransformer.**—If the primary ($a$) winding and the secondary ($b$) winding of the usual transformer are connected in series, the combination is described as an autotransformer. Whereas, the ratio of transformation as a regular transformer is $A/B$, the ratio as an autotransformer is $(A + B)/B$. If the two windings of Fig. 53 are connected in series, we have the autotransformer of Fig. 55. The primary potential is no longer $V_a$ but $V_a + V_b$. The direction of the currents in the windings is shown to conform with the equation $AI_a + BI_b = 0$. The impedance $Z_a$ can be transferred to the secondary side by multiplying it by the square of the ratio of transformation. Thus

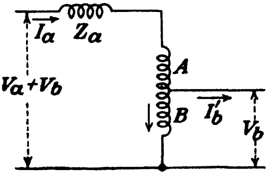



FIG. 55.

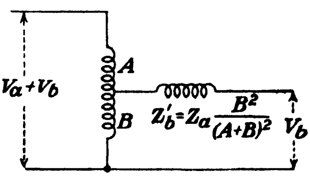



FIG. 56

we have Fig. 56, in which the equivalent transformer impedance is

$$Z_b' = \frac{B^2}{(A + B)^2} Z_a.$$

The impedance $Z_a$ is measured by applying potential to the high-tension terminals and short-circuiting the low-tension terminals, and $Z_b'$ is measured by applying potential to the low-tension terminals and short-circuiting the high-tension terminals. It is written $Z_b'$ to distinguish it from $Z_b$, which is the equivalent impedance of the usual transformer referred to the $b$ side. Note also in Fig. 55 that the low-tension current is $I_b'$, the prime being used to distinguish this current from the current $I_b$ in the $b$ winding.

**72. Three-phase Arrangements of Two-winding Transformers.** One of the important assumptions made in the analysis is that the vector sum of the primary and secondary ampere-turns in each transformer is zero. When transformers, *having the same ratio of transformation*, $A/B$, are used in 3-phase combinations, it follows from this assumption that

$$\left.\begin{aligned} AI_{a0} + BI_{b0} &= 0 \\ AI_{a1} + BI_{b1} &= 0 \\ AI_{a2} + BI_{b2} &= 0 \end{aligned}\right\} \qquad (222)$$

where the subscripts $a$ and $b$ indicate the primary and secondary. Thus, if there is current of one sequence in the primary winding, there is a corresponding current of the same sequence in the secondary winding. Also, if there is no zero-sequence current in one winding, there can be no zero-sequence current in the other winding. The ratio, $-A/B$, between the corresponding symmetrical components of the primary and secondary currents equals the ratio $-A/B$, between the primary and secondary currents themselves. The relations in equations (222) are true only when all the transformers of the group have the same ratio of transformation.

The relation between the primary and secondary terminal-potentials of a single transformer is

$$V_a - I_aZ_a = \frac{A}{B}V_b$$

If this transformer is *one* of a group of *three* transformers, its primary and secondary terminal potentials, its primary current, and its equivalent impedance can all be resolved into zero-, positive-, and negative-sequence components. Hence, when this transformer is one of a 3-phase group of transformers, all three of which have the *same ratio of transformation*, $A/B$,

$$V_{a0} - (I_{a0}Z_{a0} + I_{a1}Z_{a2} + I_{a2}Z_{a1}) = \frac{A}{B}V_{b0} \qquad (223)$$

$$V_{a1} - (I_{a0}Z_{a1} + I_{a1}Z_{a0} + I_{a2}Z_{a2}) = \frac{A}{B}V_{b1} \qquad (224)$$

$$V_{a2} - (I_{a0}Z_{a2} + I_{a1}Z_{a1} + I_{a2}Z_{a0}) = \frac{A}{B}V_{b2} \qquad (225)$$

where the subscripts $a$ and $b$ indicate the primary and secondary. If the ratios of transformation $A/B$ of all the transformers in the 3-phase group are *not* the same, the ratios should also be resolved into symmetrical components. The right-hand sides of equations (223), (224), (225) may then be written[1]

[1] Electrical Engineering, *Am. Inst. Elec. Eng.*, March, 1935, p. 350.

$$\left(\frac{A}{B}\right)_0 V_{b0} + \left(\frac{A}{B}\right)_1 V_{b2} + \left(\frac{A}{B}\right)_2 V_{b1}$$
$$\left(\frac{A}{B}\right)_0 V_{b1} + \left(\frac{A}{B}\right)_1 V_{b0} + \left(\frac{A}{B}\right)_2 V_{b2}$$
$$\left(\frac{A}{B}\right)_0 V_{b2} + \left(\frac{A}{B}\right)_1 V_{b1} + \left(\frac{A}{B}\right)_2 V_{b0}$$

where $(A/B)_0$, $(A/B)_1$, and $(A/B)_2$ are the zero-, positive-, and negative-sequence components of the ratio of transformation of the first of the group of the three transformers.

**73. *Y-Y* Connection.**—When transformers are connected *Y-Y*, the neutral on the primary side is usually connected to the neutral of the source of power. Without a neutral connection there

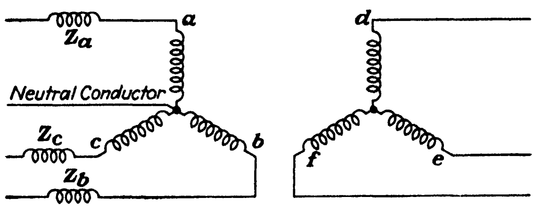


FIG. 57

can be no third harmonic in the exciting current. The *absence* of third harmonic in the exciting current *introduces* a third harmonic in the core flux, so that the peak value of the emf induced by the core flux may become dangerously high. It is customary to provide each transformer of such a group with a *third* winding, or tertiary. The tertiaries are connected in Δ and thus form a path for the third harmonic in the exciting current. Three-winding transformers are discussed subsequently.

When transformers are connected *Y-Y*, it is convenient to replace the actual transformers by their equivalent series impedances and perfect transformers connected *Y-Y*. The impedances can be transferred from one side to the other of the perfect transformers by multiplying them by the square of the ratio of the line potentials. The impedances can then be treated as parts of the line impedances on the side to which they are transferred. The replacement of the actual transformers by perfect transformers with equivalent series impedances is indicated in Fig. 57 in which $Z_a$, $Z_b$, and $Z_c$ are the equivalent

impedances of the actual transformers on the primary side. The subscripts $a$, $b$, and $c$ now indicate the *three primary phases*. If a number of $Y$-$Y$ banks are in parallel, $Z_a$, $Z_b$, $Z_c$ are the resulting impedances of the paralleled transformers in each of the three branches of the $Y$, and are computed as if the equivalent impedances themselves were in parallel. This implies that the parallel connection is not complete until the primary neutrals of *all* the banks of transformers are joined and are connected to the source of power; also, the secondary neutrals are joined but are not necessarily connected to the source of power. The joining of the secondary neutrals is not essential, however, if each bank consists of *three identical* transformers.

In general, if the equivalent impedances $Z_a$, $Z_b$, and $Z_c$ are not *alike* and if the symmetrical components of $Z_a$ are $Z_{a0}$, $Z_{a1}$ and $Z_{a2}$, the solution for the symmetrical components $I_{a0}$, $I_{a1}$, and $I_{a2}$ of the primary currents is obtained from equations (223), (224), and (225), page 191. If all the transformers are identical and, in the case of paralleled banks, if none has been removed, the impedances $Z_a$, $Z_b$, and $Z_c$ are the same and have neither positive- nor negative-sequence components. In this case equations (223), (224), and (225) reduce to

$$V_{a0} - I_{a0}Z_{a0} = \frac{A}{B}V_{b0} \tag{226}$$

$$V_{a1} - I_{a1}Z_{a0} = \frac{A}{B}V_{b1} \tag{227}$$

$$V_{a2} - I_{a2}Z_{a0} = \frac{A}{B}V_{b2} \tag{228}$$

That is to say, the transformers have the same impedances, $Z_{a0}$, to zero-, positive-, and negative-sequence currents. Note that zero-sequence current can flow through the transformers only when the neutrals of *both* primary and secondary are either grounded or connected to the neutral conductor of the system.

**74. Δ-Δ Connection.**—When the transformers are connected Δ-Δ as in Fig. 58, it is usually best to replace the actual transformers with equivalent series impedances and perfect transformers in $Y$-$Y$ as in Fig. 57. The general relations among the impedances in Fig. 57 and the equivalent impedances of the actual transformers are

$$\left.\begin{aligned} Z_a &= \frac{Z_{ca}\,Z_{ab}}{Z_{ab} + Z_{bc} + Z_{ca}} \\ Z_b &= \frac{Z_{ab}\,Z_{bc}}{Z_{ab} + Z_{bc} + Z_{ca}} \\ Z_c &= \frac{Z_{bc}\,Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}} \end{aligned}\right\} \tag{229}$$

where $Z_{ab}$, $Z_{bc}$, $Z_{ca}$ are the equivalent impedances of the actual transformers in Fig. 58. These impedances may have different *vector* values, either because the transformers are not identical or because one or more transformers have been removed from a group of several banks of Δ-Δ transformers in parallel. In the latter case, the impedances $Z_{ab}$, $Z_{bc}$, $Z_{ca}$ are the resulting impedances of the paralleled transformers on each of the three sides of the delta. If the transformers are identical, $Z_a$, $Z_b$, and $Z_c$ are each one-third of the actual equivalent impedances. Although the currents in the actual transformers may contain a zero-sequence component, the *line* currents on both the primary and secondary sides can contain *only* positive-sequence and negative-sequence components. That is, the Δ-Δ connection completely blocks the flow of zero-sequence current through the transformers.

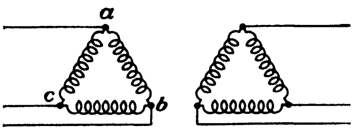


FIG. 58.

If the transformers in the Δ-Δ group are not identical, zero-sequence currents may flow in the deltas. Under these conditions it cannot be assumed that the Δ-Δ group is replaced by a $Y$-$Y$ group when computing the zero-sequence current. Consequently, the replacement of a Δ-Δ group of transformers by an equivalent $Y$-$Y$ group is limited, in general, to positive and negative sequence *only*. When the transformers are connected Δ-Δ, the zero-sequence components of both primary and secondary terminal potentials are zero and equation (223), page 191, reduces to

$$I_{a0}Z_{a0} + I_{a1}Z_{a2} + I_{a2}Z_{a1} = 0$$

where $I_{a0}$, $I_{a1}$, and $I_{a2}$ are the symmetrical components of the current in one of the actual transformers and $Z_{a0}$, $Z_{a1}$, and $Z_{a2}$ are the symmetrical components of the equivalent impedance of one of the actual transformers of the Δ-Δ group. If the

transformers of the group are identical, both $Z_{a1}$ and $Z_{a2}$ are zero and hence there can be no zero-sequence current in the deltas.

**Illustration.**—A group of six equal transformers is connected in two Δ-Δ banks in parallel. The equivalent impedance of one transformer, on the primary side, is $Z$. It becomes necessary to remove one of the transformers on phase $b$-$c$, leaving two transformers on each of the phases $a$-$b$ and $c$-$a$. Thus,

$$Z_{ab} = \frac{Z}{2}, \qquad Z_{bc} = Z, \qquad Z_{ca} = \frac{Z}{2}$$

Replace the actual transformers by perfect transformers with series impedances. If the series impedances are then transferred to a position in series with the line conductors on the primary side, their values by equations (229) are

$$Z_a = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} + 1 + \frac{1}{2}} Z = \frac{1}{8} Z$$
$$Z_b = \frac{\frac{1}{2} \times 1}{\frac{1}{2} + 1 + \frac{1}{2}} Z = \frac{1}{4} Z$$
$$Z_c = \frac{1 \times \frac{1}{2}}{\frac{1}{2} + 1 + \frac{1}{2}} Z = \frac{1}{4} Z$$

The symmetrical components of $Z_a$ are

$$Z_0 = \tfrac{5}{24} Z$$
$$Z_1 = -\tfrac{1}{24} Z$$
$$Z_2 = -\tfrac{1}{24} Z$$

The load connected to the secondary of the transformers is balanced and has an impedance $Z_{M1}$ to positive-sequence currents and an impedance of $Z_{M2}$ to negative-sequence currents. The values of both $Z_{M1}$ and $Z_{M2}$ are per $Y$-phase referred to the primary side. As shown in Chap. VI, this load has the characteristic constants of an induction motor. If the potentials applied to the transformers on the primary side are balanced, we can write for phase $a$

$$V_1 = V = I_1 Z_0 + I_2 Z_2 + I_1 Z_{M1}$$
$$V_2 = 0 = I_2 Z_0 + I_1 Z_1 + I_2 Z_{M2}$$

where $V$ is the line-to-neutral potential on the primary side and $I_1$ and $I_2$ are the positive- and negative-sequence components

of the line currents on the primary side. There is no zero-sequence line current.

Let

$$
\begin{aligned}
V &= 1.0 \\
Z &= 0.06 + j0.36 \\
Z_{M1} &= 0.90 + j0.40 \\
Z_{M2} &= 0.05 + j0.20
\end{aligned}
$$

The values of the impedance constants are based on the following assumptions. The rated current of the motor is 1.0 unit per $Y$-phase. With balanced potentials of the rated value, the motor current is

$$\frac{1.0}{0.90 + j0.40} = 0.927 - j0.412 \text{ unit per } Y \text{ phase}$$

The total rated volt-ampere input of the motor is six times the rated volt-ampere input of a single transformer. Each transformer has an equivalent impedance of $0.01 + j0.06$ unit based on its own rated volt-ampere input and rated potential. Since the transformers are connected in $\Delta$, their rated potentials are $\sqrt{3}$ units, and their rated currents are each $1/2\sqrt{3}$ unit. Thus, the equivalent impedance of one transformer, based on the potential to neutral and the rated line current of the induction motor, is $2(\sqrt{3})^2(0.1 + j0.06) = 0\ 06 + j0.36$.

The zero-, positive-, and negative-sequence components of $Z_a$ are

$$
\begin{aligned}
Z_0 &= 0.0125 + j0.075 \\
Z_1 = Z_2 &= -0.0025 - j0.015
\end{aligned}
$$

Therefore

$$
\begin{aligned}
1.0 &= I_1(0.9125 + j0.475) - I_2(0.0025 + j0.015) \\
0 &= I_2(0.0625 + j0\ 275) - I_1(0.0025 + j0.015)
\end{aligned}
$$

That is, the positive- and negative-sequence components of the line current in the $a$ conductor are

$$
\begin{aligned}
I_1 &= 0.863 - j0.449 \\
I_2 &= 0.048 - j0.022
\end{aligned}
$$

The positive- and negative-sequence components of the current in the transformer connected between lines $b$-$c$ are

$$I_{bc1} = -j\frac{1}{\sqrt{3}}I_{a1} = -0.259 - j0.498$$

$$I_{bc2} = j\frac{1}{\sqrt{3}}I_{a2} = 0.0127 + j0.0277$$

In addition to these positive- and negative-sequence components of the current, there is a zero-sequence component in the transformer current, which is determined by the following equation

$$V_{a0} - (I_0Z_0 + I_1Z_2 + I_2Z_1) = V_{b0} \qquad (A)$$

where $V_{a0}$ and $V_{b0}$ are the zero-sequence components of the transformer terminal potentials on the primary side ($a$) and on the secondary side ($b$), referred to the primary; $I_0$, $I_1$, and $I_2$ are now the zero-, positive-, and negative-sequence components of the *transformer* currents on the primary side; and $Z_0$, $Z_1$, and $Z_2$ are the symmetrical components of the *transformer* equivalent impedances referred to the primary. Since the transformers are connected in Δ on both the primary and secondary sides, $V_{a0} = 0$ and $V_{b0} = 0$. If equation ($A$) is written for the transformer $b$-$c$, the currents $I_1$ and $I_2$ [equation ($A$)] are $I_{bc1}$ and $I_{bc2}$, the impedances $Z_0$, $Z_1$, and $Z_2$ [equation ($A$)] are $\frac{2}{3}Z$, $\frac{Z}{6}$, and $\frac{Z}{6}$. That is equation ($A$) becomes,

$$\frac{2}{3}ZI_0 + \frac{Z}{6}I_1 + \frac{Z}{6}I_2 = 0$$

From which

$$I_0 = -\tfrac{1}{4}(I_1 + I_2)$$
$$= 0.062 + j0.118$$

Thus, the current in the single transformer which is connected across the line conductors $b$-$c$ is

$$I_{bc} = I_0 + I_{bc1} + I_{bc2}$$
$$= -0.184 - j0.352 \text{ unit}$$

Since the rated current for one transformer is $1/2\sqrt{3}$ unit, the current $I_{bc}$ in the transformer $b$-$c$ is 1.37 times its rated value. The current $I_{ab}$ in one of the two transformers connected across the line conductors $a$-$b$ is

$$I_{ab} = \tfrac{1}{2}(I_0 + I_{bc1}\underline{/120} + I_{bc2}\underline{/-120})$$
$$= 0.32 + j0.059 \text{ unit}$$

This current $I_{ab}$ is 1.13 times its rated value.

**75. Δ-Y Connection.**—When the transformers are connected Δ-Y, as in Fig. 59, an arrangement of perfect transformers with

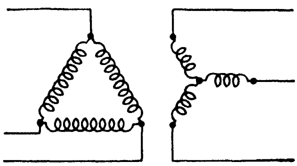



FIG. 59.

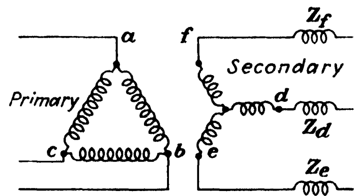



FIG. 60.

series impedances may be used to replace the actual transformers, as in Fig. 60.

The primary windings are $ab$, $bc$, and $ca$ while the corresponding secondary windings are $f$, $d$, and $e$. The vector diagrams showing the relation between the positive-sequence potentials to neutral and the positive-sequence line currents on both primary and

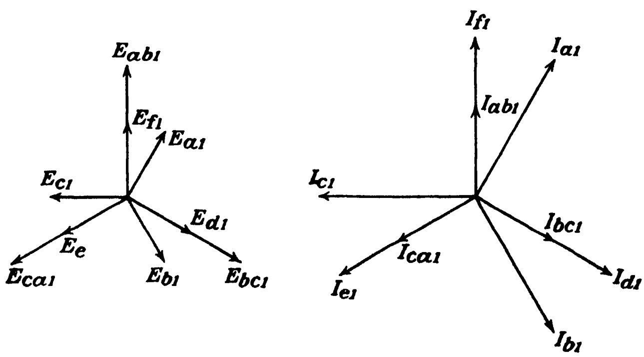



FIG. 61.

secondary sides are in Fig. 61. The corresponding vectors for the negative-sequence potentials and currents are in Fig. 62. Note that the positive-sequence line-to-neutral potential $E_{a1}$ on the Δ side lags by 30 degrees the positive sequence line-to-neutral potential $E_{f1}$ on the $Y$-side, and that the positive-sequence line current $I_{a1}$ on the Δ-side also lags by 30 degrees the positive-sequence line current $I_{f1}$ on the $Y$ side. However, it is more

convenient to compare the potentials and currents for phases *a* and *d* rather than for phases *a* and *f*, since the desired phase relationships are then 90 degrees. Thus $E_{d1}$ lags $E_{a1}$ by 90 degrees, and similarly $I_{d1}$ lags $I_{a1}$ by 90 degrees. The negative-sequence vector diagrams Fig. 62, show that $E_{d2}$ leads $E_{a2}$ by 90 degrees and that $I_{d2}$ leads $I_{a2}$ by 90 degrees. Thus the phase relations of the positive- and negative-sequence line currents on the primary side are not the same as the phase relations of the corresponding currents on the secondary side. In the symmetrical *Y*-*Y* and Δ-Δ systems of transformation, the phase relations of the corresponding sequence components of the

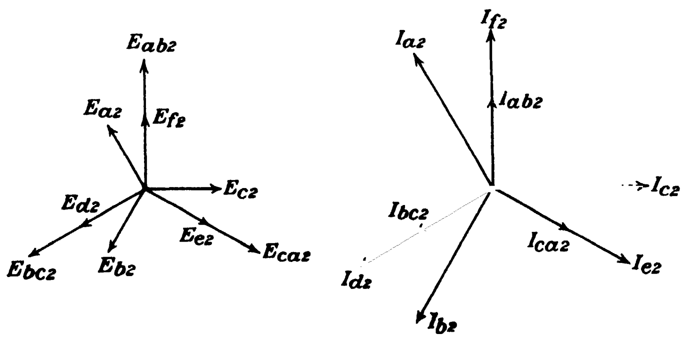



FIG 62

line currents are the same on both sides of the transformers. Although there can be no zero-sequence component in the line current on the Δ side, zero-sequence current may flow in the primary windings on this side if the neutral of the secondaries is grounded. In general, if the impedances of the transformers are not *alike* and if the symmetrical components of the equivalent impedance $Z_d$ on the secondary are $Z_{d0}$, $Z_{d1}$, and $Z_{d2}$, the zero-sequence component of the potential drop through the transformers on the secondary side is

$$V_{d0} = I_{d0}Z_{d0} + I_{d1}Z_{d2} + I_{d2}Z_{d1} \tag{230}$$

Zero-sequence current can flow in the *Y*-connected secondaries only if the neutral is grounded.

**Illustration.**—We will compare the line currents taken by a single-phase load supplied from the secondary of a *Y*-*Y* transformation with the line currents when the same load is supplied

from the secondary of a Δ-$Y$ transformation. The transformers are arranged as in Figs. 57 and 59. Assume that the primary and secondary line potentials are equal. In each case the single-phase load is connected across the secondary terminals *ef* and requires a current $I$.
Hence,

$$I_{d1} + I_{d2} = 0$$

where

$$I_{d1} = j\frac{1}{\sqrt{3}}I$$

A similar relation holds for the components of the *primary* current in the $Y$-$Y$ transformation,

$$I_{a1} + I_{a2} = 0$$

With the Δ-$Y$ transformation, however, the relation for the components of the *primary line* current $I_a$ is

$$-jI_{a1} + jI_{a2} = 0$$

Hence, if the positive- and negative-sequence components of the secondary currents each has a magnitude $|I|/\sqrt{3}$ amperes, the magnitudes of the primary line currents with the $Y$-$Y$ transformation are $|I_a| = 0$ and $|I_b| = |I_c| = |I|$, while with the Δ-$Y$ transformation the line currents are $|I_a| = |2I|$ and $|I_b| = |I_c| = |I|$.

Let us also consider the effect of unsymmetrical line impedances in both the primary and secondary sides. On the primary (Δ) side (Fig. 60) the symmetrical components of these impedances between the points $g$ and $a$ are $Z_{a0}$, $Z_{a1}$, $Z_{a2}$. On the secondary ($Y$) side the symmetrical components of the impedances between the terminal $d$ of the perfect transformer and a point $h$ are $Z_{d0}$, $Z_{d1}$ and $Z_{d2}$. Thus the impedances between $d$ and $h$ include the equivalent impedances of the transformers themselves, which may or may not be equal. We shall assume that the impedances have been adjusted to the condition of a ratio of transformation of unity between the line potentials on the primary and secondary sides. This is accomplished by multiplying the actual impedance values by the square of the actual ratio of line potentials. If $V_g$ and $V_h$ are the potentials to neutral at the points $g$ and $h$ on the primary and secondary sides,

$$-j[V_{g1} - (I_{a1}Z_{a0} + I_{a2}Z_{a2})] - (I_{d1}Z_{d0} + I_{d2}Z_{d2}) = V_{h1}$$
$$j[V_{g2} - (I_{a2}Z_{a0} + I_{a1}Z_{a1})] - (I_{d2}Z_{d0} + I_{d1}Z_{d1}) = V_{h2}$$

If the values of $I_{a1}$ and $I_{a2}$ in terms of $I_{d1}$ and $I_{d2}$ are substituted in these equations,

$$-jV_{g1} - I_{d1}(Z_{a0} + Z_{d0}) - I_{d2}(-Z_{a2} + Z_{d2}) = V_{h1}$$
$$jV_{g2} - I_{d2}(Z_{a0} + Z_{d0}) - I_{d1}(-Z_{a1} + Z_{d1}) = V_{h2}$$

Thus, zero-sequence components of impedance may be transferred from primary to secondary in the Δ-$Y$ transformation just as they may be transferred in the $Y$-$Y$ or in the Δ-Δ transformation whereas positive- and negative-sequence components

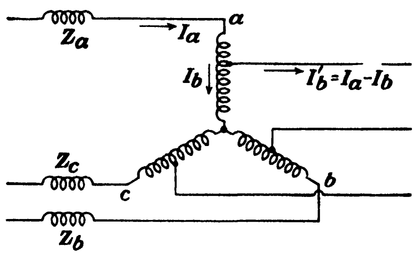



FIG. 63

of impedance may be transferred from primary to secondary only after being rotated 180 degrees.

**76. Autotransformer.**—The last case to consider is that of autotransformers. They may be connected either in $Y$ or in Δ, but in power networks the $Y$ connection would probably be used. The actual transformers may be replaced by perfect transformers with series impedances, as in Fig. 63. If desired, the impedances may be transferred from the high-tension to the low-tension side by multiplying them by the square of the ratio of transformation, $\frac{B^2}{(A + B)^2}$, as in Fig. 56, page 190. This case is essentially the same as the $Y$-$Y$ transformation. Zero-sequence power current can flow only when the neutral of the transformers is grounded and there are also grounds on both the high- and low-tension sides. In this case the transformers on the three phases will undoubtedly be identical and their impedances to zero-, positive-, and negative-sequence currents will also be identical.

In all these cases of two-winding transformers, the impedances of the transformers to positive-, negative-, and zero-sequence current—if there is zero-sequence current—are identical if the transformers on each of the three phases are identical.

**77. Three-winding Transformers.**—By following the same procedure used in analyzing two-winding transformers, we arrive at the following equations when the transformer has three windings.

$$V_a = I_a r_a + jI_a X_{aa} + jI_b X_{ab} + jI_c X_{ac} + E_a' \quad (231)$$
$$V_b = I_b r_b + jI_b X_{bb} + jI_c X_{bc} + jI_a X_{ab} + E_b' \quad (232)$$
$$V_c = I_c r_c + jI_c X_{cc} + jI_a X_{ac} + jI_b X_{bc} + E_c' \quad (233)$$

Note that all the reactances, both self and mutual, are due to flux that is not confined wholly to the magnetic core and that $E_a'$, $E_b'$, and $E_c'$ are the potentials due to the core flux. If the net number of ampere-turns is assumed to be negligibly small,

$$AI_a + BI_b + CI_c = 0 \quad (234)$$

Since the potentials $E_a'$, $E_b'$, and $E_c'$ are due to the flux in the magnetic core,

$$\frac{E_a'}{A} = \frac{E_b'}{B} = \frac{E_c'}{C} \quad (235)$$

It is now our object to reduce each of the equations (231), (232), and (233) to a form which is similar to that of equation (218). As a first step, let us eliminate the tertiary current, $I_c$, from equations (231) and (232) by means of equation (234). As an abbreviation, substitute $Z_{aa}$ for $r_a + jX_{aa}$,

$$V_a = I_a\left(Z_{aa} - j\frac{A}{C}X_{ac}\right) + jI_b\left(X_{ab} - \frac{B}{C}X_{ac}\right) + E_a' \quad (236)$$

$$V_b = I_b\left(Z_{bb} - j\frac{B}{C}X_{bc}\right) + jI_a\left(X_{ab} - \frac{A}{C}X_{bc}\right) + E_b' \quad (237)$$

If each of the three windings has the same number of turns, $E_a'$ and $E_b'$ will be equal and

$$V_a - V_b = I_a(Z_{aa} - jX_{ac} - jX_{ab} + jX_{bc}) - I_b(Z_{bb} - jX_{bc} - jX_{ab} + jX_{ac}) \quad (238)$$

This result suggests that the impedance which should be associated with the primary ($a$) winding is $(Z_{aa} - jX_{ac} - jX_{ab} + jX_{bc})$ and the impedance which should be associated with the secondary ($b$) winding is $(Z_{bb} - jX_{bc} - jX_{ab} + jX_{ac})$. If the equations for the other pairs of windings, $b$-$c$ and $c$-$a$, are treated in a similar way, these suggestions are confirmed and the impedance which should be associated with the tertiary ($c$) winding is

$$(Z_{cc} - jX_{ca} - jX_{bc} + jX_{ab}).$$

With these results as a guide, it will be observed that, if

$$jI_a\frac{A}{B}\left(X_{ab} - \frac{A}{C}X_{bc}\right)$$

is added to and subtracted from equation (236), and if

$$jI_b\frac{B}{A}\left(X_{ab} - \frac{B}{C}X_{ac}\right)$$

is likewise added to and subtracted from equation (237), we may rewrite these equations as follows

$$V_a = \left(Z_{aa} - j\frac{A}{C}X_{ac} - j\frac{A}{B}X_{ab} + j\frac{A^2}{BC}X_{bc}\right)I_a + \left[j\frac{A}{B}\left(X_{ab} - \frac{A}{C}X_{bc}\right)I_a + j\left(X_{ab} - \frac{B}{C}X_{ac}\right)I_b + E_a'\right] \quad (239)$$

$$V_b = \left(Z_{bb} - j\frac{B}{C}X_{bc} - j\frac{B}{A}X_{ab} + j\frac{B^2}{AC}X_{ac}\right)I_b + \left[j\frac{B}{A}\left(X_{ab} - \frac{B}{C}X_{ac}\right)I_b + j\left(X_{ab} - \frac{A}{C}X_{bc}\right)I_a + E_b'\right] \quad (240)$$

By a similar process, equation (233) may be put into the corresponding form

$$V_c = \left(Z_{cc} - j\frac{C}{A}X_{ac} - j\frac{C}{B}X_{bc} + j\frac{C^2}{AB}X_{ab}\right)I_c + \left[j\left(\frac{C}{B}X_{ab} - \frac{A}{B}X_{bc}\right)I_a + j\left(\frac{C}{A}X_{ab} - \frac{B}{A}X_{ac}\right)I_b + E_c'\right] \quad (241)$$

The important fact to note in these equations is that the terms which are in the brackets [ ] are in the ratio of $A:B:C$. Thus, these terms may be represented by three potentials, $E_a$, $E_b$, and $E_c$, of which it is true that

$$\frac{E_a}{A} = \frac{E_b}{B} = \frac{E_c}{C} \tag{242}$$

That is, we may assume that, although they are not due to the actual flux within the magnetic core, these potentials, $E_a$, $E_b$, and $E_c$, are those which would be produced in the windings of a perfect transformer having the same number of turns as the actual transformer. The equations may now be written in the much abbreviated form

$$V_a = I_aZ_a + E_a \tag{243}$$
$$V_b = I_bZ_b + E_b \tag{244}$$
$$V_c = I_cZ_c + E_c \tag{245}$$

where, for example,

$$Z_a = Z_{aa} - j\frac{A}{C}X_{ac} - j\frac{A}{B}X_{ab} + j\frac{A^2}{BC}X_{bc}$$

These equations together with equations (234) and (242), can be used to solve problems involving three-winding transformers when the exciting current can be neglected. Observe carefully the composition of the impedances in these three equations. The resistance of $Z_a$, for example, is the actual resistance of the primary ($a$) winding. The reactances are due to flux which is only in part confined to the actual magnetic core. $X_{aa}$ is the self-reactance of winding $a$. $\frac{A}{C}X_{ac}$ would be the mutual reactance between $a$ and $c$ if the turns in $c$ were made equal to those in $a$. Similarly, $\frac{A}{B}X_{ab}$ would be the mutual reactance between $a$ and $b$ if the turns in $b$ were made equal to those in $a$. Finally, $\frac{A^2}{BC}X_{bc}$ would be the mutual reactance between $b$ and $c$ if the turns in both $b$ and $c$ were made equal to those in $a$. It is interesting to point out that, if the sum of the mutual reactances $\frac{A}{C}X_{ac}$ and $\frac{A}{B}X_{ab}$ is greater than $\frac{A^2}{BC}X_{bc}$, as it may be, the impedance $Z_a$ will be capacitive. It is impossible, however, for more than one of the impedances $Z_a$, $Z_b$, $Z_c$ to be capacitive.

We have now proved that, if the exciting current is neglected, the actual three-winding transformer is equivalent to and may be

replaced by a *perfect* three-winding transformer, having the same turns as the actual transformer, together with an impedance connected in series with each of the three windings (see Fig. 64). This conception of the three-winding transformer is very useful. The values of the impedances are determined from short-circuit data, just as in the case of a two-winding transformer. If the secondary (b) alone is short-circuited and potential $V_a$ is applied to the primary,

$$V_a = I_a Z_a + E_a$$
$$0 = I_b Z_b + E_b$$

The impedance $Z_{ab}$ measured at the primary terminals is

$$Z_{ab} = \frac{V_a}{I_a} = Z_a + \frac{A^2}{B^2} Z_b \qquad (246)$$

Similarly,

$$Z_{ac} = \frac{V_a}{I_a} = Z_a + \frac{A^2}{C^2} Z_c$$

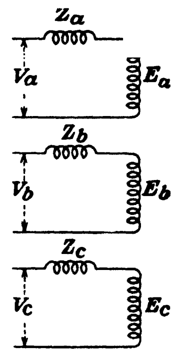


FIG. 64.

The impedance $Z_{bc}$ measured at the secondary terminals when the winding $C$ alone is short-circuited is

$$Z_{bc} = \frac{V_b}{I_b} = Z_b + \frac{B^2}{C^2} Z_c$$

The order of the subscripts is significant, the first subscript indicating the side on which the measurements are made; thus

$$Z_{ab} = \frac{A^2}{B^2} Z_{ba}$$

It is interesting to note that, while $Z_a$ is not the leakage impedance of the primary with respect to the secondary [equation (239) and (243)], $Z_{ab}$ is the equivalent impedance of primary and secondary and thus corresponds to $Z_a$ in equation (218). This is readily proved by substituting in equation (246) the values of $Z_a$ and $Z_b$ indicated in equations (239) and (240). The distinction between the impedances used in analyzing two-winding transformers and three-winding transformers should always be recognized.

The solution of these three equations for $Z_{ab}$, $Z_{bc}$, $Z_{ca}$ gives

$$\left.\begin{aligned} Z_a &= \frac{1}{2}\left(Z_{ab} + Z_{ac} - \frac{A^2}{B^2}Z_{bc}\right) \\ \frac{A^2}{B^2}Z_b &= \frac{1}{2}\left(Z_{ab} - Z_{ac} + \frac{A^2}{B^2}Z_{bc}\right) \\ \frac{A^2}{C^2}Z_c &= \frac{1}{2}\left(-Z_{ab} + Z_{ac} + \frac{A^2}{B^2}Z_{bc}\right) \end{aligned}\right\} \quad (247)$$

Multiplying by the squares of the ratios of turns has the effect of reducing the impedances to the primary side. When the impedances are transferred to the primary side, it should be assumed that the turns in the secondary and tertiary windings are altered to equal those in the primary winding. The potentials across the *perfect* transformer windings are then identical and the windings may be connected in parallel. The circuit of Fig. 64, would then become the circuit shown in Fig. 65, where $E$ is the common potential across the *perfect* transformers. The terminal potential of the primary, for example, is that from $a$ to 0. The circuit of Fig. 65 is the usual representation of the three-winding transformer.

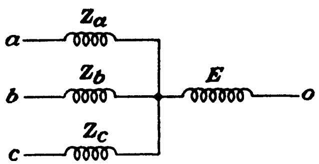


Fig. 65

**78. Three-phase Arrangements of Three-winding Transformers.**—In order to simplify the solution of problems which involve 3-phase arrangements of three-winding transformers, assume that the transformers in each 3-phase group are identical. The advantage of this assumption is apparent when we write the characteristic equations for the transformers in terms of the symmetrical components of potential and current. Consider a 3-phase group of transformers. If the impedances of the primary ($a$) windings are equal, the impedances of the secondary ($b$) windings are equal, and the impedances of the tertiary ($c$) windings are equal, the potential equations (243), (244), and (245) may be written for each of the symmetrical components of potential and current taken separately. For example,

$$V_{a0} = I_{a0}Z_a + E_{a0} \quad (248)$$

$$V_{b0} = I_{b0}Z_b + E_{b0} \quad (249)$$

$$V_{c0} = I_{c0}Z_c + E_{c0} \quad (250)$$

Similar equations may be written for the positive-sequence components and for the negative-sequence components. Furthermore, if the ratios of transformation for all the transformers in the group are equal,

$$AI_{a0} + BI_{b0} + CI_{c0} = 0 \tag{251}$$

$$\frac{E_{a0}}{A} = \frac{E_{b0}}{B} = \frac{E_{c0}}{C} \tag{252}$$

Similar equations may be written for the positive-sequence components and for the negative-sequence components.

It is often convenient to reduce the transformer impedances, $Z_a$, $Z_b$, $Z_c$, to a common base potential. This is accomplished by multiplying the actual impedances by the square of the ratio of the *base* potential to the *rated* potential of the winding. The line-to-neutral potential on either the primary or the secondary side it is usually chosen as the base potential.

The five fundamental equations [(234) and (242) to (245) inclusive] should form the foundation on which to build the analysis of problems involving three-winding transformers. It is customary to represent the actual transformers by equivalent circuits consisting of perfect transformers with series impedances (see Figs. 64 and 65). The perfect transformers are usually omitted from the diagrams although their inclusion would often aid in obtaining a clearer conception of the problem. These equivalent circuits are useful in visualizing the more or less complicated relations described by the equations. However, the equivalent circuits are merely graphical representations of the conditions set forth in the fundamental equations. The equivalent circuits themselves are not fundamental, and any question as to their appropriateness can always be decided by comparing the solution they determine with the solution which results from the equations. Equivalent circuits have been developed to aid in solving a variety of problems involving other types of coupled circuits, such as transmission lines, induction machines, and synchronous machines.

The solutions of problems involving power networks are sometimes obtained by constructing the actual network in miniature (network analyzer) and measuring the potentials and currents in the miniature network at the points where they are

desired. The transformers in the actual power network are represented by their equivalent circuits in the miniature network.

**79. ΔΔΔ Connection.**—As is customary in 3-phase arrangements of three-winding transformers, we will assume that the tertiary windings are connected in Δ in all the following cases. If the primaries and secondaries are also connected in Δ, no zero-sequence current can flow through the transformers, or their impedance to zero-sequence current is infinite. Their impedances to positive- and negative-sequence currents are equal and can be determined from the same equivalent circuit. The equivalent circuit for one phase for both positive and negative sequence is in Fig. 66. It is best to reduce the impedances of the primary, secondary, and tertiary windings to a common base potential, usually the line-to-neutral potential on either the primary or the secondary side. A balanced load, having an impedance $Z$ per phase, when reduced to the common base potential, is connected across the tertiary.

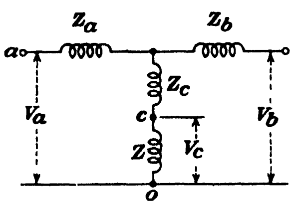


FIG 66

In Fig. 66 the base potential is the primary line-to-neutral potential, or $1/\sqrt{3}$ of the rated primary potential, and hence the impedances $Z_a$, $Z_b$, $Z_c$ in Fig. 66 are each one-third the impedances $Z_a$, $\frac{A^2}{B^2}Z_b$, $\frac{A^2}{C^2}Z_c$, [equations (247), page 206], which are determined from short-circuit measurements on the transformers. The equivalent circuit (Fig. 66) replaces the transformer in both the positive- and negative-sequence networks.

**80. ΔYΔ Connection.**—If the primaries are in Δ and the secondaries in $Y$, the positive- and negative-equivalent circuits are the same as in the ΔΔΔ case. If the secondary neutral is not grounded, the impedance to zero sequence is infinite, since no zero-sequence current can flow through the transformers. If the secondary neutral is grounded, zero-sequence current can flow in both the primary and tertiary windings but there can be no zero-sequence current in the line conductors on the *primary* side. Since the primary and tertiary windings are both connected in Δ, the zero-sequence components of the primary and tertiary terminal potentials are both zero. The zero-sequence equivalent circuit of the transformers is shown in

Fig. 67. The impedance to zero sequence measured at the secondary terminals of the transformers is

$$Z_0 = Z_b + \frac{Z_a Z_c}{Z_a + Z_c}$$

where all the impedances are reduced to a common base potential. If the secondaries are grounded through an impedance $Z_g$, the effect of $Z_g$ may be included by adding $3Z_g$ to the impedance, $Z_0$, of the transformer to zero sequence. The actual value of this grounding impedance is given on the secondary side. The value of $Z_g$ must therefore be referred to the same base potential to which the transformer impedances are referred before being combined with the transformer impedances. Since, in the equivalent circuit, the primary and tertiary impedances are in parallel, the ratio of the zero-sequence currents in the primary and tertiary windings is inversely as their impedances and the current in the tertiary referred to the common base potential is

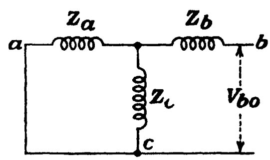



FIG. 67

$$I_{c0} = \frac{Z_a}{Z_a + Z_c} I_{b0}$$

**81. $YY\Delta$ Connection.**—If the primaries and secondaries are both connected in $Y$, the positive- and negative-sequence equivalent circuits are the same as for the $\Delta\Delta\Delta$ connection and the $\Delta Y\Delta$ connection. If the primary alone is grounded, no zero-sequence current flows in the secondary and the impedance to zero sequence measured at the primary terminals is $Z_0 = Z_a + Z_c$, just as it would be with a two-winding transformer. If the secondary alone is grounded, the impedance to zero sequence measured at the secondary terminals is $Z_0 = Z_b + Z_c$. If both primary and secondary are grounded, the zero-sequence equivalent circuit of the transformers alone is as shown in Fig. 68. The impedances and potentials in Fig. 68 are referred to a common-base potential. The equivalent circuit (Fig. 68) is the same as that shown in Fig. 66 except that the zero-sequence

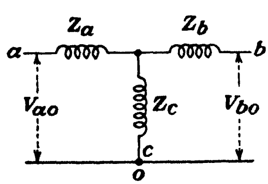



FIG. 68.

terminal potential of the tertiary is zero. The impedance measured at the secondary terminals of the zero-sequence equivalent circuit (Fig. 68) now depends upon the impedance to zero sequence of the primary circuit to which the transformers are connected. If the impedance to zero sequence of the primary circuit is $Z$, the impedance to zero sequence measured at the secondary terminals is

$$Z_0 = Z_b + \frac{(Z_a + Z)Z_c}{Z_a + Z + Z_c}$$

The impedance $Z$ is referred to the same base potential as that to which the transformer impedances are referred. It includes

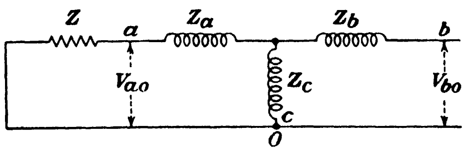


FIG. 69.

the effect of the grounding impedance on the primary side which is assumed to be independent of the grounding impedance on the secondary side. The equivalent circuit of the transformers together with the impedance $Z$ of the primary circuit, is shown in Fig. 69. There is an interesting modification of this case in which the primary and secondary neutrals are connected

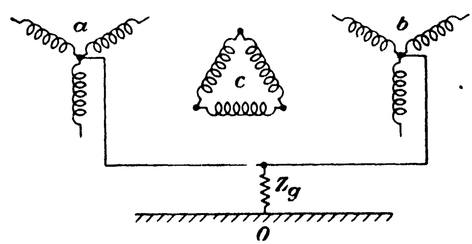


FIG. 70

and then grounded through the same impedance $Z_g$. The circuit arrangement is shown in Fig. 70. As before, the impedance to zero sequence of the primary circuit to which the transformers are connected is $Z$, but now $Z$ does not include the effect of the grounding impedance $Z_g$. The point $O$ is in the earth beyond the effect of the grounding impedance. If there are no ground

wires, the current in the grounding impedance is $3(I_{a0} + I_{b0})$. In the following analysis the impedances have their actual values, that is, they are not referred to a common base potential. The fundamental equations are

$$V_{a0} = E_{a0} + I_{a0}Z_a + (I_{a0} + I_{b0})3Z_g$$

also

$$\begin{aligned} V_{a0} &= -I_{a0}Z \\ V_{b0} &= E_{b0} + I_{b0}Z_b + (I_{a0} + I_{b0})3Z_g \\ V_{c0} &= 0 = E_{c0} + I_{c0}Z_c \end{aligned}$$

These, together with equations (251) and (252), determine the equivalent circuit. Combining the first two equations

$$0 = E_{a0} + I_{a0}(Z + Z_a + 3Z_g) + I_{b0}3Z_g \tag{253}$$

Collecting the terms of the third equation,

$$V_{b0} = E_{b0} + I_{b0}(Z_b + 3Z_g) + I_{a0}3Z_g \tag{254}$$

By means of equation (251) the fourth equation may be rewritten,

$$E_{c0} = \frac{AI_{a0} + BI_{b0}}{C}Z_c$$

Since $E_{c0} = \frac{C}{A}E_{a0} = \frac{C}{B}E_{b0}$, we may eliminate $E_{a0}$ and $E_{b0}$ from equations (253) and (254) and obtain

$$0 = \left(\frac{A^2}{C^2}Z_c + Z_a + Z + 3Z_g\right)I_{a0} + \left(\frac{AB}{C^2}Z_c + 3Z_g\right)I_{b0} \tag{255}$$

$$V_{b0} = \left(\frac{B^2}{C^2}Z_c + Z_b + 3Z_g\right)I_{b0} + \left(\frac{AB}{C^2}Z_c + 3Z_g\right)I_{a0} \tag{256}$$

As determined by these equations, the impedance to zero-sequence current is

$$\frac{V_{b0}}{I_{b0}} = Z_b + \frac{B^2}{C^2}Z_c + 3Z_g - \frac{\left(\frac{AB}{C^2}Z_c + 3Z_g\right)^2}{Z_a + \frac{A^2}{C^2}Z_c + Z + 3Z_g}$$

The equivalent circuit to which these equations apply is shown in Fig. 71. Note that the current, $I_{a0} + I_{b0}$, in the third branch of this network is not the tertiary current. The tertiary current may be obtained, however, by means of equation (251). Note

that the impedances have their actual values and are *not* referred to a common base potential. The equivalent circuit may be modified by referring the primary potential equation (255) to the secondary side by multiplying the equation by $B/A$ and by referring the primary ($a$) current in equations (255) and (256) to the secondary ($b$) side, by multiplying $I_{a0}$ by $A/B$ and at the same time multiplying the coefficients of $I_{a0}$ by $B/A$.

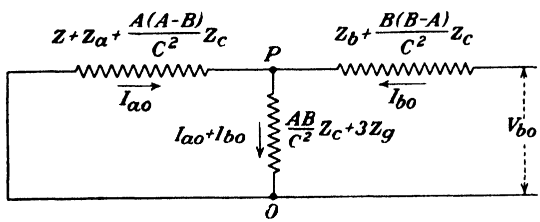



FIG. 71.

When this is done, the equivalent circuit becomes that in Fig. 72. The impedances in this figure have their actual values. Note, however, that the multipliers, $B^2/A^2$ and $B^2/C^2$, have the effect of referring to the secondary ($b$) side the primary and tertiary impedances and the impedance of the primary network to which the transformer is connected. There is, an infinite variety of equivalent circuits which have the proper secondary terminal

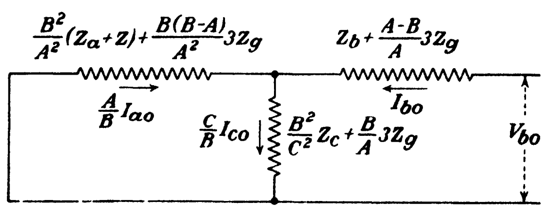



FIG. 72.

impedance. There is another interesting modification of this case in which the primary and secondary neutrals are connected but are not grounded. That is, the grounding impedance $Z_g$ is infinite. The equivalent circuit of Fig. 71 then reduces to a *through* impedance $Z + Z_a + Z_b + \frac{(A - B)^2}{C^2}Z_c$. An analysis based on the fundamental equations confirms this result. The impedances in this expression have their actual values.

**82. Interconnected Star-delta.**—Next we shall consider the so-called zigzag or interconnected star-delta. The arrangement of the transformer windings is shown in Fig. 73. The windings parallel to each other are on the same magnetic core. The primary ($a$) and secondary ($b$) windings, which have the same

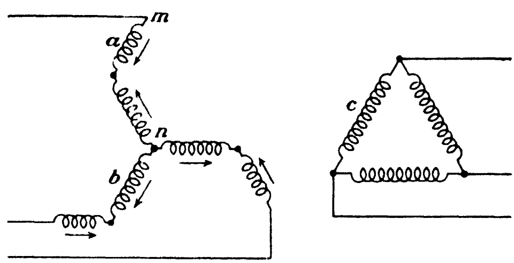



FIG. 73

number of turns, are interconnected in star, while the tertiary windings are connected in $\Delta$. The star connection is made so that the potential from $m$ to $n$ is $\sqrt{3}$ times the potential of the primary ($a$) winding. The analysis is easily made if the actual transformers are replaced by their equivalent perfect transformers with series impedances as in Fig. 64. This replacement has been made in Fig. 74. The tertiary impedance may be

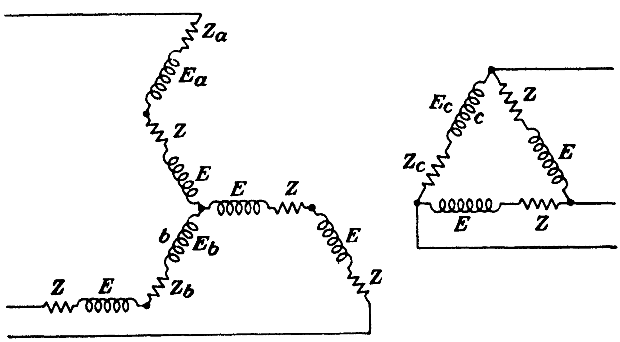



FIG. 74.

transferred to the star side by multiplying it by $(\sqrt{3}E_a/E_c)^2$. Thus the impedance to positive- or to negative-sequence current on the star side is $Z_a + Z_b + 3\frac{A^2}{C^2}Z_c$, where the impedances have their actual values. If the neutral of the star is grounded, zero-sequence current flows in opposite directions in the $a$ and $b$

windings which are on the same core. Since these windings have the same number of turns, the condition of zero total ampere-turns is realized without the tertiary carrying any current. Hence the impedance to zero-sequence current is $Z_a + Z_b$. This is the impedance measured at the terminals of the *a* winding when the *b* winding is short circuited.

Since the tertiary winding carries no zero sequence current, it may be omitted without affecting the zero-sequence operation. We then have a transformer which has a high impedance to both positive and negative sequence since it takes only exciting current for each of these sequences. However, the transformer has a low impedance ($Z_a + Z_b$) to zero sequence. The windings

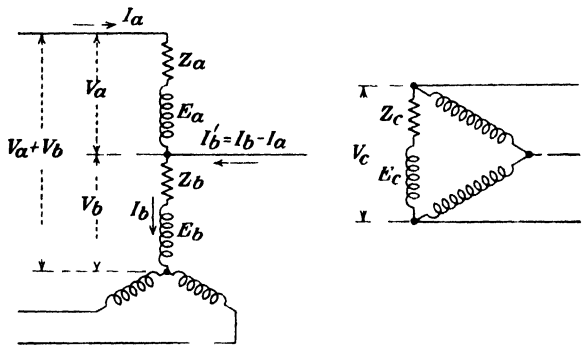



FIG. 75 —Only one of the three transformers is analyzed.

are commonly arranged on a 3-phase core and the transformer is referred to as a *grounding transformer*.[1]

**83. Autotransformer.**—We have noted in the discussion of two-winding transformers that the autotransformer is best considered as a regular transformer with its windings connected in series. This same point of view is equally useful in the case of three-winding autotransformers. Thus the actual autotransformers may be replaced by impedances and perfect transformers, as in Fig. 75, in which the nomenclature corresponds to that used with regular transformers. The high-tension potential is now $V_a + V_b$, while the low-tension current, $I_b'$, is $I_b - I_a$. The potentials across the perfect transformer windings are $E_a$, $E_b$, and $E_c$ (see Fig. 64) and the impedances have the values indicated by equations (247). We shall now

[1] WELCH, A. U., Induction Grounding Device, *Gen. Elec. Rev.*, **37**, 398.

determine the equivalent circuit for positive and negative sequence. The fundamental equations, in terms of positive- or of negative-sequence components of potential and current, are

$$\begin{aligned} V_a &= E_a + I_a Z_a \\ V_b &= E_b + I_b Z_b \\ V_c &= E_c + I_c Z_c \\ &= -I_c Z \end{aligned} \tag{257}$$

where $Z$ is the phase impedance of a balanced $\Delta$ load connected across the tertiary. Also

$$AI_a + BI_b + CI_c = 0$$

and

$$\frac{E_a}{A} = \frac{E_b}{B} = \frac{E_c}{C} \tag{258}$$

If the tertiary potentials and current are eliminated from equations (257) and (258), the expressions for the terminal potentials of the primary and secondary windings become

$$V_a = \left[ Z_a + \frac{A^2}{C^2}(Z_c + Z) \right] I_a + \frac{AB}{C^2}(Z_c + Z) I_b \tag{259}$$

$$V_b = \left[ Z_b + \frac{B^2}{C^2}(Z_c + Z) \right] I_b + \frac{AB}{C^2}(Z_c + Z) I_a \tag{260}$$

The high-tension potential is

$$V_a + V_b = \left[ Z_a + \frac{A(A + B)}{C^2}(Z_c + Z) \right] I_a + \left[ Z_b + \frac{B(A + B)}{C^2}(Z_c + Z) \right] I_b$$

If the high-tension and low-tension potentials are written in terms of the high-tension current, $I_a$, and the low-tension current, $(I_b - I_a)$,

$$V_a + V_b = \left[ Z_a + Z_b + \frac{(A + B)^2}{C^2}(Z_c + Z) \right] I_a + \left[ Z_b + \frac{B(A + B)}{C^2}(Z_c + Z) \right] (I_b - I_a) \tag{261}$$

$$V_b = \left[Z_b + \frac{B^2}{C^2}(Z_c + Z)\right](I_b - I_a) + \left[Z_b + \frac{B(A + B)}{C^2}(Z_c + Z)\right]I_a \quad (262)$$

Refer the low-tension potential and current to the high-tension side. The low-tension potential, $V_b$, referred to the high-tension side is $\frac{A + B}{B}V_b$ and the low-tension current, $(I_b - I_a)$, referred to the high-tension side is $\frac{B}{A + B}(I_b - I_a)$. If these changes are made, equations (261) and (262) become

$$V_a + V_b = \left[Z_a + Z_b + \frac{(A + B)^2}{C^2}(Z_c + Z)\right]I_a + \left[\frac{A + B}{B}Z_b + \frac{(A + B)^2}{C^2}(Z_c + Z)\right]\frac{B}{A + B}(I_b - I_a) \quad (263)$$

and

$$\frac{A + B}{B}V_b = \left[\frac{(A + B)^2}{B^2}Z_b + \frac{(A + B)^2}{C^2}(Z_c + Z)\right]\frac{B}{A + B}(I_b - I_a) + \left[\frac{A + B}{B}Z_b + \frac{(A + B)^2}{C^2}(Z_c + Z)\right]I_a \quad (264)$$

The equivalent circuit to which equations (263) and (264) apply is shown in Fig. 76.

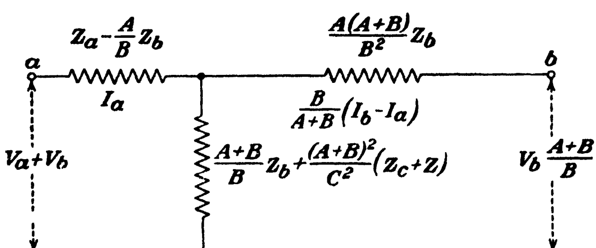



FIG. 76

The equivalent circuit for zero-sequence current differs from Fig. 76 in two particulars. First, the zero-sequence component of the terminal potential of the tertiary is zero, or $Z = 0$. Second, if the neutral of the transformers is grounded through a grounding impedance $Z_g$, the impedance $Z_b$ of the secondary should be

increased by $3Z_g$ (see Fig. 77). If these changes are made, the equivalent circuit of Fig. 76 for positive or negative sequence becomes the equivalent circuit of Fig. 78 for zero sequence. In Fig. 78 the potentials $V_{ao}$ and $V_{bo}$ are measured from the transformer terminals $a$ and $b$ to the earth side of the grounding impedance $Z_g$.

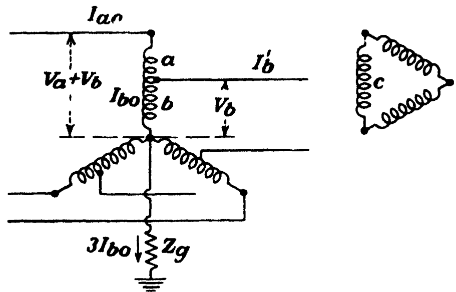



FIG. 77

Another form of the zero-sequence equivalent circuit that is of interest is derived from equations (261) and (262) before the low-tension potential and current are referred to the high-tension side. This equivalent circuit is shown in Fig. 79.

The advantage the network of Fig. 79 possesses is that we can quickly determine the effect of removing the neutral ground.

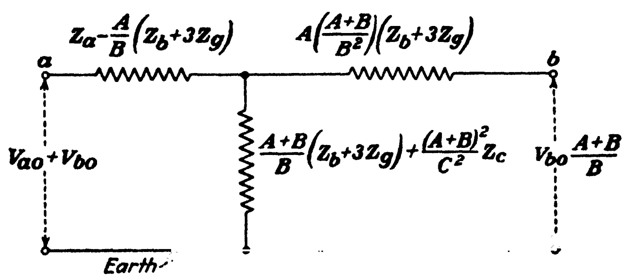



FIG. 78

If the grounding impedance, $Z_g$, becomes infinite, the secondary ($b$) carries no zero-sequence current and the impedance between $a$ and $b$ becomes $Z_a + \frac{A^2}{C^2}Z_c$. This is the impedance measured at the terminals of the primary ($a$) when the tertiary ($c$) is short-circuited. In this connection it is important to note that

the *same* zero-sequence current flows in the high- and low-tension circuits to which the transformers are connected. Thus, in combining the impedances to zero sequence of the high-tension and low-tension circuits, the actual values of the impedances should be combined, whereas, in combining the impedances of the high-tension and low-tension circuits to positive-sequence

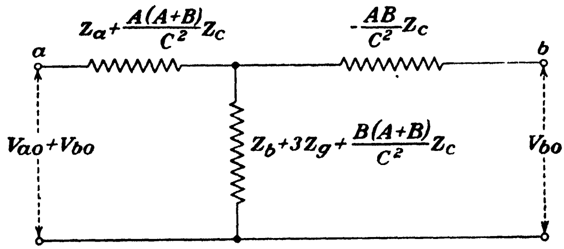
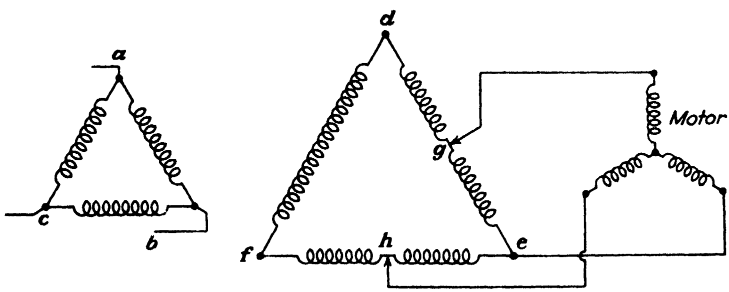



FIG. 79.

or to negative-sequence current, the impedances of one circuit must first be referred to the other circuit before being combined with it.

**Illustration.**—The following problem illustrates the application of the method of symmetrical components to an unsymmetrical combination of two-winding and three-winding transformers. A 3-phase induction motor *M* receives power from a 3-phase

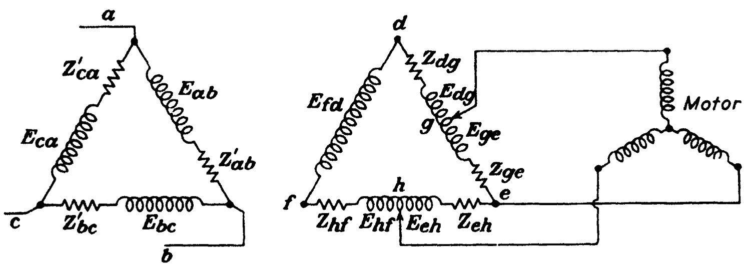



Actual Transformers

FIG. 80.

circuit *a-b-c* through three Δ-Δ-connected transformers (Fig. 80). Two of these transformers have tap points, *g* and *h*, on the secondary side so that a reduced potential can be applied to the motor while starting. Thus the transformer *c-a* is a two-winding transformer, while each of the other two transformers is a three-winding transformer. Each of the three transformers

can be replaced by a perfect transformer with suitable impedances (Fig. 81). Consider the secondary sides of the transformers. The motor receives power from a Δ system of potentials $V_{ge}$, $V_{eh}$, and $V_{hg}$. The impedances in the branches of this Δ are $Z_{ge}$, $Z_{eh}$, and $(Z_{hf} + Z_{dg})$. In writing the positive- and negative-sequence potential equations it can be assumed that these impedances have been transferred to the lines between the transformers and the motor by the method indicated in equations (229), page 194. If the impedances are equal, they need only




Perfect Transformers and Impedances
FIG. 81.

be divided by 3 when they are transferred to the lines. In this problem the impedances in the three Δ branches can be made equal by transferring from the primary to the secondary side of the two-winding transformer an impedance

$$(Z_{ge} - Z_{hf} - Z_{dg}).$$

The impedance from $h$ to $g$, by way of the secondary of the two-winding transformer, then equals

$$[Z_{hf} + (Z_{ge} - Z_{hf} - Z_{dg}) + Z_{dg}] \qquad \text{or} \qquad Z_{ge}.$$

Thus the impedances in the Δ branches $ge$, $eh$, and $hg$, are equal if $Z_{ge} = Z_{eh}$. This equality is true if the two three-winding transformers are identical. If the ratio of transformation of primary turns ($ac$) to secondary turns ($fd$) of the Δ-Δ bank is $k_a$, the remaining impedance $Z_p$ on the primary side of the two-winding transformer is $Z_p = Z_{ca}' - k_a^2(Z_{ge} - Z_{hf} - Z_{dg})$. The symmetrical components of $Z_p$ are

$$Z_{p0} = \frac{1}{3}(Z_p + Z_{ab}' + Z_{bc}')$$

and, if the two three-winding transformers are identical, when $Z_{ab}'$ equals $Z_{bc}'$,

$$Z_{p1} = \tfrac{1}{3}(Z_p - Z_{ab}')$$
$$Z_{p2} = \tfrac{1}{3}(Z_p - Z_{ab}')$$

The final circuit arrangement is indicated in Fig. 82. In Fig. 82 each of the impedances $Z$ equals $Z_{ge}/3$. Assume that the ratio of the emfs of the perfect transformers is $E_{de}/E_{ge} = k_b$. The primary currents are $I_{ab}$, $I_{bc}$, and $I_{ca}$. The corresponding secondary currents are $I_{ge}$, $I_{eh}$, and $I_{hfdg}$. The symmetrical components of $I_{ca}$ are $I_0'$, $I_1'$ and $I_2'$. The symmetrical components of $I_{hg}$ are $I_0''$, $I_1''$ and $I_2''$. If $V_1$ is the positive-sequence com-

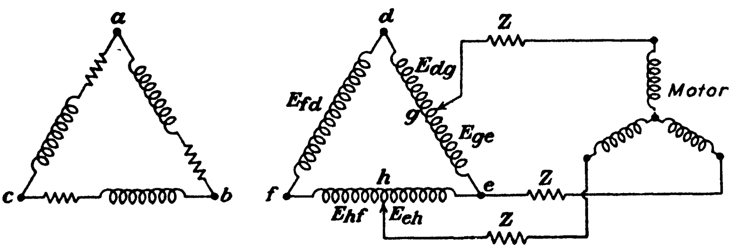



FIG. 82.

ponent of the fall in primary potential $V_{ca}$, the positive-sequence component of the emf fall in potential in the perfect transformer $c$-$a$ is

$$V_1 - (I_0'Z_1 + I_1'Z_0 + I_2'Z_2)$$

The subscript $p$ has been omitted from the impedances $Z_0$, $Z_1$, $Z_2$ to simplify the notation. It follows that the positive-sequence component of the fall in line-to-neutral emf $E_e$ on the secondary side is

$$j\frac{1}{k_a k_b \sqrt{3}}[V_1 - (I_0'Z_1 + I_1'Z_0 + I_2'Z_2)]$$

If the impedance of the motor per $Y$ phase to positive-sequence current is $Z_{M1}$,

$$j\frac{1}{k_a k_b \sqrt{3}}[V_1 - (I_0'Z_1 + I_1'Z_0 + I_2'Z_2)] = -j\sqrt{3}I_1''(Z + Z_{M1}) \qquad (1)$$

The positive-sequence component of the current $I_{en}$ is $-j\sqrt{3}I_1''$. If the potentials applied on the primary side are balanced and the impedance of the motor to negative-sequence current is $Z_{M2}$, the negative-sequence potential equation is

$$-j\frac{1}{k_a k_b\sqrt{3}}[0 - (I_0'Z_2 + I_1'Z_1 + I_2'Z_0)] = j\sqrt{3}I_2''(Z + Z_{M2}) \tag{2}$$

In writing the zero-sequence potential equation, it *cannot* be assumed that the impedances $Z_{ge}$, $Z_{eh}$, $Z_{hg}$ are transferred to the line. Since the zero-sequence components of both the primary and secondary terminal potentials are zero

$$\frac{1}{k_a}[0 - (I_0'Z_0 + I_1'Z_2 + I_2'Z_1)] + 3I_0''Z = 0 \tag{3}$$

In equation (3), the zero-sequence component of the emf fall in potential in the perfect transformer *c-a* is

$$0 - I_0'Z_0 - I_1'Z_2 - I_2'Z_1$$

The zero-sequence component of the emf fall in potential in the secondary *f-d* of the perfect transformer *c-a* is

$$\frac{1}{k_a}[-I_0'Z_0 - I_1'Z_2 - I_2'Z_1]$$

Each of the other secondaries *d-e* and *e-f* has the same zero-sequence component fall in emf. Thus in the zero-sequence system, the emfs from $f$ to $d$, from $d$ to $e$, and from $e$ to $f$ are zero-sequence components. It is important to observe that, while in the positive- and negative-sequence systems the fall in emf from $h$ to $g$ by way of $f$ and $d$ equals in magnitude the fall in emf from $g$ to $e$ or the fall in emf from $e$ to $h$, this is not true in the zero-sequence system. In the zero-sequence system, if $E$ is the fall in emf from $g$ to $e$, the fall in emf from $e$ to $h$ is also $E$, but the fall in emf from $h$ to $g$ is $[2(k_b - 1) + k_b]E$. Therefore, in the zero-sequence system the emfs from $e$ to $h$, from $h$ to $g$, and from $g$ to $e$ are not zero-sequence components since they are not equal. The zero-sequence component of the fall in emf in the perfect transformer *f-d* plus the fall in potential $3I_0''Z$ in the series impedance $3Z$ equals the zero-sequence component of the terminal potential $V_{fd}$.

It should be observed that the positive directions of current and of fall in potential have been chosen in equations (1), (2), and (3) so that the *total* ampere-turns for each transformer can be equated to *zero*. Thus, for the two-winding transformer *c-a*,

$$k_a(I_0' + I_1' + I_2') + (I_0'' + I_1'' + I_2'') = 0 \qquad (4)$$

For the three-winding transformer *a-b*,

$$k_a(I_0' + I_1'\angle{-120°} + I_2'\angle{120°}) + \frac{k_b - 1}{k_b}(I_0'' + I_1'' + I_2'') + \frac{1}{k_b}(I_0'' + I_1''\angle{-120°} + I_2''\angle{120°}) = 0 \qquad (5)$$

For the three-winding transformer *b-c*,

$$k_a(I_0' + I_1'\angle{120°} + I_2'\angle{-120°}) + \frac{k_b - 1}{k_b}(I_0'' + I_1'' + I_2'') + \frac{1}{k_b}(I_0'' + I_1''\angle{120°} + I_2''\angle{-120°}) = 0 \qquad (6)$$

Note that $(k_b - 1)/k_b$ is the ratio of the turns between $d$ and $g$ to the total turns between $d$ and $e$, and that $1/k_b$ is the ratio of the turns between $g$ and $e$ to the total turns between $d$ and $e$. If equations (5) and (6) are added,

$$k_a(2I_0' - I_1' - I_2') + 2I_0'' + \frac{2k_b - 3}{k_b}(I_1'' + I_2'') = 0 \qquad (7)$$

If equation (5) is subtracted from equation (6),

$$k_a(I_1' - I_2') + \frac{1}{k_b}(I_1'' - I_2'') = 0 \qquad (8)$$

The following simplified relations are derived from equations (4), (7), and (8).

$$k_ak_bI_1' + I_1'' = 0 \qquad (9)$$

$$k_ak_bI_2' + I_2'' = 0 \qquad (10)$$

$$k_aI_0' + I_0'' - k_a(k_b - 1)(I_1' + I_2') = 0 \qquad (11)$$

On eliminating $I_0''$, $I_1''$, and $I_2''$ from equations (1), (2), (3), (9), (10), and (11),

$$\begin{aligned} V_1 &= I_0'Z_1 + I_1'(Z_0 + 3k_a^2k_b^2(Z + Z_{M1})) + I_2'Z_2 \\ 0 &= I_0'Z_2 + I_1'Z_1 + I_2'(Z_0 + 3k_a^2k_b^2(Z + Z_{M2})) \\ 0 &= I_0'(Z_0 + 3k_a^2Z) + I_1'(Z_2 - 3k_a^2(k_b - 1)Z) \\ &\quad + I_2'(Z_1 - 3k_a^2(k_b - 1)Z) \end{aligned} \qquad (12)$$

Consider the following numerical example. The total volt-ampere rating of the induction motor equals three times the volt-ampere rating of one of the transformers. The ratio of transformation, $k_a$, equals 10. The motor is started at one-third its rated potential, or $k_b = 3$. At starting, the motor is a static circuit (without moving parts) and the impedances of the motor to positive and negative sequence are equal. Assume that $Z_{M1} = Z_{M2} = 0.18$ per unit. That is, with 0.18 rated potential applied to the motor, the starting current has its rated full-load value. We shall choose as the *unit* of *potential* the rated primary potential $E$ of the transformers, and as the *unit* of *current* the rated primary current $I$ of the transformers. Designate the winding $ab$ (Fig. 80) as $a$, the winding $dg$ as $b$, and the winding $ge$ as $c$. The turn ratios are

$$\frac{A}{B} = \frac{3k_a}{2} = 15$$

$$\frac{A}{C} = 3k_a = 30$$

$$\frac{B}{C} = 2$$

Assume that the short-circuit impedance $Z_{ab}$ is 0.08 per unit, that the short-circuit impedance $Z_{ac}$ is 0.12 per unit, and that the short-circuit impedance $Z_{bc}$ is 0.10 per unit. The short-circuit impedances $Z_{ab}$ and $Z_{ac}$ are both measured on the primary side and are thus expressed in terms of the rated primary potential and current. Since the units of potential and current on the primary side are $E$ and $I$, the impedance $Z_{ab}$ is $0.08\frac{E}{I}$ ohms and the impedance $Z_{ac}$ is $0.12\frac{E}{I}$ ohms. The short-circuit impedance $Z_{bc}$ is measured by applying potential to the $b$ winding while the $c$ winding is short-circuited. The rated potential of the $b$-winding is $\frac{B}{A}E = \frac{E}{15}$ and the rated current is $\frac{A}{B+C}I = 10I$. Thus, the short-circuit impedance $Z_{bc}$ is $0.10\frac{E}{150I}$ ohms, and

$$\frac{A^2}{B^2}Z_{bc} = 0.10\frac{15^2E}{150I} = 0.15\frac{E}{I} \text{ ohms}$$

In order to simplify the following computations, assume that the *angles* of all the impedances are *equal*. *Numerical* addition then gives the same result as *vector* addition. By equations (247), page 206,

$$Z_a = \frac{0.08 + 0.12 - 0.15}{2}\frac{E}{I}$$

$$= 0.025\frac{E}{I} \text{ ohms}$$

$$15^2 Z_b = \frac{0.08 - 0.12 + 0.15}{2}\frac{E}{I}$$

$$= 0.055\frac{E}{I} \text{ ohms}$$

$$30^2 Z_c = \frac{-0.08 + 0.12 + 0.15}{2}\frac{E}{I}$$

$$= 0.095\frac{E}{I} \text{ ohms}$$

Refer to Fig. 81 and note that the secondary winding of the two-winding transformer is equivalent to windings $b$ and $c$ connected in series. Thus, the short-circuit impedance $Z_{ca}'$ of the two-winding transformer (Fig. 81) is

$$Z_{ca}' = Z_a + \frac{A^2}{(B + C)^2}(Z_b + Z_c)$$

$$= \left[0.025 + 10^2\left(\frac{0.055}{15^2} + \frac{0.095}{30^2}\right)\right]\frac{E}{I}$$

$$= 0.060\frac{E}{I} \text{ ohms}$$

Note that

$$Z_{dg} = Z_{hf} = Z_b \qquad \text{and} \qquad Z_{ge} = Z_c$$

The impedance $Z_p$ is

$$Z_p = \left[0.060 - 10^2\left(\frac{0.095}{30^2} - 2\frac{0.055}{15^2}\right)\right]\frac{E}{I} = 0.098\frac{E}{I} \text{ ohms}$$

Note that

$$Z_{ab}' = Z_{bc}' = Z_a = 0.025\frac{E}{I} \text{ ohms}$$

The symmetrical components of $Z_p$ are

$$Z_0 = 0.049\frac{E}{I} \text{ ohms}$$

$$Z_1 = Z_2 = 0.024\frac{E}{I} \text{ ohms}$$

The impedance $Z$ is $Z_{ye}/3$ or

$$Z = \frac{1}{3}\frac{0.095}{30^2}\frac{E}{I} \text{ ohms}$$

Assume that the motor is $Y$-connected as indicated in Fig. 82. The rated potential per phase is $E/10\sqrt{3}$ and the rated current per phase is $10\sqrt{3}I$. The starting impedance of the motor is $0.18\frac{E}{300I}$ ohms. If the numerical values are substituted in equations (12) and it is noted that $V_1$ equals the rated potential $E$,

$$\begin{aligned} I &= 0.024I_0' + 1.76I_1' + 0.024I_2' \\ 0 &= 0.024I_0' + 0.024I_1' + 1.76I_2' \\ 0 &= 0.060I_0' + 0.003I_1' + 0.003I_2' \end{aligned}$$

where $I$ is the rated current of *one* transformer on the *primary* side. The solution of these equations gives

$$\begin{aligned} I_0' &= -0.028I \\ I_1' &= 0.568I \\ I_2' &= -0.0074I \end{aligned}$$

The zero-sequence unbalance factor for current is

$$\frac{0.028}{0.568} = 0.049.$$

The negative-sequence unbalance factor for current is

$$\frac{0.0074}{0.568} = 0.013.$$

Thus the unsymmetry in the circuit causes little unsymmetry in the currents.

# CHAPTER VI

## INDUCTION MACHINES

## TERMINAL CHARACTERISTICS

The following chapters will be devoted to analyzing the performance of induction and synchronous machines when they are operated under unbalanced conditions. The induction machine will be considered first since it is the simpler of the two. In the present chapter we shall determine the effect it produces upon the circuit to which it is connected and in the following chapter we shall analyze the performance of the machine itself as determined by its internal reactions. In all of this discussion, unless specifically noted to the contrary, we shall adopt the simple theory which assumes that the resistances and leakage reactances of the stator and rotor windings are constant; and that the mutual air-gap flux is distributed sinusoidally around the periphery and is proportional to the net magnetomotive force of the stator and rotor currents which produces it. Whereas both the stator and rotor may be equipped with symmetrical polyphase windings, it is *essential* that *either* the stator *or* rotor should be so equipped. The reason for this requirement is discussed in Chap. VII (see page 276). For the present it suffices to note that the performance of a machine which has unsymmetrical polyphase windings on *both* sides of the air gap cannot, in general, be analyzed by the method of symmetrical components.[1]

Moreover in order to simplify the problem, it will be assumed in this chapter and the one following, that both the stator and rotor have symmetrical windings and that the unsymmetry exists *outside* of their terminals. Again it should be observed that, if the circuits outside of *both* the stator and rotor terminals are unsymmetrical, this method of symmetrical components cannot be applied rigorously.[1] A motor for which all of the preceding assumptions hold true is described as ideal.

Heretofore we have discussed the behavior of static circuits when operating under unsymmetrical conditions. If the circuits

[1] See footnote, p. 276.

are symmetrical, that is, if the different phases are alike so that one is indistinguishable from another, the ratio of the positive sequence applied potential to the positive sequence current equals the ratio of the negative sequence applied potential to the negative sequence current (see page 93). Thus it may be said that the impedance of a symmetrical static circuit, *i.e.*, the zero sequence impedance, is the same to positive-sequence potentials as to negative-sequence potentials. It should be noted, however, that, when there are mutual reactances present in the symmetrical circuit, the impedances to zero-sequence potential and, in the case of 4-phase circuits, to second-order potentials both differ from the impedance to positive- or negative-sequence potentials. This difference is due wholly to the mutual reactances between the phases. In induction or in synchronous machines the same principles apply to the stator and rotor windings individually. The operation of the machine, however, is to a great extent controlled by the mutual reactances between the stator and rotor phases and these reactances are variable, owing to the motion of the rotor. Since it is being assumed that the mutual air-gap flux is sinusoidally distributed, the mutual reactance between a stator and a rotor phase varies as the cosine of the electrical angular displacement between these phases. When the motor is running at constant speed this electrical displacement varies directly with time. Owing wholly to the action of this *variable* mutual inductance, the ratio of the positive-sequence applied potential to the positive-sequence current is not equal to the ratio of the negative-sequence applied potential to the negative sequence current, as in the case of static circuits. Since the windings of both stator and rotor are symmetrically arranged, there will be none but zero-sequence components of self- and mutual impedances and thus the impedances of the motor to zero-, positive-, and negative-sequence components of applied potential, although different, should all be treated as zero-sequence quantities. That is to say, if symmetrical potentials are applied to the motor, the ratio of the potential vector across any phase to the current vector in that phase is the same for all the phases. These impedances of the motor to zero-, positive- and negative-sequence components of applied potential will be represented by the letters $Z_0$, $Z_1$, and $Z_2$, or, when needed, a further identifying subscript will be used and we may write

$Z_{m0}$, $Z_{m1}$ and $Z_{m2}$. In this case, however, it is important to note that the numerical subscript does not have the same significance that has been assigned to it previously. In applying the Sequence Rule in the process of multiplication, each of these impedances must be treated as a *zero sequence* quantity. The vector values of these impedances can be obtained either by direct measurement or by calculation. The latter method is used in Chap. VII. Again it is important to emphasize that the term "impedance to zero, positive or negative sequence" refers to the impedance of one phase of a *balanced* circuit, whereas the term "zero,- positive-, or negative-sequence impedance" refers to one of the symmetrical components of a system of unbalanced impedances.

If symmetrical potentials are applied to the motor, the stator currents also are symmetrical and have the same phase order as the potentials, and their vector values are completely determined by the applied potentials and the slip. If saturation effects are ignored and if the core losses are assumed to vary as the square of the flux density, the stator current is directly proportional to the applied potential for any given value of the slip. As the slip $s$ varies from 0 to 2, the relation of the current and potential is graphically represented by the well known circle diagram, Fig. 83. This diagram is drawn to scale for representative motor constants. The locus of the end of the current vector approximates a circle and is commonly drawn as such. The current vector is shown for the conditions of no load ($s = 0$, very nearly), full load, ($s = s_1$), blocked rotor ($s = 1$), and synchronous speed backward ($s = 2$). Notice that the current vector changes little as the slip increases from 1 to 2. In fact, it is often assumed that the current for a slip 1 equals the current for a slip 2.

The direction of rotation of the magnetic field in the air gap is determined by the phase order of the applied potentials. If the air gap is uniform and space harmonics are neglected, the magnetic field that is due to symmetrical currents in a symmetrical polyphase winding is constant and rotates in such a direction as to be always opposite the windings of any phase at the moment the current in that phase is passing through its maximum value. Thus the progression of the field coincides with the progression of the maximum values of the phase currents.

The direction of rotation of the field is from the *a* phase toward the *b* phase when the current in the *b* phase lags the current in the *a* phase, that is, for positive-sequence currents. The direction of rotation is reversed with negative-sequence currents and is then from the *b* phase toward the *a* phase. Let us assume that positive-sequence potentials are applied to the motor and that it is running with a slip $s_1$. Let us also assume that the positive-sequence current taken by the motor is represented by

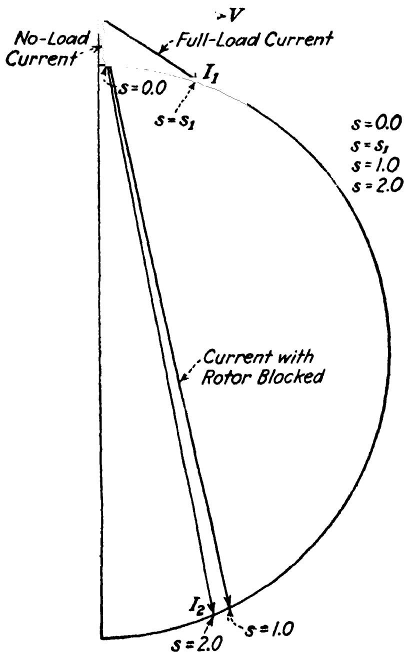



FIG. 83.

the vector $I_1$, Fig. 83. If, while the speed of the motor is maintained without change, negative-sequence potentials of the same value are applied in place of the positive-sequence potentials, the slip with respect to the negative sequence field will be $2 - s_1$, and the vector current will assume the value $I_2$ (very nearly). In this illustration the impedance of the motor to positive-sequence potentials is $Z_1 = V/I_1$, while the impedance to negative-sequence potentials is $Z_2 = V/I_2$ (very nearly). The impedance to negative-sequence potentials is often taken as the blocked impedance,

$$Z_b = \frac{V}{I_b}.$$

For the purposes of computation it is frequently better to express these characteristic ratios in the admittance form. Representative per unit values of admittance are

No load ($s = 0$)

$$Y = 0.04 - j0.4$$

Full load

$$Y = 0.9 - j0.42$$

Blocked ($s = 1$)

$$Y = 1.0 - j5.0$$

Synchronous speed backward ($s = 2$)

$$Y = 0.9 - j5.1$$

Accordingly for these values, the admittance to positive sequence varies from $0.04 - j0.4$ per unit at no load to $0.9 - j0.42$ per unit at full load. The admittance to negative sequence is essentially constant and has a value of $0.9 - j5.1$ per unit. We shall discuss the zero-sequence admittance in Chap. VII (see page 324).

**Superposition Principle.**—It is a general and important principle in the solution of linear circuits that the current in any branch equals the sum of the component currents produced in that branch by the individual action of the various potentials that are applied to the circuit. A linear circuit is one for which, if a sinusoidal potential is applied to it, the input current is also sinusoidal and is proportional to the applied potential in the sense that, if the applied potential is doubled, the input current is also doubled. Furthermore, the scope of this principle can be extended if the applied potentials are resolved into components and these components are impressed on the circuit either singly or in combination. The manner in which this general method of analysis is applied depends entirely upon the problem and upon the experience of the investigator. The guiding principle should be to choose the components of the applied potentials, or the combinations of these components, so that the solutions for the components of the currents are recognized as being familiar in form.

**84. Effect of Unbalanced Applied Potentials.**—In our problem, unbalanced potentials are applied at the terminals *a*, *b*, and *c*

of a symmetrical 3-phase induction motor while it is running at some given speed. The magnitudes of these potentials fix the vector values of $V_{ab}$ and $V_{ac}$, for example. It should be noted that these *two* potential vectors completely determine the stator and rotor currents. The superposition theorem may be applied to this problem in the following manner:

First, compute the component currents in the stator and rotor windings when the potential $V_{ab}$ is impressed across the terminals *a-b* and the terminals *b-c* are short-circuited, *i.e.*, $V_{bc} = 0$. Second, compute the component currents in the stator and rotor windings when the terminals *a-b* are short-circuited, *i.e.*, $V_{ab} = 0$, and the potential $V_{bc}$ is impressed across the terminals *b-c*. The resulting currents in the stator and rotor windings are the sums of their respective components. To one who is familiar only with the operation of the motor under balanced conditions, this method of solution would seem to substitute for one difficult problem two other problems equally difficult. In fact, these component conditions are but special cases of the general unbalanced problem. This manner of applying the Superposition Principle is thus of no value. However, if the investigator should prove, as has been done, that any unbalanced system of 3-phase line potentials is equivalent to the sum of two component systems of balanced line potentials of opposite phase order, he could then apply the Superposition Theorem successfully, since the solution for the current due to the application of each component system would then be in familiar form. Let the line potentials be resolved into their positive- and negative-sequence components, as shown diagrammatically in Fig. 14*a*, *b*, page 54. If the negative-sequence potentials are short-circuited, the positive-sequence potentials will produce positive-sequence currents which depend upon $E_{ab1}$ and the slip *s*. Likewise, if the positive-sequence potentials are short-circuited, the negative-sequence potentials will produce negative-sequence currents which depend upon $E_{ab2}$ and the slip, 2-*s*, of the rotor with respect to the negative-sequence rotating field.

In this discussion we have reviewed more of the fundamental theory than is necessary for the development of our particular investigation. In fact, we may look upon the motor as being a device of unknown construction whose only visible parts are three electrical terminals and a rotating shaft. In regard to

the characteristics of this unknown device, we may determine that, if balanced 3-phase sinusoidal potentials are impressed on the three terminals and the speed of the shaft is maintained constant, the input currents are also sinusoidal, balanced, and of the same frequency and phase order as the applied potentials. Furthermore, if the potentials are increased in magnitude, the currents are likewise increased and in the same proportion. The vector relation of these potentials and currents determines the impedance of the unknown device to positive sequence. If the speed of the shaft is maintained constant both in magnitude and in direction, while the phase order of the balanced sinusoidal potentials *alone* is reversed, the currents are again sinusoidal, balanced, and of the same frequency as the potentials. Again, if the potentials are increased in magnitude, the currents are likewise increased in the same proportion. The vector relation of the new potentials and the new currents determines the impedance of the unknown device to negative sequence. It will also be noted that these impedances to positive and to negative sequence have different values for different values of the shaft speed. Again, it is important to observe that positive-sequence potentials produce no negative-sequence currents and that negative-sequence potentials produce no positive-sequence currents. The fact that the currents have the same wave form as, and are proportional to, the applied potentials shows that the unknown device has linear characteristics and that the Superposition Theorem may be applied. Fortescue has shown how this may be done in polyphase networks, to which class the unknown device belongs. The entire analysis in this chapter is built upon the foundation developed by Fortescue. As an analogous case, consider a copper conductor which is maintained at a constant temperature. We know that, if a direct potential is impressed on the ends of this conductor, a direct current will flow in it, and that the current is proportional to the applied potential. For circuit theory nothing more needs be known about the phenomenon of conduction. Nevertheless, this terminal characteristic, which we know as Ohm's Law, forms one of the corner stones of electric-circuit theory. In the induction motor the internal reactions are well understood and are subject to quite exact computation. This computation will be made in the following chapters, but

for the present we shall consider only the terminal characteristics of the motor and the effects that are produced in the circuit external to the motor. It should be borne in mind that actual terminal measurements will show that a motor does not have exactly linear characteristics and, therefore, that any computations made on this basis will be in some error.

Let $E_1$ and $E_2$ be the positive- and negative-sequence components of the star potential from the terminal $a$ to the neutral $n$, and let $Y_1$ and $Y_2$ be the admittances of the motor per star phase to positive and negative sequence for the given value of slip. The positive-sequence current flowing in at the terminal $a$ is:

$$I_1 = E_1 Y_1$$

The negative-sequence current flowing in at the terminal $a$ is:

$$I_2 = E_2 Y_2$$

Note that both $Y_1$ and $Y_2$ are treated as zero-sequence quantities in applying the sequence rule. It should be observed that, since the admittance $Y_2$ to negative sequence is approximately five times the admittance $Y_1$ to positive sequence at full load and is from twelve to fifteen times the admittance $Y_1$ at no load, the ratio of the negative-sequence current to the positive-sequence current is from five to fifteen times as great as the ratio of the negative-sequence component of the applied potentials to the positive-sequence component of the applied potentials. That is to say, an induction motor is quite sensitive to any unbalance in its terminal potentials.

The total vector volt-ampere input supplied to the motor is

$$\begin{aligned} \text{Total } (\underset{\frown}{E}I) &= 3(\underset{\frown}{E}_1 I_1 + \underset{\frown}{E}_2 I_2) \\ &= 3(\underset{\frown}{E}_1 E_1 Y_1 + \underset{\frown}{E}_2 E_2 Y_2) \\ &= 3(|E_1|^2 Y_1 + |E_2|^2 Y_2) \end{aligned} \qquad (265)$$

This determines the active and reactive powers and thus the power factor.

The total pulsating power supplied to the motor is

$$\begin{aligned} \text{Total } S &= 3(E_1 I_2 + E_2 I_1) \\ &= 3(E_1 E_2 Y_2 + E_2 E_1 Y_1) \\ \text{Total } |S| &= 3(|E_1| \times |E_2| \times |Y_1 + Y_2|) \end{aligned} \qquad (266)$$

The magnitude of this double frequency power pulsation is approximately proportional to the double frequency torque that

is developed in the motor owing to the unbalanced conditions. This pulsating torque may produce undesirable vibration (see Art. 104).

**Illustration.**—Consider a motor which at full load operates at 0.9 power factor. The per unit full-load admittance is $Y = 0.9 - j0.436$ per phase. Also assume that the per unit admittance to negative sequence is $Y_2 = 0.9 - j4.50$. This value of per unit admittance indicates an equivalent per unit resistance of about 0.04 and an equivalent per unit reactance of about 0.21, with the rotor blocked. If the line potentials $V_{ab}$, $V_{bc}$, and $V_{ca}$ have per unit values of $\sqrt{3}$, $0.8\sqrt{3}$, and $\sqrt{3}$, respectively, the positive- and negative-sequence components of the per unit star potential $V_{an}$ are

$$V_{a1} = 0.929 + j0 \qquad \text{and} \qquad V_{a2} = 0.129 + j0.$$

The positive-sequence potential is thus 92.9 per cent of its rated value. If the slip is assumed to have its full-load value so that $Y_1 = 0.9 - j0.436$, the positive- and negative-sequence components of the per unit current flowing in at the terminal $a$ are

$$\begin{aligned} I_{a1} &= 0.929(0.9 - j0.436) \\ &= 0.836 - j0.405 \\ I_{a2} &= 0.129(0.9 - j4.50) \\ &= 0.116 - j0.581 \end{aligned}$$

Note that, whereas the unbalance in the applied potentials is 0.129/0.929 or 0.139, the unbalance in the currents is

$$\frac{\sqrt{0.116^2 + 0.581^2}}{\sqrt{0.836^2 + 0.405^2}} \qquad \text{or} \qquad 0.639$$

The line currents are

$$\begin{aligned} I_a &= 0.952 - j0.986 \\ I_b &= -0.324 - j0.130 \\ I_c &= -0.628 + j1.116 \end{aligned}$$

The vector diagram is in Fig. 84.

The total vector volt-ampere input is

$$\begin{aligned} \text{Total } (\underline{E}I) &= 3[0.929^2(0.9 - j0.436) - 0.129^2(0.9 - j4.5)] \\ &= 3(0.791 - j0.451) \text{ units} \end{aligned}$$

The unit of volt-amperes is the volt-ampere input per phase at full load. The power factor is 0.868.

The magnitude of the pulsating power is

$$|S| = 3 \times 0.929 \times 0.129 \times |0.9 - j0.436 + 0.9 - j4.5|$$
$$= 3 \times 0.63 \text{ units}$$

The ratio of the maximum value of the power pulsation to the active power is $0.63/0.791 = 0.797$.

**85. Single-phase Operation of Three-phase Motor.**—Apply balanced potentials to the terminals $a$, $b$, and $c$ of a 3-phase induction motor until the motor has attained its normal speed for the connected shaft load. Then disconnect the terminal $a$ from the electric circuit so that the motor receives single-phase power through the terminals $b$ and $c$. The analysis is simplest if the equations are written for the $a$ phase which is carrying no current. Thus, if $I_1$ and $I_2$ are the positive- and negative-sequence components of $I_a$,

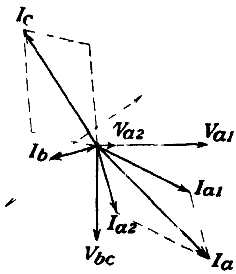


FIG. 84

$$I_1 + I_2 = 0 \tag{267}$$

Furthermore, if $V_1$ and $V_2$ are the positive- and negative-sequence components of the potential $V_{an}$, and if the potential $V_{bc}$ is taken as $V$, we may write [equation (37) and (38), page 55]

$$V_1 - V_2 = \frac{jV}{\sqrt{3}} \tag{268}$$

If $Z_1$ and $Z_2$ are the impedances of the motor per star phase to positive and negative sequence for the slip at which it is operating,

$$I_1Z_1 - I_2Z_2 = \frac{jV}{\sqrt{3}} \tag{269}$$

Thus the positive-sequence current is

$$I_1 = \frac{jV}{\sqrt{3}(Z_1 + Z_2)} \tag{270}$$

Since $V/\sqrt{3}$ is the phase potential to neutral of the circuit to which the motor is connected, the positive-sequence current,

when operating single phase, equals this potential divided by the sum of the impedances of the motor per star phase to positive and to negative sequence. Since the single-phase line current is $-j\sqrt{3}$ times the positive-sequence current [see equation (45), page 64], the line current $I$, is

$$I = \frac{V}{(Z_1 + Z_2)} \tag{271}$$

Thus the line current taken by a single-phase motor equals the line potential divided by the sum of the impedances of the motor per star phase to positive and to negative sequence.

It seems particularly interesting to the author that the foregoing analysis should be in terms of the components of a nonexistent current and that the equations are written for a winding of the motor, which, since it carries no current, may also be nonexistent.

The potential between the terminal $a$ of the motor and the line conductor from which the terminal was disconnected is $3V_2$, if the line potentials are balanced (see Fig. 14*b*, page 54). Hence the impedance to negative sequence can be measured easily at normal saturation, since this impedance equals the potential $3V_2$ divided by $\sqrt{3}$ times the single-phase current. The phase of the impedance can be determined by means of a wattmeter.

The ratio of the negative- to the positive-sequence terminal potentials, or the unbalance factor, is

$$\frac{V_2}{V_1} = -\frac{Z_2}{Z_1} \tag{272}$$

Representative per unit values of the impedances to positive and to negative sequence are

$Z_1 = 3.0\underline{/84°}$ at no load

$= 1.0\underline{/26°}$ at a slip equal to that when the motor receives balanced 3-phase power and delivers its rated load.

$Z_2 = 0.2\underline{/76°}$

Thus the magnitude of the unbalance factor varies from approximately 0.07 at no load to 0.20 at full load.

The vector volt-ampere input is

$$
\begin{aligned}
\text{Total } (\underline{E}I) &= 3(\underline{V}_1 I_1 - \underline{V}_2 I_2) \\
&= \underline{V}I \\
&= \frac{|V|^2}{(\overline{Z}_1 + \overline{Z}_2)}
\end{aligned}
\qquad (273)
$$

If $Z_1 = 1\underline{/26^\circ}$ and $Z_2 = 0.20/\underline{78^\circ}$

$$\text{Total } \underline{E}I = |V|^2(0.731 - j0.493)$$

The power factor is 0.829. Had the motor been connected to a balanced 3-phase circuit, the power factor would have been cos 26°, or 0.899, for the same value of the slip. Since it is a single-phase circuit, the power pulsation equals the magnitude of the vector-volt ampere input.

When the rotor is blocked, the impedances of the motor to positive- and negative-sequence currents are equal. Thus the positive- and negative-sequence components of the potential across the open phase are equal and opposite, and the potential across this phase is zero. Furthermore, it will be observed that the potential across each of the other phases is equal to one-half of the applied single-phase potential. When blocked, the motor acts like a reactance coil connected across the single-phase supply, the open terminal of the motor corresponding to the mid-point of the coil. If torque is now applied to the shaft, the motor will speed up and as a consequence the impedance to positive sequence will increase considerably and the impedance to negative sequence will decrease slightly. Owing to this change in the motor impedances, the positive-sequence component of the potential across the open phase will become greater than the negative-sequence component and a potential will appear across the open phase. When the motor has reached its full-load speed, the ratio of these impedances in a typical case is

$$\frac{Z_1}{Z_2} = -\frac{1\underline{/26^\circ}}{0.2\underline{/76^\circ}} = -5\underline{/-50^\circ}$$

and the ratio of the sequence components of the potential across the open phase $a$ is

$$\frac{V_1}{V_2} = -5\underline{/-50^\circ}$$

The vector diagram is in Fig. 85. The triangle of the line potentials is shown dotted. Owing to the rotation and consequent change in the impedances to positive and negative sequence, the potentials across the terminals of the motor now form a polyphase system which is somewhat out of balance. If the motor is allowed to run idle at approximately synchronous speed, the impedance to positive sequence increases still further and the

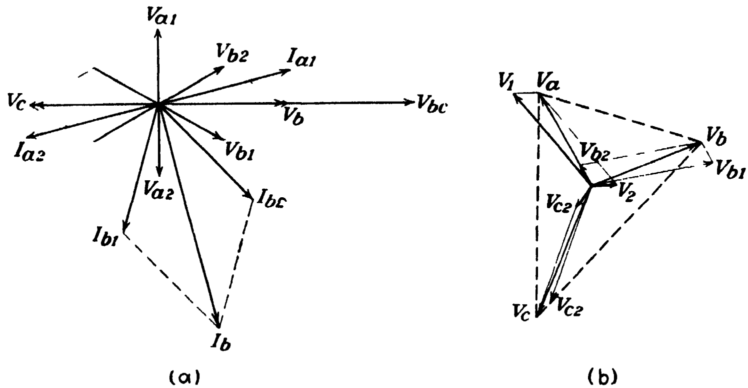



FIG. 85.—(*a*) Rotor blocked. (*b*) Rotor running at full-load speed.

ratio of the sequence components of the potential across the open phase becomes

$$\frac{V_1}{V_2} = -\frac{3\underline{/84^\circ}}{0.2\underline{/76^\circ}} = -15\underline{/8^\circ}$$

Since the positive- and negative-sequence currents are numerically equal, the single-phase motor is equivalent in some respects to two identical polyphase motors whose *stator* windings are connected in *series* in such a way as to produce oppositely rotating fields and whose rotors are rigidly coupled mechanically. If the slip of the rotors with respect to one field is $s$, the slip with respect to the other field is 2-$s$. Thus the impedance measured at the terminals of one motor is $Z_1$ and at the terminals of the other motor is $Z_2$. If balanced *line* potentials of $V$ volts are applied to this combination, the potentials across each motor will be proportional to the terminal impedance of that motor. These potentials are equal at standstill, when $s = 1.0$, and there is no net torque on the rotors. If the rotors are given a spin, the slip with respect to the field (1) which rotates in the direction

of the spin becomes less than the slip with respect to the other field (2). Under this condition $Z_1$ becomes greater than $Z_2$ and the potential $V_1$ across one motor becomes greater than the potential $V_2$ across the other motor.

The motor may also be operated when the terminals $b$ and $c$ are connected together and the line potential is impressed between them and the terminal $a$ In this case (see page 56),

$$V_{a1} = V_{a2} = \tfrac{1}{3}V \tag{274}$$

or the positive- and negative-sequence components of the star-phase potential, each equals one-third the applied potential $V$. It follows that

$$I_1 = \frac{V}{3Z_1}$$
$$I_2 = \frac{V}{3Z_2} \tag{275}$$

The unbalance factor for potential is now 100 per cent while the unbalance factor for current is approximately 500 per cent at full-load slip and is 1500 per cent at no load.

The single-phase line current is

$$I = I_1 + I_2 = \tfrac{1}{3}V(Y_1 + Y_2) \tag{276}$$

The vector volt-ampere input is

$$\text{Total } \underset{\smile}{E}I = \tfrac{1}{3}|V|^2(Y_1 + Y_2) \tag{277}$$

If $Z_1 = 1\underline{/26°}$ and $Z_2 = 0.20\underline{/78°}$,

$$\text{Total } \underset{\smile}{E}I = \tfrac{1}{3}|V|^2(1.93 - j5.33)$$
$$= V^2(0.646 - j1.78)$$

The power factor is now 0.341.

It should be evident that, if the motor is operated in this manner, it is in some respects equivalent to two identical motors, whose stator windings are connected in *parallel* so as to produce oppositely rotating fields and whose rotors are rigidly coupled mechanically. The line to neutral potentials applied to the motors are balanced and equal $V/3$.

There is another way in which a three-phase motor may receive single-phase power. The diagram of connections is in Fig. 86$a$. An autotransformer or "balance coil" is connected across the terminals $b$ and $c$, and the single-phase potential is applied across

the terminal $a$ of the motor and the mid-point $d$ of the autotransformer. Since the actual autotransformer may be replaced by a perfect transformer and a series impedance, the circuit of Fig. 86$a$ is equivalent to that in Fig. 86$b$, in which the autotransformer is perfect and its resistance and leakage reactance are represented by the circuit $de$. The impedance $Z_{de}$ is the ratio of the potential $V_{bd}$ applied to the autotransformer to the current input, when the terminals $b$ $c$ are short-circuited. As in Fig. 86$a$ the single-phase potential is applied across the points $a$

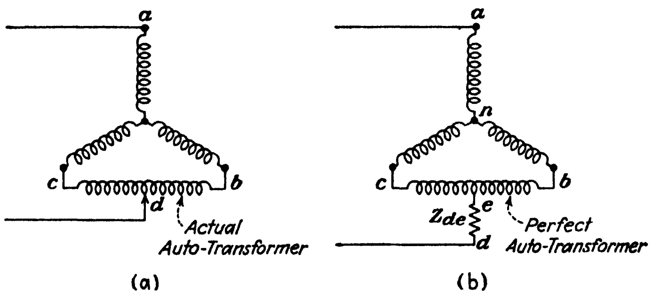



FIG 86

and $d$. Let us first determine the relation between the potential $V_{ae}$ and the positive- and negative-sequence components of the potential $V_{an}$.

$$V_{ae} = V_{an} + V_{nb} + \tfrac{1}{2}V_{bc}$$

Since $V_{bc} = V_{bn} - V_{cn}$ and since in both the positive- and negative-sequence systems $V_{bn} + V_{cn} = -V_{an}$,

$$V_{ae} = 1.5(V_{a1} + V_{a2}) \tag{278}$$

where $V_{a1}$ and $V_{a2}$ are the positive- and negative-sequence components of $V_{an}$. Furthermore, since $d$ is the mid-point of the autotransformer, the currents $I_{bd}$ and $I_{cd}$ are equal (exciting current neglected). That is to say, the currents $I_{bn}$ and $I_{cn}$ are equal, and are therefore each $-0.5I_{an}$. Under this condition, the positive- and negative-sequence components of $I_{an}$ are also equal [see page 35, Chap. I, noting that $m$, equation (11), equals $-0.5$].

Thus

$$I_{a1} = I_{a2}$$

If this condition is inserted in equation (278),

$$V_{ac} = 1.5I_{a1}(Z_1 + Z_2)$$

If to $V_{ac}$ is added the drop through the leakage impedance of the autotransformer,

$$V_{ad} = I_{a1}[1.5(Z_1 + Z_2) + 2Z_{de}]$$

or

$$I_{a1} = \frac{V_{ad}}{1.5(Z_1 + Z_2) + 2Z_{de}} \tag{279}$$

and, since $I_{a1} = I_{a2}$, the line current supplied to the motor is

$$I_a = \frac{V_{ad}}{0.75(Z_1 + Z_2) + Z_{de}} \tag{280}$$

**86. Single-phase Operation of Two-phase Motor.**—Apply a potential $V$ to the $a$ phase and leave the $b$ phase open. Let $I_1$ and $I_3$ be the positive- and negative-sequence components of $I_a$ and let $V_1$ and $V_3$ be the corresponding components of $V_a$. Then since there is no current in the $b$ phase,

$$I_b = -jI_1 + jI_3 = 0$$

and since

$$V_1 + V_3 = V$$

it follows that

$$I_1 = \frac{V}{(Z_1 + Z_3)}$$

Since the line current $I$ equals $2I$,

$$I = \frac{2V}{(Z_1 + Z_3)} \tag{281}$$

That is, the line current equals the line potential divided by one-half the sum of the impedance of the motor to positive sequence and the impedance of the motor to negative sequence.

**87. Three-phase Motor Supplied with Power through an Unsymmetrical Feeder.**—Let the zero-, positive-, and negative-sequence components of the line impedance in phase $a$ be $Z_0$, $Z_1$, and $Z_2$. Impress *balanced* potentials on the feeder and let $V_1$ be the impressed potential to neutral for phase $a$. Assume that the impedances of the motor to positive and negative

sequence are $Z_{m1}$ and $Z_{m2}$. The positive- and negative-sequence potential equations are

$$\left.\begin{aligned} V_1 &= I_1Z_0 + I_2Z_2 + I_1Z_{m1} \\ 0 &= I_2Z_0 + I_1Z_1 + I_2Z_{m2} \end{aligned}\right\} \tag{282}$$

Since the potentials impressed on the feeder are balanced, their negative-sequence component is zero. The first two terms on the right-hand side in each equation represent the potential drop in the feeder Solving these equations for the positive- and negative-sequence components of the current,

$$\left.\begin{aligned} I_1 &= \frac{V_1}{Z_0 + Z_{m1} - \dfrac{Z_1Z_2}{Z_0 + Z_{m2}}} \\ I_2 &= \frac{V_1}{Z_2 - \dfrac{(Z_0 + Z_{m1})(Z_0 + Z_{m2})}{Z_1}} \end{aligned}\right\} \tag{283}$$

The ratio of the negative- to the positive-sequence current is

$$\frac{I_2}{I_1} = -\frac{Z_1}{Z_0 + Z_{m2}} \tag{284}$$

Note that the degree of unbalance is independent of the load on the motor, since $Z_{m2}$ is practically independent of the speed of the motor.

**Illustration.**— Assume that the *a* phase of the feeder has a *resistance* of 0.15 unit, that the resistance of each of the other phases is zero, and that the feeder has no reactance. Also assume that the impedance of the motor to negative sequence is $0.20\underline{/78^\circ}$ unit. The zero- and positive-sequence components of the feeder resistance are each 0.05 unit. Thus the unbalance factor for current is

$$\left|\frac{I_2}{I_1}\right| = 0.232$$

There is another interesting case of this type, that of a 2-phase motor which receives power over a 2-phase, 3-wire circuit. Consider a 100-hp., 440-volt motor which is operating at the end of such a circuit. This motor has a full-load equivalent impedance of $3.99 + j1.70$ vector ohms per phase and a blocked

impedance of $0.303 + j0.720$ vector ohm per phase. Let the transmission circuit be the one whose constants were calculated on page 121. If balanced potentials, having a value of 440 volts, are applied to this circuit and the motor is operating at its normal full-load slip,

$$440 = I_1(0.502 + j0.330) + I_3(-0.137 + j0.194) + I_1(3.99 + j1.70)$$

$$0 = I_3(0.502 + j0.330) + I_1(0.137 - j0.194) + I_3(0.303 + j0.720)$$

The ratio of the negative- to the positive-sequence current is

$$\frac{I_3}{I_1} = -\frac{0.137 - j0.194}{0.805 + j1.05}$$

The magnitude of this ratio is 0.180. That is, the currents supplied to the motor are unbalanced 18.0 per cent. The solution for the positive-sequence current gives

$$I_1 = 82.0 - j37.2$$
$$= 90.0 \text{ amperes}$$

Thus

$$I_3 = 16.2 \text{ amperes}$$

The vector volt-ampere input to the motor is

$$P_0 + jQ_0 = 2(\mathcal{Z}_1|\,I_1\,|^2 + \mathcal{Z}_3|\,I_3\,|^2)$$
$$= 2[(3.99 - j1.70)90.0^2 + (0.303 - j0.720)16.2^2]$$
$$= 64.66 - j27.92 \quad (\text{kw.} - j \text{ kvar.})$$

Thus the motor operates at a power factor of 0.919. Had the motor been operating with *balanced* applied potentials and at the same speed, the power factor would have been 0.920.

The pulsating power input to the motor is

$$S_0 = 2|\,I_1\,| \times |\,I_3\,| \times |\,Z_1\,| \times |\,Z_3\,| \times |\,Y_1 + Y_3\,|$$
$$= 14.3 \text{ kw.}$$

Thus the maximum value of pulsation in power input to the motor is 22.2 per cent of the average power input.

**88. Three-phase Motor in Parallel with Unbalanced Load at the End of a Symmetrical Feeder.**—Let $Z_0$ be the impedance of the feeder per conductor. This is a zero-sequence quantity. Let $Y_{m1}$ and $Y_{m2}$ be the admittances of the motor to positive and

negative sequence. The admittance form is chosen since the motor is to be in parallel with the unbalanced load. This load consists of a delta combination of single-phase loads of such values that the zero-, positive-, and negative-sequence components of the admittance of the *b-c* branch are $Y_0$, $Y_1$, and $Y_2$. Also let $V_1$ and $V_2$ be the positive- and negative-sequence components of the star phase potential, $V_{an}$, at the load. We shall assume that the potentials applied to the feeder are balanced and that the star-phase potential from *a* to neutral is $E_1$. There is no negative-sequence component in these impressed potentials. The potential equations for phase *a* are

$$\begin{aligned} E_1 &= I_1 Z_0 + V_1 \\ 0 &= I_2 Z_0 + V_2 \end{aligned} \tag{285}$$

where $I_1$ and $I_2$ are the positive- and negative-sequence components of the line current $I_a$. This line current is the sum of the currents flowing to the motor and the unbalanced load. Thus by equation (63), page 86.

$$\begin{aligned} I_1 &= V_1 Y_{m1} + 3V_1 Y_0 - 3V_2 Y_2 \\ I_2 &= V_2 Y_{m2} + 3V_2 Y_0 - 3V_1 Y_1 \end{aligned} \tag{286}$$

Therefore

$$E_1 = V_1[1 + Z_0(Y_{m1} + 3Y_0)] - 3V_2 Z_0 Y_2 \tag{287}$$

and

$$0 = V_2[1 + Z_0(Y_{m2} + 3Y_0)] - 3V_1 Z_0 Y_1 \tag{288}$$

The unbalance factor for potential is

$$\frac{V_2}{V_1} = \frac{3Z_0 Y_1}{1 + Z_0(Y_{m2} + 3Y_0)} \tag{289}$$

The unbalance factor for potential is thus independent of the load on the motor since it does not depend upon its admittance to positive sequence. The numerator of this ratio [equation (289)] is fixed by the impedance of the feeder and by the positive-sequence component of the load admittance. The latter depends upon the load unbalance. The denominator depends upon the feeder impedance, the total admittance of all the loads in parallel and the admittance of the motor to negative sequence. The motor admittance is approximately proportional to the size of the motor. When the magnitude of the denominator [equation

(289)] is computed it will be observed that the effect of the motor on the unbalance factor is greatest when the angle of the feeder impedance equals the angle of the impedance of the motor to negative sequence. Furthermore, the denominator is larger for inductive than for condensive loads. The ratio of the unbalance factors for potential with and without the motor, assuming that the admittances of the loads do not change, is

$$\frac{1 + 3Z_0Y_0}{1 + 3Z_0Y_0 + Z_0Y_{m2}} \tag{290}$$

Note that, if a load having an admittance equal to the combined parallel admittance, $3Y_0$, of the actual loads were *equally* distributed in Δ on the three phases, the ratio of the potential drop in the feeder to the star-phase potential at the load would be $3Z_0Y_0$. Also, if $Y_{m1}$ is the full-load admittance of the motor per star phase, the ratio of the potential drop in the feeder, owing to the motor current alone, to the positive-sequence star-phase potential at the load is $Z_0Y_{m1}$. If the kilovolt-amperes taken by the actual load equals the kilovolt-ampere rating of the motor, these two potential drops, $3Z_0Y_0$ and $Z_0Y_{m1}$, are equal. Note, however, that $Z_0Y_{m2}$ is approximately five times as great as $Z_0Y_{m1}$

**Illustration.**—Let

$$Z_0 = 0.10/45^\circ \text{ unit}$$
$$3Y_0 = 1.0/\underline{-30^\circ} \text{ unit}$$
$$Y_{m2} = 5/\underline{-78^\circ} \text{ units}$$

$$\text{Ratio of unbalance factors} = \frac{1 + 0.10/\underline{15^\circ}}{1 + 0.10/\underline{15^\circ} + 0.5/\underline{-33^\circ}}$$
$$= \frac{1.097 + j0.026}{1.517 - j0.247}$$

Magnitude of ratio = 0.714

If the kilovolt-ampere capacity of the motor is twice as great, $Y_{m2}$ equals $10/\underline{-78^\circ}$ and the magnitude of the ratio (290) of the unbalance factors is 0.547. Thus, the addition of an induction motor in parallel with an unbalanced load will reduce the unbalance in the line potentials due to the effect of the unbalanced load. All polyphase induction motors connected to a distribution system reduce the unbalance in the line potentials caused by the

unsymmetrical arrangement of the connected single-phase loads. This balancing effect benefits the distribution system at the expense of the induction motors, for, as we shall see in Chap. VII (page 286), the heating losses in the motors are increased so that their kilovolt-ampere capacity is reduced. The solution of the foregoing equations (287) and (288) for the positive- and negative-sequence components of the star-phase potential, $V_{an}$, at the load gives

$$V_1 = \frac{1 + Z_0(3Y_0 + Y_{m2})}{[1 + Z_0(3Y_0 + Y_{m1})][1 + Z_0(3Y_0 + Y_{m2})] - 9Z_0^2Y_1Y_2}E_1 \tag{291}$$

and

$$V_2 = \frac{3Y_1Z_0}{[1 + Z_0(3Y_0 + Y_{m1})][1 + Z_0(3Y_0 + Y_{m2})] - 9Z_0^2Y_1Y_2}E_1 \tag{292}$$

**89. Effect of Short Circuit at Two of the Motor Terminals.**— Suppose that a single-phase load of admittance $Y$ is connected in parallel with the motor as described in the preceding section. The zero-, positive- and negative-sequence components of this admittance are each $Y/3$, and the equations for the positive- and negative-sequence components of the star-phase potential at the motor are

$$V_1 = \frac{1 + Z_0(Y + Y_{m2})}{1 + Z_0(2Y + Y_{m1} + Y_{m2}) + Z_0^2(Y(Y_{m1} + Y_{m2}) + Y_{m1}Y_{m2})}E_1 \tag{293}$$

and

$$V_2 = \frac{YZ_0}{1 + Z_0(2Y + Y_{m1} + Y_{m2}) + Z_0^2[Y(Y_{m1} + Y_{m2}) + Y_{m1}Y_{m2}]}E_1 \tag{294}$$

If the admittance, $Y$, of the single-phase load is now allowed to increase indefinitely, the condition of a single-phase short circuit at the motor terminals will be approached. The limiting values of $V_1$ and $V_2$ are

$$V_1 = V_2 = \frac{E_1}{2 + (Y_{m1} + Y_{m2})Z_0} \tag{295}$$

The following is an alternative method for solving this problem: Refer to Fig. 87. Let $I_a$ represent the line current flowing from $a$ to $b$ and $I_b$ the current in the motor flowing from $b$ to $n$. These two currents are *equal*, but the positive- and negative-sequence components of $I_a$ *do not equal* the corresponding components of $I_b$. The components of $I_a$ are determined by the three line currents, while the components of $I_b$ are determined by the currents in the stator windings of the induction motor. Owing to the current in the short circuit, the currents in the motor phases $c$ and $d$ are not the same as the currents in the corresponding phases of the feeder. Let $V_1$ and $V_2$ represent the positive- and negative-sequence components of the potential, $V_{bn}$, at the motor. Owing to the short circuit, these components of potential are equal, or

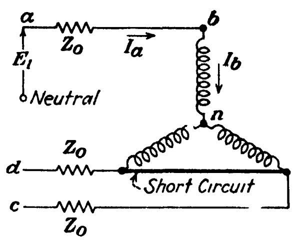



FIG. 87

$$V_1 = V_2$$

The potential equations applying to the circuit on the left of the short circuit are

$$\begin{aligned} E_1 - I_{a1}Z_0 &= V_1 \\ 0 - I_{a2}Z_0 &= V_2 \end{aligned} \qquad (296)$$

The potential equations applying to the circuit on the right of the short circuit, *i.e.*, to the motor alone, are

$$\begin{aligned} I_{b1} &= V_1Y_1 \\ I_{b2} &= V_2Y_2 \end{aligned}$$

Noting that $V_1 = V_2$ and that $I_{a1} + I_{a2} = I_{b1} + I_{b2}$, the equations for the positive- and negative-sequence components of the star-phase potential, $V_{bn}$, at the motor reduce to

$$V_1 = V_2 = \frac{E_1}{2 + (Y_{m1} + Y_{m2})Z_0} \qquad (297)$$

**Illustration.**—Assume the line impedance, $Z_0$, to be $0.15/45°$ unit and the admittances of the motor to positive and negative sequence to be $Y_{m1} = 1/\underline{-26°}$ (slip equals its full-load value,) and $Y_{m2} = 5/\underline{-78°}$. Thus

$$V_1 = V_2 = \frac{E_1}{2.77 - j0.36}$$

or

$$| V_1 | = 0.358 | E_1 |$$

The total vector volt-ampere input supplied to the motor is $3|V_1|^2(Y_{m1} + Y_{m2})$, or $3|E_1|^2(0.248 - j0.682)$. Before the short circuit occurs and for the full-load value of the slip, the vector volt-ampere input is $3|E_1|^2(0.688 - j0.335)$. Thus the short circuit reduces the input to the motor, for the same value of the slip, to 36 per cent of its former value. The power factor is reduced from 0.90 to 0.34 by the short circuit.

**90. Three-phase Motor Supplied from Two-phase Circuit through Scott-connected Transformers.**—This problem is particularly interesting for the reason that it involves the relations that exist between the symmetrical components of a 2-phase system and of a derived 3-phase system. The arrangement of the circuits is shown diagrammatically in Fig. 88. The phases of the 2-phase circuit are $a$ and $b$ while the terminals of the derived 3-phase circuit are $c$, $d$, and $e$. The transformer on the $b$ phase has a transformation ratio $k$, *i.e.*, $V_b = kV_{de}$ on open circuit. The transformer on the $a$ phase has a transformation ratio $\frac{2k}{\sqrt{3}}$, *i.e.*, $V_a = \frac{2k}{\sqrt{3}}V_{cf}$ on open circuit. The point $f$ is midway between $d$ and $e$ so that $V_{df} = V_{fe}$. For these ratios of transformation and this location of the point $f$, the 3-phase potentials on open circuit are balanced if the 2-phase potentials are also balanced. Furthermore, the 3-phase potential to neutral, $V_{cn}$, is two-thirds of the transformer potential, $V_{cf}$. The foregoing relations hold for both phase orders of the 2-phase potentials. Thus if $V_{a1}$ is the positive-sequence component of the 2-phase potential on open circuit, the corresponding component of the 3-phase potential to neutral, $V_{c1}$, is $V_{a1}/\sqrt{3}k$. Also, $V_{c2} = V_{a3}/\sqrt{3}k$. Note that the subscript 3 indicates a negative-sequence component in the 2-phase system while the subscript 2 indicates the negative-sequence component in the 3-phase system.

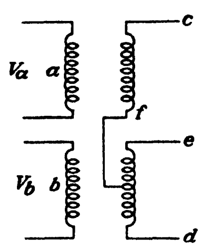


FIG. 88.

In analyzing the performance of power transformers, the exciting current is commonly neglected and the relation between the currents in the windings is determined by equating the ampere

turns on the primary side to the ampere-turns on the secondary side. Thus, for the $a$ transformer, the relation of the currents is

$$I_a = \frac{\sqrt{3}}{2k} I_c \underline{/0^\circ}$$

Since the $b$ transformer has two secondary windings, the relation of the currents is

$$I_b = \frac{1}{2k}(I_{df} + I_f)$$

When the 3-phase currents are balanced,

$$I_b = \frac{\sqrt{3}}{2k} I_{df} \underline{/\pm 30^\circ}$$

where the positive (+) sign is used for the positive-sequence phase order and the negative (−) sign, for the negative-sequence phase order.

Therefore, since balanced currents on the 3-phase side require balanced currents on the 2-phase side, the positive-sequence current $I_{c1}$ equals $\frac{2k}{\sqrt{3}} I_{a1}$. Similarly the negative-sequence current $I_{c2}$ equals $\frac{2k}{\sqrt{3}} I_{a2}$.

The general relations between the terminal potentials of the transformers on the 2-phase and 3-phase sides are expressed as follows

$$\left.\begin{aligned} V_a - I_a Z_a &= \frac{2k}{\sqrt{3}} V_{cf} \\ V_b - I_b Z_b &= 2k(V_{df} + I_{df} Z_{df}) \\ V_b - I_b Z_b &= 2k(V_{fe} + I_{fe} Z_{fe}) \end{aligned}\right\} \tag{298}$$

where $Z_a$ is the equivalent impedance of the $a$ transformer referred to the 2-phase side; $Z_b$ is impedance assigned to the 2-phase winding of the $b$ transformer, and is referred to the 2-phase side (see page 205). Also $Z_{df}$ and $Z_{fe}$ are the impedances assigned to the two halves of the 3-phase winding of the $b$ transformer and are referred to the 3-phase side. The impedance $Z_a$ is measured by applying potential to the $a$ winding and short-circuiting $c$-$f$. If $Z_{bd}$ is the impedance measured at the terminals of $b$ when $d$-$f$ is short-circuited, $Z_{be}$ is the impedance measured at the

terminals of $b$ when $f$-$c$ is short-circuited and $Z_{de}$ is the impedance measured at the terminals $d$-$f$ when $f$-$c$ is short-circuited, the following values of $Z_b$, $Z_d$, and $Z_e$ are obtained. [Refer to equations (247), page 206.]

$$Z_b = \tfrac{1}{2}(Z_{bd} + Z_{be} - 4k^2 Z_{de})$$
$$Z_d = \frac{1}{8k^2}(Z_{bd} - Z_{be} + 4k^2 Z_{de})$$
$$Z_e = \frac{1}{8k^2}(-Z_{bd} - Z_{be} + 4k^2 Z_{de})$$

The currents $I_a$ and $I_b$ are assumed to flow into the transformers in the directions in which *drops* in potential $V_a$ and $V_b$ are taken. The currents $I_{df}$ and $I_{fe}$ are assumed to flow in the 3-phase load circuit in the direction in which the drops in potential $V_{df}$ and $V_{fe}$ are taken. This convention is commonly used when the transformers are to be replaced by their equivalent networks, as we shall do.

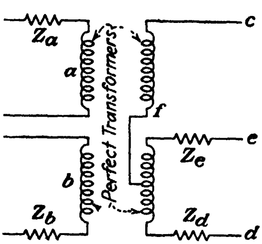


FIG. 89

An actual transformer is equivalent to the combination of a perfect transformer and a network of impedances which are determined by the core losses and the resistances and reactances of the transformer windings. If the exciting current is neglected, the branches of this network are combinations of the resistances and leakage reactances of the transformer windings. From this point of view the actual circuit diagram in Fig. 88 is equivalent to the circuit diagram in Fig. 89, in which the impedances and currents are as defined in equations (298). The terminal potentials on the primary sides of the perfect transformers are $(V_a - I_aZ_a)$ and $(V_b - I_bZ_b)$. The corresponding terminal potentials on the secondary sides are

$$V_{ef}, \quad (V_{df} + I_{df}Z_{df}), \qquad \text{and} \qquad (V_{fe} + I_{fe}Z_{fe})$$

The relation between the positive-sequence potentials on the 2-phase and 3-phase sides of the perfect transformers are the same as for the actual transformers on open circuit. Thus

$$V_{a1} - (I_aZ_a)_1 = k\sqrt{3}[V_{e1} + (I_eZ_e)_1]$$

If the drops in potential due to the leakage impedances are written in terms of their symmetrical components, this expression reduces to

$$V_{a1} - I_{a1}Z_{a0} - I_{a3}Z_{a2} = k\sqrt{3}(V_{c1} + I_{c1}Z_{c0} + I_{c2}Z_{c2}) \quad (299)$$

Similarly the negative-sequence equation is

$$V_{a3} - I_{a3}Z_{a0} - I_{a1}Z_{a2} = k\sqrt{3}(V_{c2} + I_{c2}Z_{c0} + I_{c1}Z_{c1}) \quad (300)$$

These equations can be written in better form for determining the positive- and negative-sequence components of the 3-phase currents by noting that

$$I_{a1} = \frac{\sqrt{3}}{2k}I_{c1} \qquad Z_{a0} = \frac{Z_a + Z_b}{2}$$

$$I_{a3} = \frac{\sqrt{3}}{2k}I_{c2} \qquad Z_{a2} = \frac{Z_a - Z_b}{2}$$

Also, if $Z_{df} = Z_{fe} = Z_d$, it follows that

$$Z_{c0} = \frac{2}{3}Z_d \qquad \text{and} \qquad Z_{c1} = Z_{c2} = -\frac{Z_d}{3}$$

Furthermore, if the impedances of the induction motor to positive and to negative sequence are $Z_{m1}$ and $Z_{m2}$

$$V_{c1} = I_{c1}Z_{m1} \qquad \text{and} \qquad V_{c2} = I_{c2}Z_{m2}$$

If these substitutions are made in equations (299) and (300),

$$\frac{V_{a1}}{k\sqrt{3}} = I_{c1}\left(\frac{Z_a + Z_b}{4k^2} + \frac{2}{3}Z_d + Z_{m1}\right) + I_{c2}\left(\frac{Z_a - Z_b}{4k^2} - \frac{Z_d}{3}\right) \quad (301)$$

and

$$\frac{V_{a3}}{k\sqrt{3}} = I_{c2}\left(\frac{Z_a + Z_b}{4k^2} + \frac{2}{3}Z_d + Z_{m2}\right) + I_{c1}\left(\frac{Z_a - Z_b}{4k^2} - \frac{Z_d}{3}\right) \quad (302)$$

These equations can readily be extended to include the impedances of the line conductors on both the 2-phase and 3-phase sides. It should be noted that, if the transformer impedances are adjusted so that $\frac{Z_a - Z_b}{4k^2} - \frac{Z_d}{3} = 0$, the transformers act like a symmetrical group, since the positive-sequence current is then proportional to the positive-sequence applied

potential and likewise the negative-sequence current is proportional to the negative-sequence applied potential.

This same conclusion in regard to symmetry will be reached if it is noted in Fig. 89 that the 3-phase side will be balanced if a portion of the equivalent impedance of transformer $a$ is transferred to the 3-phase side so that it will result in there being equal impedances in each of the 3-phase line conductors. Since the ratio of transformation of the $a$ transformer is $2k/\sqrt{3}$, there will remain on the 2-phase side an impedance $Z_a - \frac{4k^2}{3}Z_d$. If this resulting impedance is equal to $Z_b$, the 2-phase side will also be balanced and the transformers will act like a symmetrical group. This balanced condition can always be obtained by adding the proper amount of resistance and reactance to $Z_a$ and $Z_b$.

**91. Three-phase Phase Converter.**—We have already seen (page 238) that, when a 3-phase induction motor is running with but two of its terminals connected to the source of power, there exists across the three terminals of the motor 3-phase potentials which are but slightly out of balance. These potentials can then be used as a source of power to supply another 3-phase motor as indicated in Fig. 90. When so used, the first motor is known as a phase converter inasmuch as the single-phase applied potential is converted into 3-phase potentials. When the terminals of the phase converter and the motor are directly connected as in Fig. 90. the corresponding terminal potentials are the same. Thus, if $V_{a1}$ and $V_{a2}$ are the positive- and negative-sequence components of the potential, $V_a$, of the phase converter and $V_{b1}$ and $V_{b2}$ are the positive- and negative-sequence components of the potential, $V_b$, of the motor,

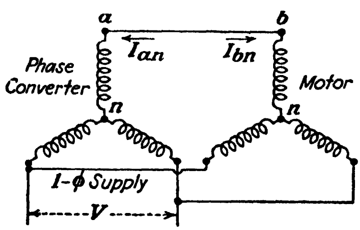



FIG. 90.

$$V_{a1} = V_{b1} = V_1$$
$$V_{a2} = V_{b2} = V_2$$

Let $Y_{a1}$ and $Y_{a2}$ be the admittances of the phase converter to positive and to negative sequence and $Y_{b1}$ and $Y_{b2}$ be the

corresponding admittances of the motor. Then the current $I_{an}$ in the phase converter is

$$I_{an} = V_1 Y_{a1} + V_2 Y_{a2}$$

Similarly the current $I_{bn}$ in the motor is

$$I_{bn} = V_1 Y_{b1} + V_2 Y_{b2}$$

The sum of these currents, $I_{an}$ and $I_{bn}$, is zero and thus

$$V_1(Y_{a1} + Y_{b1}) + V_2(Y_{a2} + Y_{b2}) = 0$$

This determines the degree of unbalance in the terminal potentials of the phase converter and motor. It is

$$\frac{V_2}{V_1} = -\frac{Y_{a1} + Y_{b1}}{Y_{a2} + Y_{b2}} \qquad (303)$$

Note that the degree of unbalance in the terminal potentials is reduced by increasing the admittance to negative sequence of either the phase converter or the motor. If the motor is disconnected from the phase converter, the degree of unbalance in the terminal potentials of the converter is

$$\frac{V_2}{V_1} = -\frac{Y_{a1}}{Y_{a2}}$$

If the phase converter is operated without shaft load, $Y_{a1}$ and $Y_{a2}$ are the no-load and blocked (approximately) admittance when operated as a motor with balanced applied potentials. If the motor has the same rating as the phase converter, its no-load and blocked admittance may be equal to $Y_{a1}$ and $Y_{a2}$. In this case the degree of unbalance in the terminal potentials of the phase converter, when the motor is operated without shaft load, is the same as when the motor is disconnected from the circuit. It will be observed that, as the shaft load on the motor is increased, that is, as $Y_{b1}$ increases, the degree of unbalance in the potentials increases. For example let,

$$\begin{aligned}
Y_{a1} &= 0.05 - j0.40 \text{ (no shaft load)}\\
Y_{a2} &= 0.9 - j4.5\\
Y_{b1} &= 0.05 - j0.40 \text{ (no shaft load)}\\
Y_{b1} &= 0.92 - j0.39 \text{ (full load)}\\
Y_{b2} &= 0.9 - j4.5
\end{aligned}$$

At no load the degree of unbalance in the potentials is

$$\left|\frac{V_2}{V_1}\right| = \left|\frac{0.05 - j0.40}{0.9 - j4.5}\right| = 0.088$$

At full load the degree of unbalance in the potentials is

$$\left|\frac{V_2}{V_1}\right| = \left|\frac{0.05 - j0.40 + 0.92 - j0.39}{2(0.9 - j4.5)}\right| = 0.136$$

The values of the positive- and negative-sequence components of the potential may now be determined by noting that the applied single-phase potential, $V$, is

$$V = -j\sqrt{3}V_1 + j\sqrt{3}V_2$$

Thus

$$V_1 = \frac{jV}{\sqrt{3}\left(1 - \frac{V_2}{V_1}\right)}$$

$$= j\frac{V}{\sqrt{3}}\frac{Y_{a2} + Y_{b2}}{Y_{a1} + Y_{b1} + Y_{a2} + Y_{b2}} \tag{304}$$

and

$$V_2 = -j\frac{V}{\sqrt{3}}\frac{Y_{a1} + Y_{b1}}{Y_{a1} + Y_{b1} + Y_{a2} + Y_{b2}} \tag{305}$$

The total vector volt-ampere input supplied to both the phase converter and the motor is

$$\begin{aligned}\text{Total }(\underline{V}I) &= 3[\underline{V}_1(I_{a1} + I_{b1}) + \underline{V}_2(I_{a2} + I_{b2})]\\ &= 3[\underline{V}_1V_1(Y_{a1} + Y_{b1}) + \underline{V}_2V_2(Y_{a2} + Y_{b2})]\end{aligned}$$

By using the preceding relations for $V_1$ and $V_2$ the expression for the vector volt-ampere input reduces to

$$\text{Total }(\underline{V}I) = |V|^2\frac{(Y_{a1} + Y_{b1})(Y_{a2} + Y_{b2})}{Y_{a1} + Y_{b1} + Y_{a2} + Y_{b2}} \tag{306}$$

For the representative values of the admittances that have been given at full load, this becomes

$$\begin{aligned}\text{Total }(\underline{V}I) &= |V|^2\frac{(0.05 - j0.40 + 0.92 - j0.39)2(0.9 - j4.5)}{0.05 - j0.40 + 0.92 - j0.39 + 1.8 - j9.0}\\ &= |V|^2(0.82 - j0.78)\end{aligned}$$

Thus the power factor of the single-phase input is 0.72. Notice that with balanced potentials applied to the motor its full-load power factor is 0.92.

**92. Methods of Balancing the Line Potentials of a Three-phase System.**—The line potentials become more nearly balanced as their negative-sequence components are reduced in magnitude. There are two general methods of effecting this reduction. In one the balancing device is connected across the line conductors and in the other it is connected in series with them. A polyphase induction motor connected to the circuit near an unbalanced load which is the cause of the unbalance in the line potentials is an example of the first method (see page 245). The impedance of the motor being much less to negative-sequence potentials than to positive-sequence potentials, the negative-sequence current demanded by the load is in part diverted from the transmission line and circulates between the motor and the load. As an extreme case, if it were possible to construct a motor having an impedance to negative sequence of zero, the negative-sequence potentials at the motor would be short-circuited. The entire negative-sequence current demanded by the load would then flow through the motor, so that the line currents supplied by the source of power would be balanced, *if* the generated emfs at the source of power were *balanced* and the line were *symmetrical.*

While it is impossible to reduce the impedance of the motor to negative sequence to zero, the negative-sequence component in the line potentials can be eliminated by connecting in series with the motor an alternating-current generator whose armature windings are connected so that their generated emfs are of negative-sequence phase order. If the magnitude and phase of these generated emfs are properly adjusted, they will so unbalance the terminal potentials of the motor that it will take a negative-sequence current equal and opposite to that required by the unbalanced load. That is, the negative-sequence current will circulate between the motor and the load and will thus be removed entirely from the line running to the source of power. If this line is *symmetrical* and the generated emfs at the source of power are *balanced,* the potentials across the line at the motor will be balanced.

It is possible to control automatically the magnitude and phase of the negative-sequence generated emfs so as always to maintain balanced line potentials. Since the negative-sequence generator carries both positive- and negative-sequence currents,

there will be both positive- and negative-sequence components in its terminal potentials. By properly adjusting the magnitude and phase of its excitation emf, the negative-sequence terminal potential can be made equal in magnitude and opposite in phase to the negative-sequence star potential of the motor. Under these conditions there will be no negative-sequence component in the line potentials at this point. If a synchronous motor is used in place of the induction motor, not only can the line potentials be balanced but the power factor can be improved.

The second general method of obtaining a balanced condition in the line potentials is by inserting the balancing device in

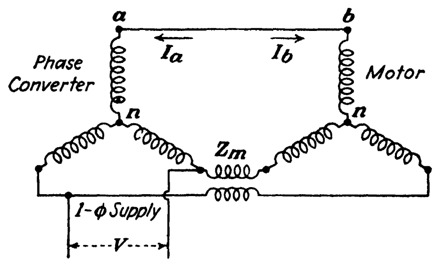



FIG. 91.

series with the line between the point at which the potentials are unbalanced and the point at which it is desired to maintain balanced potentials. The balancing device may be either an unsymmetrical arrangement of static impedances or a negative-sequence generator. In the latter case perfect balance can be obtained by adjusting the magnitude and phase of the negative-sequence potential introduced by the generator.

As an example of the effect of an unsymmetrical-static circuit in reducing the unbalance in the line potentials, consider the arrangement in Fig. 91.[1] The mutual impedance, $Z_m$, is connected between a phase converter and an induction motor. The connection is made so that the mutual reactance is negative. Let us first consider a general case in which there is an unsymmetrical arrangement of line impedances between the phase converter and the motor. The symmetrical components of these impedances are $Z_0$, $Z_1$, and $Z_2$. The admittances of the phase converter to positive and to negative sequence are $Y_{a1}$

[1] This method of reducing the unbalance is due to R. R. Lawrence.

and $Y_{a2}$ and of the motor, $Y_{b1}$ and $Y_{b2}$. The positive- and negative-sequence components of the current in the phase converter are $I_{a1}$ and $I_{a2}$, and the corresponding components of the current in the unsymmetrical line and the motor are $I_{b1}$ and $I_{b2}$. The Kirchhoff Equations which apply to this circuit are

$$\left.\begin{aligned} I_{a1} + I_{a2} + I_{b1} + I_{b2} &= 0 \\ V_{a1} &= I_{b1}Z_0 + I_{b2}Z_2 + V_{b1} \\ V_{a2} &= I_{b2}Z_0 + I_{b1}Z_1 + V_{b2} \end{aligned}\right\} \qquad (307)$$

Single-phase applied potential $V = -j\sqrt{3}V_{a1} + j\sqrt{3}V_{a2}$. We may also write

$$\begin{aligned} I_{a1} &= V_{a1}Y_{a1} & \qquad I_{b1} &= V_{b1}Y_{b1} \\ I_{a2} &= V_{a2}Y_{a2} & \qquad I_{b2} &= V_{b2}Y_{b2} \end{aligned}$$

Substitute the values of $I_{a1}$, $I_{a2}$, $I_{b1}$, and $I_{b2}$ given in the last four relations in each of the first three equations.

Eliminate $V_{a1}$ and $V_{a2}$ from the first three equations and obtain the ratio of the negative-sequence to the positive-sequence component of the potential applied to the motor. Thus

$$\frac{V_{b2}}{V_{b1}} = -\frac{Y_{a1} + Y_{b1} + Y_{b1}(Y_{a1}Z_0 + Y_{a2}Z_1)}{Y_{a2} + Y_{b2} + Y_{b2}(Y_{a2}Z_0 + Y_{a1}Z_2)} \qquad (308)$$

By combining the second, third, and fourth equations

$$j\frac{V}{\sqrt{3}} = V_{b1}[1 + Y_{b1}(Z_0 - Z_1)] - V_{b2}(1 + Y_{b2}(Z_0 - Z_2)) \qquad (309)$$

The values of $V_{b1}$ and $V_{b2}$ can be obtained from equations (308) and (309).

We shall now determine the effect on the unbalance factor, $V_2/V_1$, of inserting the mutual reactance $Z_m$, as in Fig. 91. In order to simplify the expressions let us assume that $Z_m$ has a negligible amount of resistance in the coils and a negligible amount of magnetic leakage, or the self- and mutual inductances are equal. Under these assumptions, the zero-, positive-, and negative-sequence components of the self-impedance between the points $a$ and $b$ are $j\frac{2x}{3}$, $-j\frac{x}{3}$, and $-j\frac{x}{3}$, where $x$ is the self-reactance of one coil. If the coils are connected in the circuit so that their mutual reactance is negative, the zero-, positive-, and negative-sequence components of the mutual reactance

between the other two lines are $-j\frac{x}{3}$, $-j\frac{x}{3}$, and $-j\frac{x}{3}$. The positive-sequence component of the line drop from $a$ to $b$ is

$$j\frac{2x}{3}I_{b1} - j\frac{x}{3}I_{b2} + j\frac{x}{3}I_{b1} - j\frac{2x}{3}I_{b2} = jxI_{b1} - jxI_{b2}$$

The first two terms are due to self-reactance and the last two terms are due to mutual reactance (see page 89). Similarly the negative-sequence component of the line drop from $a$ to $b$ is

$$j\frac{2x}{3}I_{b2} - j\frac{x}{3}I_{b1} + j\frac{x}{3}I_{b2} - j\frac{2x}{3}I_{b1} = jxI_{b2} - jxI_{b1}$$

The unbalance factor for the motor can be written

$$\frac{V_{b2}}{V_{b1}} = -\frac{Y_{a1} + Y_{b1} + Y_{b1}(jxY_{a1} - jxY_{a2})}{Y_{a2} + Y_{b2} + Y_{b2}(jxY_{a2} - jxY_{a1})} \tag{310}$$

At this point it is interesting to note that, if a condenser of reactance $(-j3x)$ is connected in the line between the points $a$ and $b$, the unbalance factor for the motor is

$$\frac{V_2}{V_1} = -\frac{Y_{a1} + Y_{b1} - jxY_{b1}(Y_{a1} + Y_{a2})}{Y_{a2} + Y_{b2} - jxY_{b2}(Y_{a1} + Y_{a2})} \tag{311}$$

**Illustration.**—If the representative numerical values of the admittances given on page 253 are inserted in the expression for the unbalance factor,

$$\frac{V_2}{V_1} = -\frac{0.97 - j0.79 - jx(0.92 - j0.39)(0.85 - j4.1)}{1.8 - j9.0 + jx(0.9 - j4.5)(0.85 - j4.1)}$$

The square of the magnitude of the unbalance factor is

$$\left|\frac{V_2}{V_1}\right|^2 = \frac{(0.97 - 4.1x)^2 + (-0.79 - 0.82x)^2}{(1.8 + 7.52x)^2 + (-9.0 + 17.7x)^2}$$
$$= \frac{1.57 - 9.25x + 17.5x^2}{84.2 + 345x + 369x^2}$$

The minimum value of this fraction is 0.00163 and occurs when $x = 0.29$; that is the product of the self-reactance of one of the coils and the full-load current is 0.29 of the rated star potential. The magnitude of the unbalance factor is $\sqrt{0.00163}$ or 0.040. If the mutual reactance is removed, the magnitude of the

unbalance factor becomes 0.136[1] or about three times as great as when the mutual reactance is present.

With a condenser of reactance $3x$ inserted in the line between $a$ and $b$, the square of the magnitude of the unbalance factor is

$$\left|\frac{V_2}{V_1}\right|^2 = \frac{1.57 - 11.11x + 24.95x^2}{84.2 - 412.9x + 525x^2}$$

The minimum value of this fraction is 0.0135 and occurs when $x = 0.157$. Thus the condenser has a reactance of 0.47 unit. The magnitude of the unbalance factor is now $\sqrt{0.0135}$ or 0.116. That is, the condenser can produce a maximum reduction in the unbalance factor of 15 per cent, while the mutual reactance can produce a maximum reduction of 71 per cent.

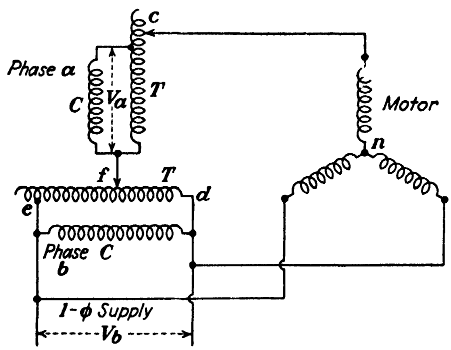



FIG. 92.

**93. Two-phase Phase Converter.**—A 2-phase phase converter can be arranged so as to provide balanced potentials for the motor as in Fig. 92.

The phase converter has symmetrical 2-phase stator windings $C$-$C$ which have autotransformers $T$-$T$ connected across them. The rotor may have a symmetrical 2-phase or 3-phase winding or may have one of the usual cage type. The tap point on the autotransformer that is connected across the $a$ phase is located so that the potential $V_{cf}$ is $k_1V_a$ at no load, where $V_a$ is the potential across the $a$ phase of the phase converter. The tap point $f$ on the autotransformer that is connected across the $b$ phase of the phase converter is located so that the potential $V_{df}$ is $k_2V_b$ at no load, where $V_b$ is the potential across the $b$ phase

[1] See p. 254.

of the phase converter. This potential $V_b$ is the same as that of the single-phase supply. We have already seen (page 190) that an autotransformer is equivalent to the combination of a perfect transformer and a series impedance which represents the

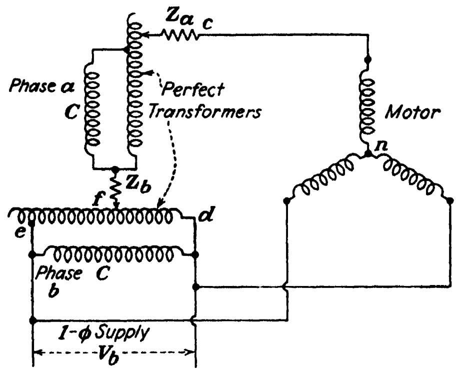



FIG. 93.

resistance and leakage reactance of the autotransformer. Thus the autotransformers in Fig. 92 can be replaced by perfect transformers and impedances $Z_a$ and $Z_b$ as in Fig. 93. Furthermore, notice that the currents in $Z_a$ and $Z_b$ are the same and that the potentials $V_{cd}$ and $V_{ce}$ would be the same if the impedance $Z_b$

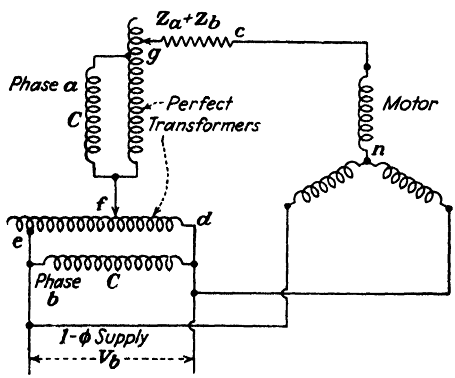



FIG. 94.

were connected directly in series with $Z_a$ as in Fig. 94. In Fig. 94 the transformers are perfect, that is, are without losses or leakage. The combined leakage impedances of the transformers can now be considered as a portion of the 3-phase trans-

mission circuit between the phase converter and the motor. Since there is no impedance in either of the other conductors, the zero-, positive-, and negative-sequence components of the impedance of the $c$ phase of this 3-phase line are each $(Z_a + Z_b)/3$.

We shall now determine the positive and negative-sequence components of the 3-phase star potentials measured from the terminals $g$, $d$, and $e$. The positive-sequence component of the star potential $V_g$ is [see equation (33), page 53]

$$V_{g1} = \tfrac{1}{3}(V_{gd} + V_{de}\underline{/60°})$$

If we adopt the conventions that $k_1V_a = V_{gf}$[1] and $k_2V_b = V_{df}$, it follows that $V_{gd} = k_1V_a - k_2V_b$. Thus

$$V_{g1} = \tfrac{1}{3}(k_1V_a - k_2V_b + V_b\underline{/60°})$$

Since $V_a = V_{a1} + V_{a3}$ and $V_b = -jV_{a1} + jV_{a3}$, where $V_{a1}$ and $V_{a3}$ are the positive- and negative-sequence components of the potential $V_a$ across the $a$ phase of the phase converter,

$$\begin{aligned} V_{g1} &= \tfrac{1}{3}[(V_{a1} + V_{a3})k_1 + (-jV_{a1} + jV_{a3})(-k_2 + 0.5 + j0.866)] \\ &= \tfrac{1}{3}V_{a1}[k_1 + 0.866 + j(k_2 - 0.5)] \\ &\qquad + \tfrac{1}{3}V_{a3}[k_1 - 0.866 - j(k_2 - 0.5)] \end{aligned}$$

Similarly, the negative-sequence component of the star potential, $V_g$, is

$$\begin{aligned} V_{g2} &= \tfrac{1}{3}(V_{gd} + V_{de}\underline{/-60°}) \\ &= \tfrac{1}{3}V_{a3}[k_1 + 0.866 - j(k_2 - 0.5)] \\ &\qquad + \tfrac{1}{3}V_{a1}[k_1 - 0.866 + j(k_2 - 0.5)] \end{aligned}$$

These expressions for the positive- and negative-sequence components can be simplified in form by writing $K$ for

$$[k_1 + 0.866 + j(k_2 - 0.5)]$$

The expressions then become

$$\begin{aligned} V_{g1} &= \tfrac{1}{3}[V_{a1}K + V_{a3}(\hat{K} - \sqrt{3})] \\ V_{g2} &= \tfrac{1}{3}[V_{a3}\hat{K} + V_{a1}(K - \sqrt{3})] \end{aligned} \tag{312}$$

We have thus determined the positive- and negative-sequence components of the 3-phase star potential in terms of the cor-

[1] At no-load $V_{gf}$ and $V_{ef}$ are identical since there is then no current in $Z_a$ and $Z_b$.

responding components of the 2-phase potentials and the positions of the tap points on the autotransformers. Note that it is only when $k_1 = 0.866$ and $k_2 = 0.5$ that the 2-phase and 3-phase potentials are simultaneously balanced. We can now write the positive- and negative-sequence potential equations that connect the phase converter and the motor.

$$V_{g1} = I_{c1}Z_0 + I_{c2}Z_2 + V_{c1}$$

and

$$V_{g2} = I_{c2}Z_0 + I_{c1}Z_1 + V_{c2}$$

But

$$Z_0 = Z_1 = Z_2 = \frac{Z_a + Z_b}{3}$$

$$I_{c1} = V_{c1}V_{c1}$$

$$I_{c2} = V_{c2}Y_{c2}$$

where $Y_{c1}$ and $Y_{c2}$ are the admittances of the motor to positive and negative sequence. In order to simplify the nomenclature, let

$$\frac{Z_a + Z_b}{3} = Z$$

Then

$$V_{g1} = V_{c1}(1 + ZY_{c1}) + V_{c2}ZY_{c2}$$

$$V_{g2} = V_{c2}(1 + ZY_{c2}) + V_{c1}ZY_{c1}$$

We shall now write the current equation that applies to the $a$ phase of the phase converter and the $c$ phase of the motor. The current in the phase converter flowing toward the point $f$ is $V_{a1}Y_{a1} + V_{a3}Y_{a3}$, where $Y_{a1}$ and $Y_{a3}$ are the admittances of the phase converter to positive sequence and to negative sequence. Thus current flowing into the autotransformer at the point $g$ is $\frac{1}{k_1}(V_{a1}Y_{a1} + V_{a3}Y_{a3})$. The current flowing into the motor at the point $c$ is $V_{c1}Y_{c1} + V_{c2}Y_{c2}$. Since the sum of the current flowing into the autotransformer at $g$ and the current flowing into the motor at $c$ is zero, the current equation is

$$\frac{1}{k_1}(V_{a1}Y_{a1} + V_{a3}Y_{a3}) + V_{c1}Y_{c1} + V_{c2}Y_{c2} = 0$$

Therefore, the necessary Kirchhoff Equations may be written in the following form

$$\left.\begin{aligned} \tfrac{1}{3}[V_{a1}\underline{K} + V_{a3}(\underline{K} - \sqrt{3})] &= V_{c1}(1 + ZY_{c1}) + V_{c2}ZY_{c2} \\ \tfrac{1}{3}[V_{a3}\underline{K} + V_{a1}(K - \sqrt{3})] &= V_{c2}(1 + ZY_{c2}) + V_{c1}ZY_{c1} \\ V_{a1}Y_{a1} + V_{a3}Y_{a3} + k_1(V_{c1}Y_{c1} + V_{c2}Y_{c2}) &= 0 \end{aligned}\right\} \quad (313)$$

$$V = -jV_{a1} + jV_{a3}$$

where $V$ is the single-phase applied potential.

When, as in this case, there is no impedance between the single-phase source and the motor terminals $d$ and $e$, we may also write

$$V = -j\sqrt{3}V_{c1} + j\sqrt{3}V_{c2}$$

These Kirchhoff equations are sufficient to determine the positive- and negative-sequence components of the potentials of the phase converter and motor in terms of the admittances of the phase converter and motor to positive and negative sequence, the leakage impedances of the autotransformers, and the positions of the tap points on these transformers. The particular problem we shall solve is to determine these tap points so that the potentials applied to the motor will be balanced, or so that $V_{c2} = 0$. For this condition the general equations become

$$\left.\begin{aligned} \tfrac{1}{3}[V_{a1}K + V_{a3}(\underline{K} - \sqrt{3})] &= V_{c1}(1 + ZY_{c1}) \\ \tfrac{1}{3}[V_{a3}\underline{K} + V_{a1}(K - \sqrt{3})] &= V_{c1}ZY_{c1} \\ V_{a1}Y_{a1} + V_{a3}Y_{a3} + k_1V_{c1}Y_{c1} &= 0 \\ V = -jV_{a1} + jV_{a3} &= -j\sqrt{3}V_{c1} \end{aligned}\right\} \quad (314)$$

From the last equation $V_{a3} = V_{a1} - \sqrt{3}V_{c1}$. If this value of $V_{a3}$ is substituted in the second and third equations and if the ratios of $V_{a1}$ to $V_{c1}$ as then determined are equated, the following relation between the circuit constants is obtained

$$\frac{2}{\sqrt{3}}\frac{Y_{c1}}{Y_{a1} + Y_{a3}}k_1^2 + \frac{Y_{a1} - Y_{a3}}{Y_{a1} + Y_{a3}}k_1 + \sqrt{3}ZY_{c1} + \frac{\sqrt{3}}{2} - j(k_2 - 0.5) = 0 \quad (315)$$

The general solution of this equation for $k_1$ and $k_2$, in terms of the admittances of the phase converter and motor and the impedances of the transformer, is too complex to write. However, we shall, determine the numerical values of $k_1$ and $k_2$ for representative values of the admittances and impedances, both for the condition of no shaft load on the motor and for rated

load. First, it should be noted that the impedance of an auto-transformer depends upon the position of the tap point with respect to the transformer terminals. Thus the value of the impedance $Z$ cannot be determined until both $k_1$ and $k_2$ are known. Since $Z$ is but one-third the sum of the transformer impedances, it is relatively unimportant and as a first approximation might be omitted. Its value can be determined from this approximate solution and then used in recomputing more exact values of $k_1$ and $k_2$.

Let us assume that the preliminary solution indicates a value of $Z$ of $0.03\underline{/75°}$, and let us use the representative values of the admittances as given on page 253. It is first necessary to compare the per unit values of admittance for the phase converter and motor. If their ratings are equal when operating under balanced conditions, we have $2E_aI_a = 3E_cI_c$ and, since their terminal potentials are equal, $E_a = \sqrt{3}E_c$, from which it follows that $I_a = \frac{\sqrt{3}}{2}I_c$. Therefore, the full-load admittance of the 2-phase phase converter is one-half that of the 3-phase motor. If the no-load and blocked admittances are proportional to the full-load admittances, the representative values of the admittances of the phase converter and motor to positive and negative sequence are (see page 253)

$$Y_{a1} = 0.025 - j0.20 \qquad Y_{c1} = 0.92 - j0.39 \text{ (full load)}$$
$$Y_{a3} = 0.45 - j2.25 \qquad Y_{c1} = 0.05 - j0.40 \text{ (no load)}$$
$$Y_{c2} = 0.90 - j4.50$$

If these values (using the full-load value of $Y_{c1}$) are substituted in equation (315),

$$(0.258 + j0.383)k_1^2 - (0.84 + j0.011)k_1 + 0.897 - jk_2 + j0.541 = 0$$

Consider first the real portion of this equation which does not include $k_2$.

$$0.258k_1^2 - 0.84k_1 + 0.897 = 0$$

The roots are

$$\frac{-0.84 \pm \sqrt{0.84^2 - 4 \times 0.258 \times 0.897}}{0.516}$$

These roots are complex and hence it is impossible to attain perfect balance at full load with a converter and motor having the assumed values of the constants. The roots can be made real either by increasing the coefficient of $k_1$ or by reducing the coefficient of $k_1^2$ in equation (315). Inspection of this equation shows that the coefficient of $k_1$ can be slightly increased by increasing the ratio of the blocked admittance to the no-load admittance of the phase converter. The maximum value of this coefficient is in the neighborhood of 0.9. However, with this value of the coefficient the roots of the equation are complex. Hence, in order to accomplish our purpose, it is necessary to reduce the coefficient of $k_1^2$. This can be done by increasing either or both the no-load and blocked admittances of the phase converter. If, for example, the blocked resistance and reactance of the phase converter are reduced to 4 per cent and 15 per cent (converter base), the blocked (or negative-sequence) admittance becomes $Y_{a3} = 0.83 - j3.11$ motor base. If this change alone is made in equation (315),

$$(0.205 + j0.268)k_1^2 - (0.88 + j0.015)k_1 + 0.897 - jk_2 + j0.541 = 0$$

The solution for $k_1$ is now

$$k_1 = \frac{0.88 \pm \sqrt{0.88^2 - 4 \times 0.205 \times 0.897}}{0.41}$$
$$= 1.67 \text{ (discard the larger root)}$$

The corresponding value of $k_2$ is 1.26.

Since the rating of the phase converter was based on a value of $k_1$ of 0.866, the current in its $a$ phase is nearly twice its full-load value. In order to determine the current in the $b$ phase of the phase converter, it is first necessary to compute the positive- and negative-sequence components of the current in the $a$ phase. By combining the second and third equations of (314), the ratio of these components is

$$\frac{I_{a3}}{I_{a1}} = -\frac{\dfrac{k_1(K - \sqrt{3})}{Y_{a1}} + 3Z}{\dfrac{k_1 \underline{K}}{Y_{a3}} + 3Z}$$

If the transformer impedances, $3Z$, are neglected, this ratio $I_{a3}/I_{a1}$ is $-2.54 - j6.2$, from which it follows that

$$I_{a1} = -1.54 - j6.2$$

Since $I_b = -jI_{a1} + jI_{a3}$,

$$I_b = \frac{(6.2 - j3.54)I_a}{-1.54 - j6.2}$$

Thus both phases of the phase converter carry about twice their rated current. If, however, the capacity of the phase converter is increased to meet this demand, both the no-load and blocked admittances will be greater since they are proportional to the

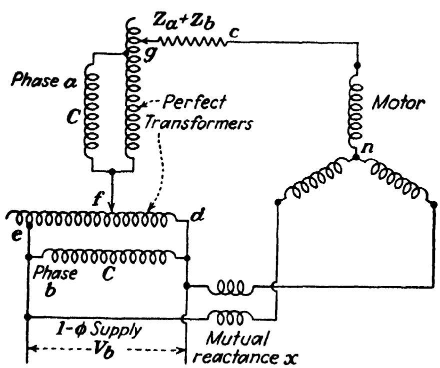



FIG. 95.

rating of the converter if their per unit values are maintained constant. A further increase in these admittances will reduce the values of $k_1$ and $k_2$, which will in turn reduce the currents taken by the converter. This reaction makes it unnecessary to double the capacity of the converter. Furthermore, it may be advisable to insert a mutual reactance between the $d$ and $e$ phases of the motor, as in Fig. 95, so as to effect a partial balance, in which case a converter of smaller capacity can be used. If this mutual reactance, $x$, has negligible resistance and no leakage, and is connected so that its sign is negative, the Kirchhoff equations become

$$\left.\begin{aligned} V_{a1}K + V_{a3}(\underline{K} - \sqrt{3}) &= 3V_{c1}[1 + (Z + jx)Y_{c1}] \\ V_{a3}\underline{K} + V_{a1}(K - \sqrt{3}) &= 3V_{c1}(Z - jx)Y_{c1} \\ V_{a1}Y_{a1} + V_{a3}Y_{a3} + k_1V_{c1}Y_{c1} &= 0 \end{aligned}\right\} \quad (316)$$

Eliminating the potentials from these equations gives the following equation for $k_1$ and $k_2$

$$\frac{2}{\sqrt{3}} \frac{Y_{c1}}{(Y_{a1} + Y_{a3})(1 + j2xY_{c1})} k_1^2 + \frac{(Y_{a1} - Y_{a3})}{Y_{a1} + Y_{a3}} k_1 + \frac{\sqrt{3}(Z - jx)Y_{c1}}{1 + j2xY_{c1}} + \frac{\sqrt{3}}{2} - j(k_2 - 0.5) = 0 \quad (317)$$

Compare this equation with equation (315) and note the effect of the mutual reactance. With the same values of admittances and transformer impedances as were used in the last computation and with a mutual reactance of 0.20, the values of $k_1$ and $k_2$ when the motor delivers its rated load are

$$k_1 = 1.17$$
$$k_2 = 0.51$$

With no load on the motor, or $Y_{c1} = 0.05 - j0.40$, the necessary values of $k_1$ and $k_2$ for obtaining balanced potentials are

$$k_1 = 0.67 \qquad k_2 = 0.48$$

Note that it is necessary to move the tap point on the *a* transformer over a much greater range than the tap point on the *b* transformer in order to obtain perfect balance.

# CHAPTER VII

# INDUCTION MACHINES

## INTERNAL REACTIONS

In Chap. VI the distribution of potential and current is determined when symmetrically wound induction machines are operated in connection with polyphase circuits that are in some respect unsymmetrical. Insofar as the induction machine itself is concerned, the analysis does not penetrate beyond the terminals and the solution always depends upon the impedance that the machine offers to positive- and to negative-sequence current. It is shown how these impedances can be determined from measurements made at the terminals of the machine, but no inquiry is made as to what relation exists between the impedances and the fundamental electrical properties of the stator and rotor windings. Except for these two terminal impedances, the machine itself was looked upon virtually as an unknown device. The next step in the analysis is to investigate the electrical reactions within the machine and the relation that the terminal impedances bear to the electrical characteristics of the windings and the speed of the rotor. In the present chapter we shall consider the case in which both the stator and the rotor windings are symmetrical, while in Chap. VIII the analysis will be extended to include cases in which either the stator or rotor winding is unsymmetrical. For the present, we shall assume that the unsymmetry which exists on either the stator or rotor side lies wholly outside the machine. It is important to realize that, in general, the method of symmetrical components cannot properly be applied if there is unsymmetry on both the stator and rotor sides.[1] This is discussed in Art. 99.

Since it is best not to encumber the analysis with relatively less important details, we shall disregard all time and space harmonics and shall assume that the potentials and currents vary sinusoidally with time and that the radial density of the air-gap flux within the machine varies sinusoidally in space

[1] See footnote, p. 276.

around the periphery of the rotor. We shall also assume that the air gap is uniform so that the flux within it is entirely independent of the relative position of the stator and rotor. The foregoing assumptions are always made in the usual analysis of the induction machine.

**94. Rotating Magnetic Field.**—In order to obtain a thorough understanding of the operation of either the induction machine or the synchronous machine, it is essential to have a clear conception of the production of a rotating magnetic field and the effects produced by it. To this end let us consider a single set of coils, one coil per pole, of unit pitch, arranged around the periphery, as in Fig. 96, and connected in series. We shall measure the distance $x$ in electrical radians from an origin at one of the coil sides to the point at which the flux density $B$ is desired. If $k_1$ is the constant of proportionality between the maximum value of the radial flux density in the air gap and the value of the instantaneous current in the coils,

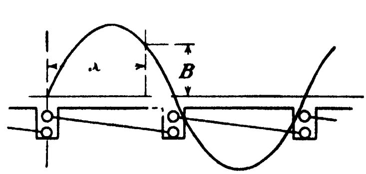


FIG. 96.

$$B = k_1\sqrt{2}I_1 \sin x \sin \omega t \tag{318}$$

where $I_1$ is the ampere or root-mean-square value of the coil current. This expression shows that, although the flux density varies sinusoidally both in space and in time, the field is a pulsating one which does not change its *position* in the air gap. The points of zero flux density always coincide with the coil sides. By applying a well-known trigonometrical transformation, the expression for the flux density becomes

$$B = k_1 \frac{I_1}{\sqrt{2}}[\cos(x - \omega t) - \cos(x + \omega t)] \tag{319}$$

The two components in equation (319) can be interpreted as rotating fields of the same constant magnitude, moving oppositely around the air gap at a speed proportional to the frequency. Each component is one-half the maximum value of the *actual* field. Thus a sinusoidally pulsating magnetic field is equivalent to two equal but oppositely rotating fields of constant magnitude. This method of analyzing a pulsating magnetic field was first

used by Ferraris to explain the operation of a single-phase induction motor.

An important generalization can be made. If there are several sets of coils, similar to the set just described, arranged about the air gap and carrying currents of the same frequency, each set will produce two oppositely rotating magnetic fields. Those which move in the positive direction are of *constant* magnitude and rotate at the *same* speed. Consequently, they combine to form a single field of constant magnitude moving at constant speed in the positive direction. Similarly, the oppositely rotating components combine to form a single field of constant magnitude which moves at constant speed in the negative direction. Whenever sinusoidal currents of the same frequency flow in a polyphase group of coils, each phase of which is arranged as in Fig. 96, there will be produced in the air gap a flux distribution consisting, in general, of but two components, each of which has a constant magnitude and both of which rotate at the same speed, the one *positively* and the other *negatively* with respect to the windings. If the two component fields have the same magnitude, the air-gap flux will pulsate sinusoidally without changing its position. If one field is zero, the air-gap flux has a constant magnitude and rotates at a constant speed. In general, if both component fields are present, the air-gap flux has a variable magnitude and rotates at a variable speed. The magnitudes of these two component fields can be controlled at will by adjusting the magnitude and relative time-phase of each of the currents. For example, consider, in addition to the coils indicated in Fig. 96, a second set of coils similarly arranged around the periphery and displaced by an angle $\alpha$ in the positive direction from the first set of coils. Let the current in the second set of coils have a value $I_2$ and lag the current in the first set of coils by an angle $\theta$. The flux density at the point $x$ radians from the origin, due to the second set of coils, is

$$B = k_2\sqrt{2}I_2 \sin(x - \alpha) \sin(\omega t - \theta)$$

or

$$B = \frac{k_2 I_2}{\sqrt{2}} [\cos(x - \omega t - \alpha + \theta) - \cos(x + \omega t - \alpha - \theta)] \quad (320)$$

Notice that the positively rotating field due to the second set of coils leads the corresponding field due to the first set of coils by

the angle $(\alpha - \theta)$, while the negatively rotating field due to the second set of coils lags the corresponding field due to the first set of coils by the angle $(\alpha + \theta)$. The space angle between the fields that are traveling in the same direction is either the difference or the sum of the space angle, $\alpha$, between the sets of coils and the time angle, $\theta$, between the currents in the coils. If the ampere-turns per pole are the same for each set of coils, that is, if $k_1I_1$ equals $k_2I_2$ and if $(\alpha + \theta)$ equals $\pi$, the two component fields that are rotating in the negative direction will be equal and in space opposition so that their sum is zero. In this case the resultant field in the gap will consist of the two positively rotating components only and will thus have a constant magnitude. If the current in the second set of coils leads the current in the first set of coils by an angle $(\pi - \alpha)$, the resultant field will be constant but will rotate in the negative direction. This result suggests the method that is employed in Chap. VIII for analyzing the performance of a motor having an unsymmetrical stator winding. In the usual form of 2-phase motor, the windings of each phase have the same number of turns per pole and are displaced in space phase by 90 degrees $(\alpha = \pi/2)$. If the phase currents are equal and are displaced in time-phase by 90 degrees $(\theta = \pm\pi/2)$, the air-gap flux will be constant and will rotate in either the positive or negative direction depending upon whether the current in the second phase lags or leads the current in the first phase. If the windings are displaced by 90 degrees but do not have the same number of turns, a single rotating field will be produced if the currents are displaced by 90 degrees and have such magnitudes that the ampere-turns per phase are equal (see Art. 119, page 345). If the ampere-turns per pole for the two sets of coils are not equal, or if the phase-displacement angles $(\theta - \alpha)$ and $(\theta + \alpha)$ are not equal to $\pi$, the air-gap flux will consist in part of a constant field that rotates in the positive direction and in part of a constant field that rotates in the negative direction.

In the usual types of 3-phase motors, the windings of each phase have the same number of turns and are displaced 120 degrees in space. If the currents are equal and displaced 120 degrees in time-phase, the field will be constant and will rotate in either the positive or negative direction, depending upon the phase order of the currents. As in the 2-phase motor, it is

always possible to choose the magnitudes and time-phase relations of the currents so that a single rotating field will be produced whatever may be the turns per phase and the relative positions of the phases.

It is sometimes convenient to represent these rotating fields by rotating vectors, whose lengths represent the magnitudes of the fields and whose positions indicate the points in the air gap at which the flux densities are maximum. Thus in Fig. 97,

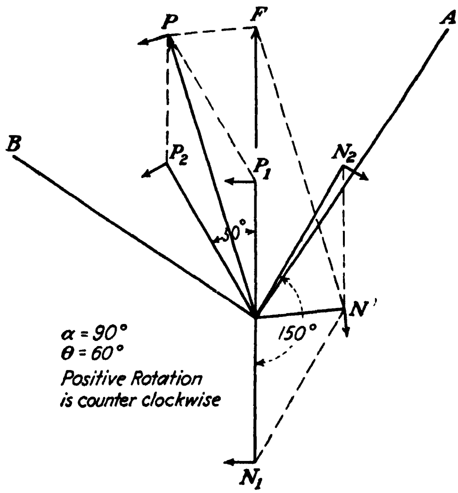



FIG. 97.

$P_1$ and $P_2$ represent the positively rotating component fields and $N_1$ and $N_2$ the negatively rotating component fields due to the currents in the first set and second set of coils at the time $t$ equals zero. Refer to equations (319) and (320). In Fig. 97 the vectors are drawn so that their sum represents the resultant field. When the conditions are such that only the positively or the negatively rotating field exists separately, the air-gap flux is represented by a *single vector of constant length.* The field is then said to be *circular.* When both the positively and negatively rotating fields are present, the resulting air-gap flux is represented by the sum of two opposing rotating vectors, as in Fig. 97. As the fields move past each other they will be in conjunction at one moment ($A$) and in opposition a quarter of a cycle later ($B$).

The terminus of the vector $F$ which represents the resultant field lies at a point whose $x$, $y$ coordinates with respect to the axes $A$ and $B$ are

$$x = P \cos \omega t + N \cos \omega t = (P + N) \cos \omega t$$
$$y = P \sin \omega t - N \sin \omega t = (P - N) \sin \omega t$$

If $\omega t$ is eliminated from these equations by squaring and adding, the equation of an ellipse results or

$$\frac{x^2}{(P + N)^2} + \frac{y^2}{(P - N)^2} = 1$$

The square of the magnitude of the vector $F$ is

$$|F|^2 = (P + N)^2 \cos^2 \omega t + (P - N)^2 \sin^2 \omega t$$

This is readily transformed into a form which is often more useful

$$|F|^2 = P^2 + N^2 + 2PN \cos 2\omega t$$

Thus the terminus of the vector F which represents the resultant field lies on an ellipse whose major and minor radii are $P + N$ and $P - N$. This type of rotating field is said to be *elliptical.* It is interesting to note that the area swept out by the vector $F$ is directly proportional to the time and therefore the velocity of the field is inversely proportional to the square of its magnitude.

It follows that two sinusoidal space distributions of flux, pulsating sinusoidally with time at the same frequency, having the same maximum value and being in both space quadrature and time quadrature, unite to form a single rotating field having a constant magnitude. This field is described as a *circular field.*

Conversely, a circular field is equivalent to two pulsating sinusoidally distributed fields of the same maximum value which are in both space quadrature and time quadrature. This method of analysis is very useful when considering the reaction of the currents in circuits which are symmetrical about axes in space quadrature. Such a condition exists, for example, in the rotor of the usual type of salient-pole synchronous machine. In Art. 120 the method is applied to the induction machine and in Art. 141 is applied to the synchronous machine.

**95. Symmetrical Polyphase Winding.**—A symmetrical polyphase winding is one in which the individual sets of coils, or

phases, are alike and are displaced one from the next by an angle $2\pi$ divided by the number of phases, that is, by the characteristic angle of an $n$-phase system. If such a winding carries either positive- or negative-sequence currents, a circular field will be produced in the air gap. If the order of the phases is chosen so that the second phase is displaced in the positive direction from the first phase, positive-sequence currents will produce a positively rotating field and negative-sequence currents will produce a negatively rotating field with respect to the winding. In either case, the field will rotate so that it lies opposite any phase when the current in that phase is passing through its maximum value. In general, if the winding carries unbalanced currents, the field in the air gap will be elliptical. This field will be due entirely to the action of the positive- and negative-sequence components of the currents, since all other sequence components of current can produce no fundamental flux, although they may produce space-harmonic components of flux. For example, in the 3-phase induction motor, zero-sequence current produces third-harmonic flux in the air gap, but produces no fundamental flux (see page 326).

**96. Stator and Rotor Circuits Symmetrical.**—Consider an induction motor whose stator and rotor windings are symmetrical. If the stator windings are connected either directly or through balanced impedances to a balanced source of potential, balanced currents will flow and a circular field will be produced in the air gap. If the rotor windings are short-circuited or closed through balanced impedances, balanced currents will be induced in the windings. These balanced currents produce another circular field in the air gap, and the field rotates synchronously with the inducing field produced by the stator currents. The frequency of the induced currents in the rotor depends upon the relative speed of the inducing field and the rotor windings and is equal to the frequency of the stator currents multiplied by the slip. The slip $s$ is defined as

$$s = \frac{n_a - n_b}{n_a}$$

where $n_a$ is the speed of the circular field proauced by the stator currents with respect to the stator winding and $n_b$ is the speed of the rotor winding with respect to the stator winding. If

the slip of the rotor winding with respect to the stator field is $s$, the slip of the stator winding with respect to the rotor field is $1/s$. If the speed of the rotor winding is less than the speed of the stator field, the slip is positive and the phase sequence of the induced rotor currents is also positive. If the speed of the rotor winding is in the same direction but greater than the speed of the stator field, the slip is negative and the phase sequence of the rotor currents is also negative.

**97. Stator Circuit Unsymmetrical, Rotor Circuit Symmetrical.** If the stator windings are connected to the balanced source of potential through unbalanced impedances, the stator currents will be unbalanced and the field they produce will be elliptical. The rotor is acted upon by the two oppositely rotating components of this elliptical field. If the rotor circuits are balanced, each of these rotating fields will induce balanced currents in the rotor windings and the frequencies of these currents will be $sf$ and $(2 - s)f$, where $f$ is the frequency of the stator current. The positive-sequence stator field induces positive-sequence currents in the rotor windings of a frequency $sf$, and the negative-sequence stator field induces negative-sequence currents in the rotor windings of a frequency $(2 - s)f$. The positive-sequence rotor currents produce a component field which rotates synchronously with the positive-sequence stator field and the negative-sequence rotor currents produce a component field which rotates synchronously with the negative-sequence stator field. Thus the component fields due to the rotor currents generate positive- and negative-sequence emfs in the stator windings.

**98. Stator Circuit Symmetrical, Rotor Circuit Unsymmetrical.** If the stator windings are connected to the balanced source of potential through *balanced* impedances, while the rotor windings are closed through *unbalanced* impedances, the circular field produced by the stator currents will induce unbalanced currents having a frequency $sf$ in the rotor windings, where $f$ is the frequency of the balanced potentials applied to the stator and $s$ is the slip of the rotor with respect to the stator field. These rotor currents will produce fields that rotate positively and negatively with respect to the rotor windings and will induce potentials of frequencies $f$ and $(1 - 2s)f$ in the stator windings. Since the stator windings are closed through balanced impedances

and there are no other potentials of the frequency $(1 - 2s)f$ acting on the stator side, the currents of this frequency induced in the stator will be balanced and will produce a circular field rotating synchronously with the inducing field due to the rotor currents. The frequencies of the balanced currents which flow in the stator windings will differ by $2sf$ and hence the phenomenon of beats will be observed in the stator current. The frequency $2sf$ of the beats being directly proportional to the slip will increase as the load on the motor increases. Furthermore, it should be noted that, since the stator currents of frequency $(1 - 2s)f$ flow in the entire distribution system to which the motor is connected, there will appear at the stator terminals potentials of frequency $(1 - 2s)f$. These potentials equal the product of the current of the same frequency, $(1 - 2s)f$, and the impedance of the system at this frequency as viewed from the terminals of the stator. Hence voltmeters connected across the stator terminals will also record beats. The behavior of a motor having an unsymmetrical rotor circuit depends not only upon the characteristic constants of the motor but also upon the impedance of the distribution system to which the motor is connected.

**99. Unsymmetrical Stator and Rotor Circuits.**—Consider the case in which the stator windings are connected to the source of balanced potentials through unbalanced impedances and the rotor windings are closed through unbalanced impedances. The unbalanced currents of frequency $f$ in the stator will induce unbalanced currents of frequencies $sf$ and $(2 - s)f$ in the rotor. The unbalanced rotor currents of frequency $sf$ will induce potentials in the stator of frequencies $f$ and $(1 - 2s)f$, while the unbalanced rotor currents of frequency $(2 - s)f$ will induce potentials in the stator of frequencies $f$ and $(3 - 2s)f$. The unbalanced currents of frequency $(1 - 2s)f$ in the stator will induce unbalanced currents of frequencies $sf$ and $(2 - 3s)f$ in the rotor, while the unbalanced currents of frequency $(3 - 2s)f$ in the stator will induce unbalanced currents of frequencies $(2 - s)f$ and $(4 - 3s)f$ in the rotor. Since this reflection and multiplication of frequency will continue, it is clear that, in general, the method of symmetrical components *cannot be applied* when there is unsymmetry on *both* sides of the air gap.[1] Another,

[1] In some cases of unbalance an approximate solution can be obtained by a method similar to that employed in Art. 144.

though somewhat different, aspect of this problem is discussed in Arts. 144, 145, pages 442, 449.

**100. Hysteresis and Eddy Current Losses. Tooth Losses.**—We have assumed that the radial flux density in the air gap varies sinusoidally around the periphery of the rotor. The core losses in the motor depend not only upon the magnitude of this air-gap flux but also upon the way in which the flux is distributed in the teeth and in the iron back of the teeth. It simplifies the solution to neglect the local leakage-flux component set up about each slot. If this be done, the flux density in the teeth becomes substantially radial, although on account of the taper the flux density is not uniform throughout the length of a tooth. If the rotating field in the air gap is circular, the maximum flux per tooth is the same for each tooth and the time variation of the flux in each tooth is sinusoidal. The core loss is then the same in each tooth. A different condition arises when the field varies *elliptically* with respect to teeth which are stationary. Let $\phi_1$ and $\phi_2$ be the maximum values of the tooth flux due to the positive- and negative-sequence components of the elliptical field. It is best to choose an origin ($x = 0$) at the point in the air gap where the flux density is zero at the moment the oppositely rotating component fields coincide. The flux $\phi_x$ in a tooth located $\pi x/P$ radians from this origin is

$$\phi_x = \phi_1 \sin\left(\frac{\pi x}{P} - \omega t\right) + \phi_2 \sin\left(\frac{\pi x}{P} + \omega t\right)$$

where $P$ is the pole pitch. Note that the frequency of the flux variation produced by each of the two fields, $\varphi_1$ and $\varphi_2$ is the same. Or

$$\phi_x = (\phi_1 + \phi_2) \sin\frac{\pi x}{P} \cos \omega t + (\phi_1 - \phi_2) \cos\frac{\pi x}{P} \sin \omega t$$

For simplicity, assume that the core loss in any tooth varies as the square of the maximum flux in that tooth. The square of the maximum tooth flux is

$$(\phi_1 + \phi_2)^2 \sin^2\frac{\pi x}{P} + (\phi_1 - \phi_2)^2 \cos^2\frac{\pi x}{P}$$

It is sometimes more convenient to write this expression as

$$\phi_1^2 + \phi_2^2 + 2\phi_1\phi_2 \cos\frac{2\pi x}{P}$$

For $x = 0$, the maximum flux is $(\phi_1 - \phi_2)$. For $x = P/2$, the maximum flux is $(\phi_1 + \phi_2)$. The average value of the square of the tooth flux taken for all the teeth is $\phi_1^2 + \phi_2^2$. Some teeth will have losses above the average and some losses below the average. For example, take as unity the loss per tooth due to a circular positively rotating field. If the field becomes elliptical, owing to the addition of a 10 per cent negatively rotating field, the maximum tooth loss will become 1.21, the minimum loss will become 0.81, and the average loss will become 1.01. This nonuniformity in the tooth-loss tends to produce local hot spots. It should be noted that the assumption as to the way tooth losses vary leads in this case to the conclusion that the total loss produced by two oppositely rotating fields equals the sum of the losses that the fields would produce individually. This is generally true of eddy-current losses due to fields which are either in space quadrature or which vary at different frequencies.

Another condition arises when the teeth are moving and the rotating field is elliptical. The flux per tooth now is

$$\phi_1 \sin\left(\frac{\pi x}{P} - s\omega t\right) + \phi_2 \sin\left(\frac{\pi x}{P} + (2 - s)\omega t\right)$$

where $s$ is the slip of the teeth with respect to the positively rotating field. When $s$ is small, the positively rotating field produces a slow variation in the tooth flux while the negatively rotating field superposes a rapid variation in the tooth flux. The tooth loss is now the same in *each* of the teeth. Since the eddy-current loss can be computed separately for each of the component fields and since the loss varies as the square of the flux density and the square of the frequency, the average eddy-current loss is proportional to

$$s^2\phi_1^2 + (2 - s)^2\phi_2^2$$

For example, if $s = 0.05$ and $\phi_2 = 0.1\phi_1$, the tooth loss is 0.04 of what the loss would be if the teeth were stationary, the field circular, and the maximum tooth flux $\phi_1$.

The *hysteresis loss*, produced by the joint action of the flux variations of two entirely different frequencies such as $sf$ and $(2 - s)f$, when $s$ is small, can be computed by the method used

when the hysteresis loop is unsymmetrical. It has been suggested that, if $B_1$ is the average value of the flux density and $B_2$ is the maximum departure from the average, the loss can be computed from the formula[1]

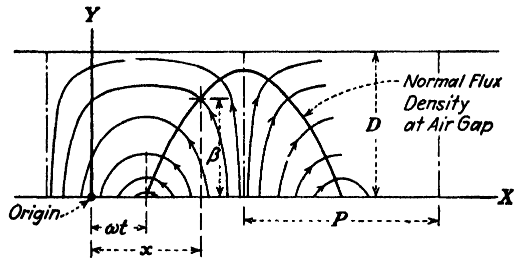
$$\text{Loss per cycle} = (a + bB_1^{1.9})B_2^{1.6}$$

It should be understood that the constants $a$ and $b$ and the exponents 1.9 and 1.6 depend not only upon the magnetic qualities of the iron but to some extent upon the values of $B_1$ and $B_2$. In this problem $B_1$ is the instantaneous value of the low-frequency flux variation while $B_2$ is the maximum value of the high-frequency flux variation. Since $B_1$ varies sinusoidally and $B_2$ is constant, the average loss depends upon the average value of $(a + bB_1^{1.9})$. If the exponent is taken as 2 instead of 1.9, this average value is $a + \frac{bB_{1m}^2}{2}$, where $B_{1m}$ is the maximum value of the low-frequency variation. For example, if the hysteresis loss per cycle for a given value of $B_2$ is three times as great when $B_1 = B_{1m}$ as when $B_1 = 0$, the average loss is twice as great as the loss $B_2$ would produce alone. Furthermore, if $B_2$ is $0.1B_{1m}$, the hysteresis loss in the teeth is $2(2 - s)0.1^{1.6}$, or 0.10 as great as the loss would be if the flux in each tooth pulsated sinusoidally with a frequency $f$ and a maximum density $B_{1m}$. Note that the resultant hysteresis loss does not equal the sum of the hysteresis losses occasioned by the independent action of the two component fields. This is generally true for hysteresis losses.

**101. Core Losses, Circular Field.**—The losses in the iron back of the teeth depend upon the way in which the flux is distributed in this region and upon its time variation. The first simplifying assumption will be that the radial flux density at a circumference which passes through the base of each tooth varies sinusoidally around the circumference. Since it is practically impossible to obtain an expression for the flux density at any point in the iron without assuming the permeability to be constant, this assumption will be made. The accuracy of the analysis will depend upon how much the permeability varies. Although the solution for the flux density can be obtained in

[1] BALL, J. D., Unsymmetrical Hysteresis Loop, *Am. Inst. Elec. Eng.*, Vol. 34, page 2693, 1915.

terms of the polar coordinates of the point in the iron, this will not be done. This solution would be expressed in terms of Bessels' Functions. If it is assumed that the cylindrical boundary surfaces of the iron are replaced by planes, rectangular coordinates can be used as in Fig. 98. The solution for the flux density can now be expressed in terms of hyperbolic functions. The pole pitch is $P$ and the depth of the iron back of the slots is




*For* $y = 0$, $\beta_y = B \sin\left(\frac{x\pi}{P} - \omega t\right)$

*$x$, $P$ and $D$ are expressed in same units*

*$\omega$ is the angular velocity (radians per second) of the rotating field*

FIG. 98.

$D$. Assume a circular rotating field whose normal density at any point on the $X$ axis, $x$ distance from the origin, is

$$B \sin\left(\frac{\pi x}{P} - \omega t\right)$$

Also assume that the normal density (in the $Y$ direction) along the boundary $y = D$ is zero, an assumption which leads to an inconsistency of minor importance because of the large value of the permeability of the iron. The flux density at any given point $(x,y)$ is derived from a potential function which satisfies Laplace's equation, provided the iron is so finely laminated that the eddy currents produced by the varying flux play no important part in determining the distribution of the flux.

$$\frac{\partial^2 V}{\partial^2 x} + \frac{\partial^2 V}{\partial^2 y} = 0$$

The potential function $V$ which satisfies this relation and the two boundary conditions is

$$V = -\frac{P}{\pi}\frac{B}{\sinh\frac{\pi D}{P}}\cosh\frac{\pi}{P}(y - D)\sin\left(\frac{\pi}{P}x - \omega t\right) \tag{321}$$

The *tangential* flux density (in the $X$ direction) is $\partial V/\partial x = B_x$, whence,

$$B_x = -\frac{B}{\sinh\frac{\pi D}{P}}\cosh\frac{\pi}{P}(y - D)\cos\left(\frac{\pi}{P}x - \omega t\right) \tag{322}$$

The *normal flux* density (in the $Y$ direction) is $\partial V/\partial y = B_y$, whence,

$$B_y = -\frac{B}{\sinh\frac{\pi D}{P}}\sinh\frac{\pi}{P}(y - D)\sin\left(\frac{\pi}{P}x - \omega t\right) \tag{323}$$

Thus the flux density at every point within the core, back of the teeth consists of two components that are in space quadrature and time-quadrature. In general, when a vector quantity varies in this manner, it is said to vary elliptically, inasmuch as its value may be represented by the rotating radius of an ellipse. The major and minor radii of the ellipse are

$$\frac{B}{\sinh\frac{\pi D}{P}}\cosh\frac{\pi}{P}(y - D)$$

and

$$\frac{B}{\sinh\frac{\pi D}{P}}\sinh\frac{\pi}{P}(y - D)$$

Note that the major radius is always parallel to the $X$ axis since $\cosh\frac{\pi}{P}(y - D)$ is greater than $\sinh\frac{\pi}{P}(y - D)$. The cycle of elliptical variation is the same for all points at the same radial distance from the air gap. The ellipse becomes flatter as the point recedes from the air gap and the ellipse finally reduces to a line, owing to the boundary condition that $B_y = 0$ when $y = D$. The ellipse which indicates the variation of the flux density in the core at the point $(x,y)$ is in Fig. 99. As the air-gap flux

moves in the positive direction (toward the right), the flux density $B$ rotates in the counterclockwise direction. At the moment when the point of zero air-gap flux density passes through $m$ (Fig. 99) or when $\omega t = \pi x/P$, the flux density at $(x,y)$ has its maximum value $\beta_m$.

Since this elliptical variation of the flux density is the sum of two sinusoidally pulsating variations acting in *space* quadrature, the eddy-current loss can be computed separately for each component variation. Little is known, about the hysteresis loss produced by a flux variation of this type. In lieu of experimental evidence, assume, as a workable approximation, that the

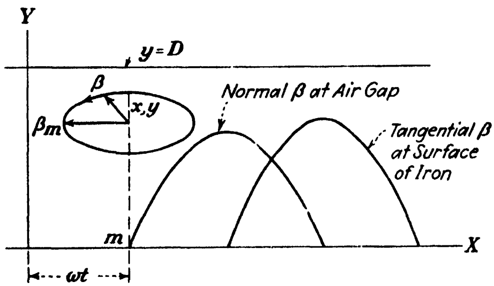



FIG. 99.

hysteresis loss due to an elliptical flux density variation can be computed with sufficient accuracy by the method used in computing the eddy-current loss. In consequence the core loss, $P_c$, per unit volume, at a given point in the core is proportional to the sum of the squares of the maximum values of the tangential and normal components of the flux density at that point. Thus

$$P_c = K_c(B_{xm}^2 + B_{ym}^2)$$

When the air-gap field is circular, the maximum values of the components are

$$B_{xm} = \frac{B \cosh \frac{\pi}{P}(y - D)}{\sinh \frac{\pi D}{P}}$$

$$B_{ym} = \frac{B \sinh \frac{\pi}{P}(y - D)}{\sinh \frac{\pi D}{P}}$$

The average value of the core loss per unit volume is

$$\frac{K_c}{D}\int_0^D (B_{xm}^2 + B_{ym}^2)dy = K_c\frac{B^2}{\theta \tanh \theta} \tag{324}$$

where $\theta = \pi D/P$.

**102. Core Losses. Elliptical Field.**—If the air-gap flux consists of a positively rotating field having a maximum normal, or radial, density $B_1$, and a negatively rotating field having a maximum radial density $B_2$, the tangential and normal components of the flux density at a point $(x,y)$ in the core are

$$B_x = -\frac{\cosh \frac{\pi}{P}(y - D)}{\sinh \frac{\pi D}{P}}\left[B_1 \cos\left(\frac{\pi x}{P} - \omega t\right) + B_2 \cos\left(\frac{\pi x}{P} + \omega t\right)\right] \tag{325}$$

$$B_y = -\frac{\sinh \frac{\pi}{P}(y - D)}{\sinh \frac{\pi D}{P}}\left[B_1 \sin\left(\frac{\pi x}{P} - \omega t\right) + B_2 \sin\left(\frac{\pi x}{P} + \omega t\right)\right] \tag{326}$$

These equations assume that the origin ($x = 0$) is at the point of zero air-gap flux density at the moment when the positively and negatively rotating fields coincide ($\omega t = 0$). Each of these rotating fields produces an elliptical variation of the flux density at points in the core. It is readily shown that the combination of these two component elliptical variations is an elliptical variation. The axes of the resultant ellipse do not coincide with the coordinate axes since the space quadrature components $B_x$ and $B_y$, of the resultant density, are no longer in time quadrature. The angle that the axes of the ellipse make with the coordinate axes depends upon the value of $x$, or upon the position of the point in the core with respect to the origin chosen. The maximum value of $B_x^2$ at a point $(x,y)$ is

$$B_{xm}^2 = \frac{\cosh^2 \frac{\pi}{P}(y - D)}{\sinh^2 \frac{\pi D}{P}}\left(B_1^2 + B_2^2 + 2B_1B_2 \cos \frac{2\pi x}{P}\right)$$

The maximum value of $B_y^2$ at the point $(x,y)$ is

$$B_{ym}^2 = \frac{\sinh^2 \frac{\pi}{P}(y - D)}{\sinh^2 \frac{\pi D}{P}} \left( B_1^2 + B_2^2 - 2B_1B_2 \cos \frac{2\pi x}{P} \right)$$

The core loss is no longer the same at all points at the same distance from the air gap. The average value of the core loss per unit volume for all values of $x$ and $y$ is

$$P_c = K_c \frac{(B_1^2 + B_2^2)}{\theta \tanh \theta} \tag{327}$$

where $\theta = \pi D/P$. Compare this with the expression for the loss when the field is circular [equation (324)]. Since the average value of the tooth loss is also proportional to $(B_1^2 + B_2^2)$, the total core loss is proportional to $(B_1^2 + B_2^2)$. If the field is circular and the maximum value of the flux density in the air gap is $B$, the total core loss is proportional to $B^2$. Here again the assumption as to the way the losses vary leads to the conclusion that the total loss produced by two oppositely rotating fields equals the sum of the losses that the fields produce individually. If the field consists of positive- and negative-sequence components, each of which has a maximum value of air-gap flux density $0.5B$, the resulting field is purely pulsating and the maximum air-gap flux density is $B$. The maximum density occurs when the oppositely rotating component fields coincide. The core loss is proportional to $[(0.5B)^2 + (0.5B)^2]$ or to $0.5B^2$. Thus the core loss in the stator due to a *rotating* field is twice the core loss due to a *pulsating* field if the maximum value of the air-gap flux density is the same in both cases. A different condition, which is of special interest, exists in the rotor when the motor is running near synchronous speed. The flux density variation in the iron back of the teeth consists of two elliptically varying components. The positive-sequence component, which is generally the greater, varies at slip frequency, while the negative-sequence component varies at $(2 - s)$ times the rated frequency applied to the stator.

In Fig. 100 let the rotating vector $F_p$ represent the flux density at a given point in the rotor due to the positive-sequence field.

This vector rotates at an average speed of $s$ times the synchronous speed. At this same given point there is also a flux density due to the negative-sequence field. This flux density is represented by the vector $F_n$, which also rotates elliptically but at an average speed of $(2 - s)$ times the synchronous speed. If $s = 0.03$, the average speed of $F_n$ is about sixty-five times the average speed of $F_p$ and thus $F_n$ makes a complete revolution while $F_p$ moves about 5.5 degrees.

The ellipticity of each of these components is the same at points equally distant from the air gap but increases as the point recedes from the air gap. The resultant flux density at the given point is represented by the sum of the two vectors $F_p$

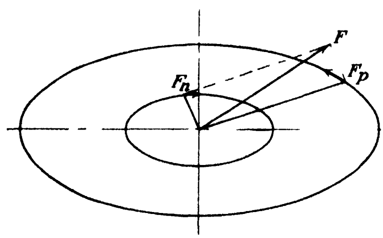



FIG 100

and $F_n$. Since the frequencies of the two elliptically varying component flux densities are different, the total eddy-current loss equals the sum of the eddy-current losses produced by the component variations independently. Thus, if the negative-sequence component of the air-gap flux is 0.10 of the positive-sequence component and the slip is 0.05, the total eddy-current loss is $[(0.05)^2(1.0)^2 + (1.95)^2(0.1)^2]$ or 0.04 of the eddy-current loss in the rotor at standstill due only to the positive-sequence component of the flux. Until something more definite is learned concerning the hysteresis loss produced by such a flux variation, this loss might be computed by following a procedure similar to that suggested for computing the hysteresis loss in the rotor teeth. Since the nature of the variation in flux density in the rotor core is somewhat different from that in the rotor teeth, we are hardly justified in drawing an exact parallel between the two cases. Perhaps no better estimate of the hysteresis loss can be made than to assume that it is the sum of the losses produced by the positive- and negative-sequence fields acting

separately. If this is done, the total loss when the negative-sequence component of the air-gap flux is 0.1 the positive-sequence component and the slip is 0.05 is

$$[(0.05)(1^{1.6}) + 1.95(0.1)^{1.6}],$$

or is 0.1 the loss that would be produced by the positive-sequence component of the air-gap flux alone when the rotor is at rest.

The determination of the flux density within the iron and of the losses produced by the time variation of this flux density is one of the difficult problems that has never been solved with the precision the problem merits. The foregoing discussion does not pretend to be an accurate analysis and is presented principally for the purpose of calling attention to the general nature of the problem, particularly in regard to the effect of unbalanced conditions on the distribution of the flux and the attendant losses.

**103. Copper Loss.**—When the circuit on the stator side is unsymmetrical, the positive- and negative-sequence currents in the stator phases have a single frequency $f$ while the corresponding component currents in the rotor phases have frequencies $sf$ and $(2 - s)f$. On the other hand, when the unsymmetry is on the rotor side, the positive- and negative-sequence currents in the stator phases have frequencies $f$ and $(1 - 2s)f$, while the corresponding component currents in the rotor have a single frequency $sf$. When the component currents in a circuit have different frequencies, the total heat loss due to the current is the sum of the losses due to the components taken separately. Thus the copper loss is the same in each phase of the rotor when the stator circuits are unsymmetrical and is the same in each phase of the stator when the rotor circuits are unsymmetrical. In order to compute the copper loss when the component currents in a circuit have the same frequency, it is first necessary to compute the resultant current in the circuit. If the resultant currents in the several phases are different, the copper losses in the phases are different and there will be local hot spots. This occurs in the stator when the stator circuits are unsymmetrical and in the rotor when the rotor circuits are unsymmetrical.

In general, if $I_1$ and $I_2$ are positive- and negative-sequence components, the average heat loss in all the phases is proportional to $|I_1|^2 + |I_2|^2$. If the positive- and negative-sequence

components are in time-phase in one of the phases, the heat loss in that phase is proportional to $|I_1|^2 + |I_2|^2 + 2|I_1| \cdot |I_2|$. As the time-phase displacement between the component currents is increased, the heat developed in this phase will decrease. When the time-phase displacement becomes 60 degrees, the current in this phase will be the same as the current in one of the other phases. With any further increase in time-phase displacement, one of the other currents will become the greatest of the three. When the time-phase displacement is 60 degrees, the loss in this phase is proportional to

$$|I_1|^2 + |I_2|^2 + |I_1| \cdot |I_2|.$$

For example, if $I_1 = 1$ unit and $I_2 = 0.5$ unit, the *average* heat loss per phase is proportional to 1.25. The *maximum* heat loss for one phase may be proportional to 2.25, while the *minimum possible* heat loss in the phase which is carrying the *greatest* current is proportional to 1.75. Thus, while the average heat loss is increased only 25 per cent by the addition of a 50 per cent negative-sequence current, the heat loss in one of the phases is increased from 75 to 125 per cent. That is, the effect of the negative-sequence current is to cause local hot spots in the windings. This is perhaps the most objectional result of operating induction motors on unbalanced-potential circuits. If the capacity of the motor is limited by the heat developed in the hottest phase of the stator, a 5 per cent unbalance in the applied potential may reduce the capacity of the motor by 20 per cent under the most favorable conditions and by 30 per cent under the most unfavorable conditions when the positive- and negative-sequence currents in one phase of the stator are in time-phase. The computation is based upon a no-load current of 35 per cent at a power factor of 10 per cent and an impedance to negative sequence of 20 per cent. Since the negative-sequence current is but 0.05/0.20 or 25 per cent of the positive-sequence current, the reduction in the capacity of the motor based on *average* heating is approximately 4 per cent.

**104. Torque.**—To determine the torque developed by the rotating fields in the air gap with the currents in the conductors, it is essential to adopt proper conventions as to the directions of positive quantities. Such conventions are illustrated in Fig. 101. The air-gap flux density $\beta$ is positive when directed

radially outward. The *rise* in generated emf $e$, due to the relative motion of the conductor and the flux density, is positive when acting from $a$ to $b$, or when $b$ is positive with respect to $a$. The current $i$ in the conductor is positive when flowing from $a$ to $b$. Both the torque $T$ and the electrical angular velocity $\omega$ of the rotating field with respect to the conductor are positive when clockwise. These conventions are equivalent to assuming that $T$ and $\omega$, $\beta$, and $e$ and $i$ are positive when directed along the positive $X$, $Y$, $Z$ coordinate axes, respectively. Since we shall assume the conductor stationary and the field moving, the *relative* motion of the conductor with respect to the field is counterclockwise when the angular velocity of the field is positive. With these conventions, the "right-hand" rule shows that the rise in generated emf is negative if both flux density and angular velocity of the field are positive. Thus,

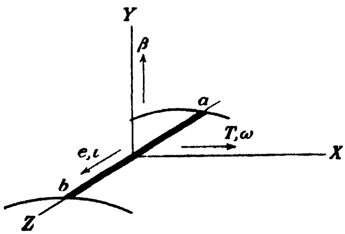



FIG. 101.

$$e = -\beta l \frac{2R}{p} \omega$$

where $l$ is the length of the conductor, $R$ is the radial distance from the shaft to the conductor, and $p$ is the number of poles. The "left-hand" rule shows that the torque $T$ developed by the field with a current $i$ in the conductor is negative when both flux density and current are positive. Thus

$$T = -\beta liR$$

It is more convenient to write the expression for the torque in terms of emf and current, whence

$$T = \frac{p}{2} \frac{ei}{\omega} \tag{328}$$

It is important to understand that $e$ is proportional to the flux density at the conductor in which the current $i$ is flowing. The electromagnetic torque is clockwise, or positive, when the product of the rise in generated emf and current in the same direction is positive, *provided* the magnetic field rotates in the clockwise, or positive, direction. If the direction of rotation

of the field alone is reversed, the direction of the torque becomes counterclockwise or negative. Therefore, it is generally true that the direction of the electromagnetic torque is the same as the direction of the relative velocity of the rotating field with respect to the conductor if the electric power generated in the conductor is positive. For example, since the electric power *generated* in the short-circuited rotor windings of a symmetrical polyphase induction machine is always positive, being equal to the copper loss in the rotor windings, the direction of the electromagnetic torque on the rotor is the same as the direction of the relative velocity of the rotating field with respect to the rotor. Where the speed of the rotor is less than the speed of the field, the electromagnetic torque acts in the direction in which the field rotates. When the speed of the rotor becomes greater than the speed of the field, the electromagnetic torque is reversed in direction.

Consider two fields which rotate positively at angular velocities $\omega_1$ and $\omega_2$. Let $e_1$ and $e_2$ be the emfs generated by these fields in one phase of the windings and let $i_1$ and $i_2$ be the currents in this phase. By following the method used in deriving equation (328), the total instantaneous torque developed in this phase is

$$T = \frac{p}{2}\left(\frac{e_1 i_1}{\omega_1} + \frac{e_2 i_2}{\omega_2} + \frac{e_1 i_2}{\omega_1} + \frac{e_2 i_1}{\omega_2}\right) \tag{329}$$

Let

$$e_1 = \sqrt{2}E_1 \sin \omega_1 t \quad \text{and} \quad i_1 = \sqrt{2}I_1 \sin(\omega_1 t + \theta_1) \tag{330}$$
$$e_2 = \sqrt{2}E_2 \sin(\omega_2 t + \alpha) \quad \text{and} \quad i_2 = \sqrt{2}I_2 \sin(\omega_2 t_1 + \alpha + \theta_2)$$

If these values of emf and current are substituted in equation (329), the expression for the torque developed in this phase is

$$\begin{aligned} T = \frac{p}{2}\bigg\{&\frac{E_1 I_1}{\omega_1}[\cos\theta_1 - \cos(2\omega_1 t + \theta_1)] \\ &+ \frac{E_2 I_2}{\omega_2}[\cos\theta_2 - \cos(2\omega_2 t + 2\alpha + \theta_2)] \\ &+ \frac{E_1 I_2}{\omega_1}[\cos((\omega_1 - \omega_2)t - \alpha - \theta_2) - \cos((\omega_1 + \omega_2)t + \alpha + \theta_2)] \\ &+ \frac{E_2 I_1}{\omega_2}[\cos((\omega_1 - \omega_2)t - \alpha + \theta_1) - \cos((\omega_1 + \omega_2)t + \alpha + \theta_1)]\bigg\} \end{aligned} \tag{331}$$

Similar expressions are obtained for the torque developed in the other phases. Since the windings are symmetrically arranged and the currents in them are balanced, the emfs and currents in the next phase to the right of the first phase will *lag* the corresponding quantities in the first phase by the characteristic angle of the system. Since the expression for the torque involves the cosines of the *sum* and *difference* of the phase angles of the emf and current, the terms involving the cosine of the *difference* of the phase angles are the same for all the phases, while the terms involving the cosine of the *sum* of the phase angles will themselves differ in time-phase by twice the characteristic angle of the system and their sum for *all* phases is zero. The *total* instantaneous torque developed by all $n$ phases is

$$T = \frac{np}{2}\left\{\frac{E_1I_1}{\omega_1}\cos\theta_1 + \frac{E_2I_2}{\omega_2}\cos\theta_2 + \frac{E_1I_2}{\omega_1}\cos[(\omega_1 - \omega_2)t - \alpha - \theta_2] + \frac{E_2I_1}{\omega_2}\cos[(\omega_1 - \omega_2)t - \alpha + \theta_1]\right\} \quad (332)$$

The sum of the other terms in the several phases is zero. The total instantaneous torque consists of a constant term and a sinusoidal term which has a frequency equal to the difference of the two frequencies corresponding to $\omega_1$ and $\omega_2$. The *average* value of the torque in all $n$ phases is

$$T_{av} = \frac{np}{2}\left(\frac{E_1I_1}{\omega_1}\cos\theta_1 + \frac{E_2I_2}{\omega_2}\cos\theta_2\right) \quad (333)$$

The *maximum* value of the total sinusoidal or pulsating torque in all $n$ phases is

$$T_S = \frac{np}{2}\left[\left(\frac{E_1I_2}{\omega_1}\right)^2 + \left(\frac{E_2I_1}{\omega_2}\right)^2 + 2\left(\frac{E_1I_2}{\omega_1}\right)\left(\frac{E_2I_1}{\omega_2}\right)\cos(\theta_1 + \theta_2)\right]^{\frac{1}{2}} \quad (334)$$

Consider two fields, one rotating positively at angular velocity $\omega_1$, the other rotating negatively at angular velocity $\omega_2$. The total instantaneous electromagnetic torque developed in any one phase is

$$T = \frac{p}{2}\left(\frac{e_1i_1}{\omega_1} - \frac{e_2i_2}{\omega_2} + \frac{e_1i_2}{\omega_1} - \frac{e_2i_1}{\omega_2}\right) \quad (335)$$

Note that the negative character of $\omega_2$ is indicated by prefixing a negative sign, $\omega_2$ itself being a positive number. The general expressions for the instantaneous generated emfs and for the currents are unchanged. If the expressions for the second emf and current are written as $e_2 = \sqrt{2}E_2 \sin(-\omega_2 t + \alpha)$ and $i_2 = \sqrt{2}I_2 \sin(-\omega_2 t + \alpha + \theta_2)$, a positive value of $\theta_2$ would indicate a lagging instead of a leading current. This would be confusing and contrary to established custom. The expression for the generated emf can always be written as $e = \sqrt{2}E \sin \omega t$, whether the field rotates in one direction or the other with respect to the conductor. Note that the sequence of the emfs generated in the different phases by the second or negatively rotating field is opposite to the sequence of the emfs generated by the positively rotating field. The emf generated by the negatively rotating field in the next phase at the right of the first *leads* the emf generated in the first phase by the characteristic angle of the system. If these effects of reversal in direction of rotation of the second field are introduced, the expression for the total instantaneous torque is

$$T = \frac{np}{2}\left\{\frac{E_1 I_1}{\omega_1}\cos\theta_1 - \frac{E_2 I_2}{\omega_2}\cos\theta_2 \right.$$
$$\left. - \frac{E_1 I_2}{\omega_1}\cos[(\omega_1 + \omega_2)t + \alpha + \theta_2] + \frac{E_2 I_1}{\omega_2}\cos[(\omega_1 + \omega_2)t + \alpha + \theta_1]\right\} \tag{336}$$

The torque consists of a constant term and a sinusoidal term which has a frequency equal to the sum of the two frequencies corresponding to $\omega_1$ and $\omega_2$.

The average value of the torque in all $n$ phases is

$$T_{av} = \frac{np}{2}\left\{\frac{E_1 I_1}{\omega_1}\cos\theta_1 - \frac{E_2 I_2}{\omega_2}\cos\theta_2\right\} \tag{337}$$

The *maximum* value of the total sinusoidal or pulsating torque in all $n$ phases is

$$T_S = \frac{np}{2}\left[\left(\frac{E_1 I_2}{\omega_1}\right)^2 + \left(\frac{E_2 I_1}{\omega_2}\right)^2 - 2\left(\frac{E_1 I_2}{\omega_1}\right)\left(\frac{E_2 I_1}{\omega_2}\right)\cos(\theta_1 - \theta_2)\right]^{\frac{1}{2}} \tag{338}$$

Note that in both cases the total pulsating torque is independent of the phase angle $\alpha$ introduced in the expression for $e_2$ for generality. When one of the frequencies is not a multiple of the other, the phase angle $\alpha$ has little significance.

When the unsymmetry in the circuit is on the stator side, the fields rotate oppositely at synchronous speed with respect to the stator and the frequency of the pulsating torque is double the supply frequency. Because of the relatively large moments of inertia of the stator and rotor, the pulsating torque produces torsional vibrations of relatively small amplitude, but even such small vibrations may be very objectionable. When the unsymmetry is on the rotor side, the fields rotate in the same direction with respect to the stator, if the slip is small, and the frequency of the pulsating torque is only twice the *slip* frequency. At light loads the frequency of the pulsating torque may be less than 0.5 cycle per second, and thus the pulsating torque may cause torsional vibrations of such great amplitude as to produce a noticeable change in the speed of the rotor. This action has an important bearing on the effect produced by damper windings on the operation of a synchronous machine.

The application of these general torque expressions to specific problems is discussed in Arts. 106, 111, 112.

It is interesting to note the effect of hysteresis and eddy-current losses upon the torque developed in an induction motor. The general relation between the power loss due to a rotating magnetic field and the torque developed in consequence of this loss is

$$T = k\frac{P}{\omega_m}$$

where $T$ is the torque, $P$ is the power loss due to rotation of the field, and $\omega_m$ is the angular velocity (mechanical) of the field with respect to the conducting and magnetic masses in which the power loss occurs. The constant $k$ depends upon the units. If $T$ is desired in pound-feet and $P$ is expressed in watts, $k$ equals 0.738. Since the electrical angular velocity $\omega$ is $\frac{p}{2}\omega_m$, where $p$ is the number of poles, the torque is usually expressed as

$$T = k\frac{p}{2\omega}P$$

In a polyphase induction motor, the torque due to the induced rotor currents in the short-circuited rotor winding is

$$T = k\frac{p}{2s\omega}nI^2R$$

where $I$ is the current in one rotor phase, $R$ is the resistance of one rotor phase, $n$ is the number of phases, $\omega$ is the electrical angular velocity of the magnetic field with respect to the stator, and $s$ is the slip of the rotor with respect to this magnetic field. The direction of the torque is in the direction of the relative motion of the magnetic field with respect to the rotor. The eddy-current and hysteresis losses produced in the mass of rotor iron augment the torque due to the currents in the rotor winding. Under normal operating conditions when the slip is small, the added torque due to these losses is also small and is customarily neglected and the net torque acting on the rotor is assumed to be proportional to the square of the rotor current and the resistance $R/s$ of the rotor branch in the equivalent circuit of the motor.

The rotating magnetic field also exerts a reaction torque on the currents in the stator conductors. Under normal operating conditions as a motor, the direction of this torque is *opposite* to the direction of rotation of the magnetic field. The magnitude of the torque is proportional to the product of the square of the stator current and the resistance of the parallel combination of the magnetizing and rotor branches in the equivalent circuit of the motor. This torque is greater than that exerted on the rotor conductors. The rotating magnetic field produces eddy current and hysteresis losses in the mass of the stator iron. These losses are assumed customarily to be the entire core loss in the motor. One effect of this core loss is to exert a torque on the stator iron in the direction of rotation of the magnetic field. Therefore, the net torque acting on the stator is the difference between the torque due to the currents in the stator conductors and the torque which acts on the stator iron due to the eddy current and hysteresis losses.

When currents flow in conductors located in slots as in the stator and rotor, there may be little direct torque acting on the conductors themselves, inasmuch as they are partially shielded by the surrounding iron from the direct action of the air-gap

flux. Nevertheless, it is true that the total torque due to the current in the conductors is correctly computed if it is assumed that the conductors are arranged in the air gap so that the entire torque acts directly upon them, provided it is also assumed that the emf generated by the rotating magnetic field in the so-arranged conductors equals the emf which is actually generated by the rotating field in the conductors when they are embedded in the slots. This torque due to the currents in the conductors, even though it acts principally upon the iron surrounding these conductors, has no relation to the torque due to the eddy-current and hysteresis losses produced in the iron by the rotating field.

**105. Vector Diagrams and Equivalent Circuits.**—The performance of an induction machine is usually computed from a vector diagram or from an equivalent circuit. It is generally conceded that satisfactory results can be obtained by either of these methods if the machine operates under balanced conditions. The theory of operation is based upon the fact that there is produced in the air gap a single magnetic field of constant magnitude rotating at constant speed.

When the circuits on either the stator or the rotor side are unsymmetrical, two magnetic fields, of constant magnitude, are produced in the air gap and rotate either at different speeds or in opposite directions. A vector diagram or an equivalent circuit can be drawn representing the conditions due to *each* of the two rotating fields. There will be a positive-sequence vector diagram and a negative-sequence vector diagram. The relation between the two diagrams is determined by the kind of unsymmetry existing on the stator side or on the rotor side. The component parts of the *average power* or *torque*, being due to potentials and currents of the *same sequence*, can be computed from the positive-sequence and negative-sequence diagrams taken separately. The *pulsating power or torque* is due to potential of one sequence and current of the other sequence. It has been shown, however, that the *magnitude* of the pulsating torque depends upon the magnitudes of the generated emfs and the currents and upon the sum or difference of the phase angles of the currents with respect to the generated emfs of the *same sequence*. Thus, it is not necessary to consider the interrelations between the emfs and currents in the positive- and negative-sequence diagrams in order to determine the magnitude alone

of the pulsating torque. The time-phase of the pulsating torque cannot be determined without first establishing the interrelation between the positive- and negative-sequence diagrams. Fortunately, in the case of the induction machine, this phase relation has little significance. With a synchronous machine, however, the phase relation of the pulsating torque produced by the action of an unsymmetrical damper winding may be of very considerable importance, and a more complete analysis must be made.

While it is assumed that the reader is familiar with the vector diagram and the corresponding equivalent circuit used when the machine is operating under balanced conditions, it is desirable to set them down and comment briefly on the various quantities appearing in them. There are two cases to consider; one in which the rotor revolves either opposite to the field or in the same direction as the field but slower, and the other in which the rotor revolves in the same direction as the field but faster. The general vector diagram and equivalent circuit for the first case are shown in Figs. 102 and 105, and those for the second case are shown in Figs. 108 and 112. It is assumed that all rotor quantities are reduced to the basis of a one-to-one ratio with the stator, so that in the two windings at standstill equal emfs will be generated by the air-gap flux and also equal stator and rotor currents will produce equal components of this flux. The values of all potentials, currents, and circuit constants are given per phase, preferably on the assumption of $Y$ connection in a 3-phase machine.

$\phi$ = resultant air gap flux

$I_\phi$ = magnetizing current necessary to produce $\phi$

$E_a = E_b$ = rise in generated emf in stator ($a$) and in rotor ($b$) owing to $\phi$, on the assumption that the rotor is stationary

$V_a$; $I_a$ = rise in terminal potential through the stator; stator current in the same direction that $E_a$ is taken

$V_b$; $I_b$ = rise in terminal potential through the rotor; rotor current in the same direction that $E_b$ is taken

$\theta_a$ = phase angle by which stator current leads rise in stator emf $E_a$

$\theta_b$ = phase angle by which rotor current leads rise in rotor emf $E_b$

$Z_\phi$ = magnetizing impedance, being the complex ratio of $-E_a$ to $I_\phi$
$r_a$ = stator resistance at actual stator frequency
$r_b$ = rotor resistance at actual rotor frequency
$x_a$; $x_b$ = stator leakage reactance (*a*); rotor leakage reactance (*b*), both being at standard frequency of supply
$s$ = slip
$\omega$ = standard angular velocity of supply

When the stator windings or rotor windings carry direct current, the current density is uniform over the cross section of the conductors and the resistance is said to have its true or ohmic value. When the windings carry alternating current, the current density is no longer uniform. In consequence of the nonuniformity in current density the resistance is *greater* than its true or ohmic value, while the inductance is less than its value calculated on the assumption that the current density is uniform.

The rotor bars of a cage winding are frequently designed to accentuate the nonuniformity in current density in order thereby to increase the starting torque without increasing the rotor copper loss at full load, at which time the rotor frequency is so low that the rotor resistance has its true value. Since the rotor resistance is usually the most important constant of the motor, it is essential that its value should be known as accurately as possible and hence the effect of frequency should not be ignored. For a single rectangular copper bar in a slot, the ratio of the alternating-current resistance to the true resistance is 2.85 if the bar is 1 in. deep and the frequency is 60 cycles. For a bar 2 in. deep, this ratio is 5.5 at 60 cycles and 7.7 at 120 cycles.

While it is theoretically correct to compute the stator and rotor reactances, $x_a$ and $x_b$, by multiplying the inductances calculated for the actual frequencies of the currents in the windings by $2\pi$ times the standard supply frequency, this is seldom done. The reactances are usually computed on the assumption that the current density in the conductors is uniform, but the reactances are measured when the rotor is blocked and the frequency has its rated value. The ratio of that portion of the actual leakage reactance, due to flux within the conductor, to the value of the reactance calculated on the assumption of uniform current density depends upon the depth of the conduc-

tor, its resistivity and the frequency. For a single rectangular copper bar in a slot, this ratio is 0.55 for a depth of 1 in. and frequency of 60 cycles, and is 0.18 for a depth of 2 in. and frequency of 120 cycles. The dependence of the motor constants, particularly those of the rotor, upon frequency is of considerable practical importance but it may make the accurate measurement of the motor constants difficult.

It is important to note that, while the potentials and currents in the rotor are shown, Fig. 102, in their proper phase relations with respect to each other, the frequency of these quantities is

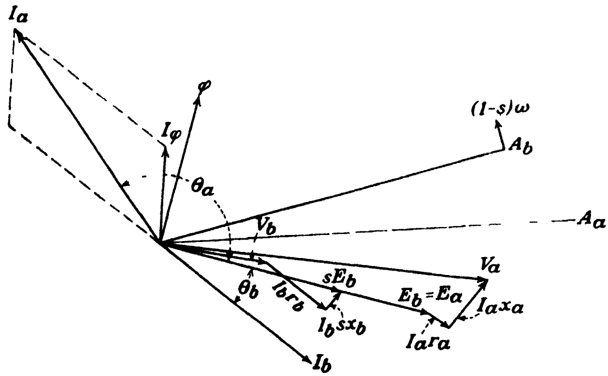



FIG. 102.

but a fraction of the stator frequency. Quantities having different frequencies are not usually shown in the same vector diagram. This apparent inconsistancy disappears if it be assumed that the axis on which the rotor vectors are projected, to obtain their instantaneous values, rotates in the positive direction at the speed of the rotor, or with an angular velocity $(1 - s)\omega$. The axis on which the stator quantities are projected to obtain their instantaneous values is stationary. These axes for rotor and stator are shown in the diagram (Fig. 102) as $A_b$ and $A_a$. All the potential and current vectors rotate synchronously at the same angular velocity $\omega$.

For purposes of computation it is sometimes convenient to resolve the stator leakage impedance drop into two components, one due to the load component $(-I_b)$ of the stator current and the other due to the magnetizing component $I_\phi$. This resolution is illustrated in Fig. 103. The potential drop $I_\phi Z_a$, being in

many cases nearly in phase with the terminal potential $V_a$, can be subtracted numerically from $V_a$ with little error.[1] It is even more accurate to subtract the leakage-reactance drop $I_\phi x_a$ and write $|V_a'| = |V_a| - |I_\phi x_a|$. $V_a'$ is usually about $0.97V_a$. This procedure gives a simple and sufficiently

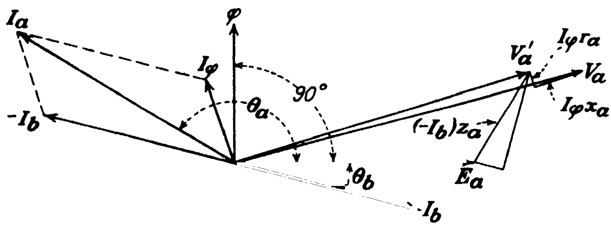



FIG. 103.

accurate means for computing the rotor current. The rotor current is

$$I_b = \frac{V_a'}{\sqrt{\left(r_a + \frac{r_b}{s}\right)^2 + (x_a + x_b)^2}} \tag{339}$$

If the applied potential is maintained constant, both the air-gap flux and the magnetizing current which produces it diminish as the load on the motor increases. Thus, $V_a'$ increases slightly as the load on the motor is increased.

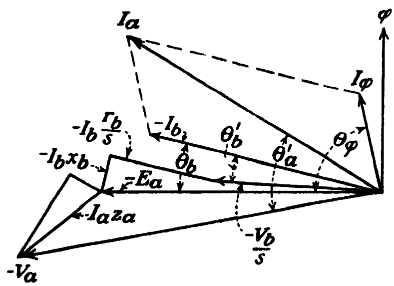



FIG 104.

If the vector diagram Fig. 102 is redrawn so that each potential is shown as a fall in potential rather than as a rise in potential, and if the stator current is shown as the sum of a load component $(-I_b)$ and a magnetizing component $(I_\phi)$, the result is the vector diagram Fig. 104. The conditions represented in Fig. 104 are the same as obtain in the equivalent circuit of the machine, Fig. 105. In Fig. 105 the subscript 1 is used to show that the

[1] LYON, W. V., Calculation of the Performance of an Induction Motor, *Elect. World*, **65**, 1168, 1915.

diagram is the positive-sequence equivalent circuit and so distinguish it from the negative-sequence equivalent circuit of Fig. 106. The arrows near the potentials and currents in Fig. 105 show the directions in which potentials and currents are taken to be positive. The core loss is $|E_a| \cdot |I_\phi| \cos \theta_\phi$. This presentation shows clearly the points of similarity and of

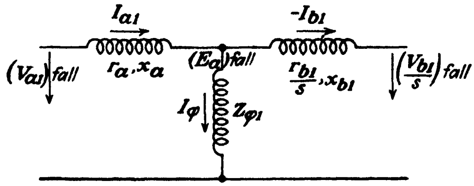



FIG 105

divergence between the vector diagram and the equivalent circuit so frequently used to represent the induction machine. Note that in the equivalent circuit the phase relation between the load component $(-I_b)$ of the stator current and the fall in potential $(-E_a)$ across the magnetizing impedance is the *same* as the phase relation between the rotor current $I_b$ and the emf $sE_b$ generated in the rotor by the air-gap flux. The most important difference is that the angle between the stator current

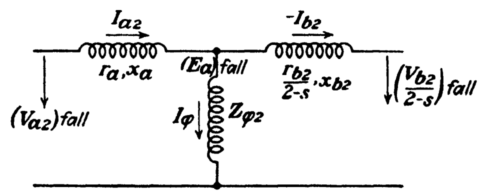



FIG. 106.

$I_a$ and the potential across the magnetizing circuit $Z_\phi$ is $(\pi - \theta_a)$ and is an angle of lag, whereas $\theta_a$ is an angle of lead. This difference must be noted if the pulsating torque is computed on the stator side.

The fall in potential through one phase of the stator is due in part to the stator resistance, in part to stator leakage flux, and in part to the mutual flux in the air gap produced by the joint action of the currents in the stator and rotor windings. This joint action is equivalent to the actions of the so-called "magnetizing current" $I_\phi$ (see Fig. 102) in the stator winding. If

the emf generated in the stator by the mutual air-gap flux is $E_a$, the ratio $(-E_a/I_\phi)$ is the magnetizing impedance $Z_\phi$ of the motor. If the magnetizing impedance is shunted by the equivalent leakage impedance of the rotor, which in the positive-sequence equivalent circuit is $r_{b1}/s + jx_{b1}$ and in the negative-sequence equivalent circuit is $r_{b2}/(2 - s) + jx_{b2}$, the current in the magnetizing impedance has its proper value and the potential across it equals that generated in the stator by the mutual air-gap flux. In this respect the equivalent circuit provides a convenient means for visualizing the interaction of the stator and rotor currents and for determining the relation between the potential applied to the stator and the stator current. It is assumed that the rotor is short-circuited so that $V_{b1}$ in Fig. 105 and $V_{b2}$ in Fig. 106 are both zero.

Given the shaft output and adequate data in regard to the electrical output of the rotor, if there is any, and also all the motor constants, the complete performance of the motor can be calculated.

The power input to the stator of a 3-phase motor is (Fig. 104)

$$3| -V_a | \cdot | I_a | \cos \theta_a'.$$

The copper loss in the stator is $3| I_a |^2 r_a$. The core loss is (Fig. 104) $3| -E_a | \cdot | I_\phi | \cos \theta_\phi$.

The power transferred from the stator to the rotor is (Fig. 104)

$$3| -E_a | \cdot | -I_b | \cos \theta_b.$$

In general, the average torque acting on the rotor conductors is [equation (333)];

$$\frac{np}{2\omega_1} | E_1 | \cdot | I_1 | \cos \theta_1$$

where $n$ is the number of phases, $p$ is the number of poles, $| E_1 |$ is the actual emf generated in one phase of the rotor $| I_1 |$ is the actual current per rotor phase, $\theta_1$ is the actual angle between $E_1$ and $I_1$, and $\omega_1$ is the angular velocity of the field with respect to the rotor. Since $E_1 = sE_b$, $I_1 = I_b$, $\theta_1 = \theta_b$, and $\omega_1 = s\omega$, the torque is more conveniently written

$$T = \frac{np}{2\omega} | E_b | \cdot | I_b | \cos \theta_b$$

The torque expressed in pound-feet is

$$T = 0.738\frac{np}{2\omega}| E_b | \cdot | I_b | \cos \theta_b$$

In the following expressions for torque the conversion factor 0.738 has been omitted. In a 3-phase motor the torque acting on the rotor conductors, written in terms of the potential and current given in Fig. 104, is

$$\frac{3p}{2\omega}| -E_a | \cdot | -I_b | \cos \theta_b$$

For large values of slip, there will also be a torque on the rotor iron proportional to the core loss in the rotor.

The copper loss in the rotor is $3| -I_b |^2 r_b$.

The shaft output is $3(1 - s)| -E_a | \cdot | -I_b | \cos \theta_b$ − (friction and windage).

The electrical output of the rotor is $3| -V_b | \cdot | -I_b | \cos \theta_b'$.

**106. Computation of Unbalanced Operation.**—When the motor has unbalanced potentials applied to the stator windings, and the rotor windings are either short-circuited or closed through balanced impedances, the complete performance can be computed from the positive- and negative-sequence equivalent circuits. If $s$ is the slip of the rotor with respect to the positive-sequence field, the slip with respect to the negative-sequence field is $2 - s$, and the positive- and negative-sequence equivalent circuits are constructed as in Figs. 105 and 106. The rotor terminal potentials $V_{b1}$ and $V_{b2}$ equal the corresponding rotor currents multiplied by the impedance of the balanced circuit connected across the rotor terminals.

Representative per unit values of the circuit constants, based on rated stator potential and current, are approximately

$r_{a1} = r_{a2} = 0.025$ at rated frequency
$r_{b1} = 0.025$ at zero frequency (positive sequence)
$r_b = 0.040$ at rated frequency
$r_{b2} = 0.060$ at twice rated frequency (negative sequence)
$x_{a1} = x_{a2} = 0.10$ at rated frequency
$x_{b1} = 0.12$ in positive-sequence diagram
$x_b = 0.10$ at rated frequency
$x_{b2} = 0.09$ in negative-sequence diagram
$Z_\phi = 0.10 + j2.8$

Total friction and windage, based on rated volt-ampere input $= 3 \times 0.01$. In assigning values to the rotor constants, it is assumed that the rotor bars are approximately 0.75 in. deep and that the rated frequency is 60 cycles.

If the slip near full load is 0.022, the impedance of the motor to positive and to negative sequence at this slip and with the rotor short-circuited are

$$Z_1 = 0.025 + j0.10 + \frac{(0.10 + j2.8)(1.14 + j0.12)}{1.24 + j2.92}$$

$$= 0.925 + j0.559$$

$$Z_2 = 0.025 + j0.10 + \frac{(0.10 + j2.8)(0.03 + j0.09)}{0.13 + j2.89}$$

$$= 0.0533 + j0.188$$

In computing the impedance to negative sequence, the magnetizing impedance is usually neglected. If this is done, the value of the impedance to negative sequence is $0.055 + j0.19$, the difference between the two values being about 1 per cent.

Knowing the impedances to positive and to negative sequence, the positive- and negative-sequence stator currents can be determined for any values of positive- and negative-sequence applied potentials.

As a concrete example, assume that $V_{a1} = 1.0$ and $V_{a2} = 0.15$. The unbalance in the potentials is purposely exaggerated and is much greater than would be met ordinarily in practice. The magnitudes of the positive- and negative-sequence stator currents are

$$|I_{a1}| = \left| \frac{1}{0.925 + j0.559} \right| = 0.926$$

$$|I_{a2}| = \left| \frac{0.15}{0.0533 + j0.188} \right| = 0.767$$

If the phase relation of the potentials is such that these component currents are in phase in one of the stator phases, the copper loss in this phase will be 285 per cent of its normal full-load value. The average copper loss in all the stator phases is 160 per cent of its full-load value.

The positive- and negative-sequence rotor currents are

$$I_{b1} = \frac{0.10 + j2.8}{1.24 + j2.92} I_{a1}$$

$$= (0.824 + j0.316) I_{a1}$$

and

$$I_{b2} = \frac{0.10 + j2.8}{0.13 + j2.89} I_{a2}$$
$$= (0.98 + j0.009) I_{a2}$$

The stator and rotor negative-sequence currents can be assumed equal without much error, and this is frequently done.

Unit torque is

$$\frac{3p}{2\omega} | V_a | \cdot | I_a |$$

where $V_a$ and $I_a$ are the rated values of the impressed potential and the stator current.

Full-load torque output is $K\frac{3p}{2\omega} | V_a | \cdot | I_a |$ where $K$ is the product of the efficiency of the motor and the power factor at the stator terminals.

The positive-sequence electromagnetic torque is

$$T_1 = \frac{3p}{2s\omega} | I_{b1} |^2 r_{b1}$$
$$= \frac{3p}{2\omega} 0.761$$

The negative-sequence electromagnetic torque is

$$T_2 = -\frac{3p}{2(2 - s)\omega} | I_{b2} |^2 r_{b2}$$
$$= -\frac{3p}{2\omega} 0.0171$$

The negative-sequence torque is only about 2 per cent of the positive-sequence torque, despite the fact that the negative-sequence current is abnormally great.

The positive- and negative-sequence torques, being in the directions of rotation of the positive- and negative-sequence fields, are opposite in direction. The *net* electromagnetic torque is $\frac{3p}{2\omega}0.744$ and the *shaft* torque is $\frac{3p}{2\omega}0.734$.

When the motor is operating under balanced conditions, the greatest torque is developed when the slip $s$ is

$$s = \frac{r_b}{\sqrt{r_a^2 + (x_a + x_b)^2}}$$

This slip is usually between 0.10 and 0.15. When the motor is operating under such unbalanced conditions that the positive- and negative-sequence currents are equal and the slip is 0.15, the ratio of the negative-sequence torque to the positive-sequence torque is $\frac{r_{b2}}{1.85}\frac{0.15}{r_{b1}}$. If the rotor conductors are so deep that $r_{b2}$ is twice $r_{b1}$, the ratio of the torques is approximately 0.16. It should be noted that, for a constant negative-sequence applied potential, the negative-sequence torque is nearly independent of the slip, or of the load on the motor, up to the point of maximum torque or breakdown. It is generally true that unsymmetry in the applied potentials causes little reduction in the torque when the slip is not greater than its full-load value.

It is interesting to note that the negative-sequence electrical input to the rotor is $3|I_{b2}|^2\frac{r_b}{2-s}$, while the negative-sequence copper loss in the rotor is $3|I_{b2}|^2 r_b$, or about twice as great as the input. Since the negative-sequence electromagnetic torque acting on the rotor is in the direction of rotation of the negative-sequence field, an equal and opposite torque must be applied to the shaft to enable the motor to turn in the positive direction. The shaft power due to this torque is $(1-s)\frac{2\omega}{p}\frac{3p}{2(2-s)\omega}|I_{b2}|^2 r_b$. The *sum* of the negative-sequence electrical input to the rotor and this shaft input equals the negative-sequence copper loss in the rotor. In the positive-sequence diagram, the *difference* of the electrical input to the rotor and the shaft output equals the copper loss in the rotor.

For the purposes of computation, the expression for the pulsating torque given in the third and fourth terms of equation (336) can be considerably simplified. If the values of the rotor emfs and currents obtained from the vector diagram or the equivalent circuit are substituted in equation (336), page 291,

$$T_S = \frac{np}{2\omega}[-|E_{b1}|\cdot|I_{b2}|\cos(2\omega t + \alpha + \theta_{b2}) + |E_{b2}|\cdot|I_{b1}|\cos(2\omega t + \alpha + \theta_{b1})] \quad (340)$$

But

$$E_{b1} = I_{b1}Z_{b1}$$

where

$$Z_{b1} = \frac{r_{b1}}{s} + jx_{b1}$$

and

$$E_{b2} = I_{b2}Z_{b2}$$

where

$$Z_{b2} = \frac{r_{b2}}{2 - s} + jx_{b2}$$

It will be observed that the magnitude of the vector difference

$$\frac{np}{2\omega}(-I_{b1}Z_{b1}I_{b2} + I_{b2}Z_{b2}I_{b1})$$

equals the magnitude (maximum value) of the pulsating torque given in equation (340). The pulsating torque is

$$T_S = \frac{np}{2\omega}|\, I_{b1} \,| \cdot |\, I_{b2} \,| \cdot |\, Z_{b1} - Z_{b2} \,| \tag{341}$$

In the numerical example the pulsating torque is

$$\begin{aligned} T_S &= \frac{3p}{2\omega}0.817 \times 0.752 \times |\, 1.11 + j0.03 \,| \\ &= \frac{3p}{2\omega}0.682 \end{aligned}$$

Thus the 120-cycle torque pulsations have an amplitude of 0.92 times the average electromagnetic torque—large enough to cause very objectionable vibrations. It should be observed that the magnitude of these torque pulsations is directly proportional to the magnitude of the negative-sequence potential applied to the stator. In the numerical example the value chosen for the negative-sequence potential is greater than would be met ordinarily in practice.

The copper loss in the stator due to the positive-sequence current is $3 \times 0.926^2 \times 0.025 = 3 \times 0.0214$.

The loss due to the negative-sequence current is

$$3 \times 0.767^2 \times 0.025 = 3 \times 0.0147.$$

The copper loss in the rotor due to the positive-sequence current is $3 \times 0.817^2 \times 0.025 = 3 \times 0.0167$.

The loss due to the negative-sequence current is

$$3 \times 0.752^2 \times 0.06 = 3 \times 0.0339.$$

The core loss in the stator iron, being approximately proportional to the sum of the squares of the positive- and negative-sequence emfs generated by the air gap fluxes, is

$$(|E_{a1}|^2 + |E_{a2}|^2)g_\phi$$

where $g_\phi$ is the conductance of the magnetizing impedance. Since

$$|E_{a1}| = |I_{b1}Z_{b1}| = 0.937$$
$$|E_{a2}| = |I_{b2}Z_{b2}| = 0.071$$

and

$$g_\phi = \frac{0.10}{7.85} = 0.0127$$

the core loss in the stator iron is $3 \times 0.0112$.

If the core loss in the rotor iron equals that in the stator iron when the rotor is at rest, the core loss in the rotor for the assumed operating conditions, if estimated as in Art. 102, page 283, is approximately 0.045 times the stator core loss. The total core loss is $3 \times 0.0117$.

The total losses are

$$3[0.0214 + 0.0147 + 0.0167 + 0.0339 + 0.0117 + 0.01]$$
$$= 3(0.1084 \text{ times the rated volt-ampere per phase}).$$

The total vector volt-ampere input supplied to the motor is

$$3[1(0.793 - j0.479) + 0.15(0.209 - j0.737)]$$
$$= 3(0.824 - j0.590)$$

The power factor at the stator terminals is 0.812.

The shaft output is $3(0.824 - 0.108)$.

The efficiency is 0.880 or 88 per cent.

This example illustrates the method by which the performance of the induction machine can be calculated when the machine is operated under any given unbalanced conditions on the stator side.

**107. Points of Maximum Loss.**—When unbalanced potentials are applied to the motor, the stator currents are unbalanced and the air-gap flux varies elliptically. This results in unequal

copper losses in the phases of the stator and unequal core losses in the stator teeth. Fortunately, maximum copper loss and maximum tooth loss do not occur at the same point. For given positive- and negative-sequence components in the applied potentials and for a given load on the motor, both the positive- and negative-sequence components of the stator currents and the corresponding components of the air-gap flux are fixed in magnitude. The greatest heating in a stator phase will occur when the positive- and negative-sequence components of the current in that phase are in time-phase. For this condition the positive- and negative-sequence components of the emf generated in the phase in question by the air-gap flux have a time-phase displacement $\theta_{a1} - \theta_{a2}$. The tooth loss at a point $\pi x/P$ radians from the origin is

$$K\left(\phi_1^2 + \phi_2^2 - 2\phi_1\phi_2 \cos \frac{2\pi x}{P}\right)$$

where $\phi_1$ and $\phi_2$ are the maximum values of the tooth flux due to the positive- and negative-sequence components of the air-gap flux, the origin being taken at the point in the gap where the flux density is zero when the component fields coincide. If a phase belt of conductors is located at the origin, the positive- and negative-sequence components of the emf generated in the belt by the air-gap flux will be in time-phase. If the belt of conductors is located at a point $\pi x/P$ radians from the origin, the positive- and negative-sequence emfs will be displaced $2\pi x/P$ in time-phase. For the condition of maximum heating loss in this phase belt, the time-phase displacement of the positive- and negative-sequence components of the generated emf is $\theta_{a1} - \theta_{a2}$. That is, $2\pi x/P$ equals $(\theta_{a1} - \theta_{a2})$ and the tooth loss at the center of the belt is

$$K[\phi_1^2 + \phi_2^2 - 2\phi_1\phi_2 \cos(\theta_{a1} - \theta_{a2})].$$

At no load, $(\theta_{a2} - \theta_{a1})$ is about 10 degrees and the tooth loss has approximately its least value. At full load, $(\theta_{a1} - \theta_{a2})$ is about 50 degrees and the tooth loss is still considerably less than its average value. Thus, when the conditions are such that the copper loss in a phase has its greatest value, the loss in the adjacent teeth is less than the average tooth loss. This

is a favorable condition since the hottest conductors are surrounded by the cooler teeth.

**108. Single-phase Motor.**—A special and interesting case of unbalance is that in which a three-phase motor operates with one of its terminals disconnected from the source of power. If $Z_1$ and $Z_2$ are the impedances of the motor to positive and negative sequence and if $I_1$ and $I_2$ are the positive- and negative-sequence components of the current in the open phase of the stator, the following relations exist (see page 235)

$$I_1 + I_2 = 0$$
$$-j\sqrt{3}I_1Z_1 + j\sqrt{3}I_2Z_2 = -jV$$

The line potential is written $-jV$ so that the expression for the positive-sequence current is

$$I_1 = \frac{V}{\sqrt{3}(Z_1 + Z_2)}$$

The positive-sequence stator current equals the normal phase potential $V/\sqrt{3}$ divided by the sum of the phase-impedances of the motor to positive and negative sequence. Since the line current taken by the motor is $\sqrt{3}|I_1|$ it equals the applied line potential divided by the sum of the phase impedances of the motor to positive and negative sequence. This relation is represented by the equivalent circuit of Fig. 107. Since the stator resistances and reactances in the positive- and negative-sequence equivalent circuits are always equal, the subscripts 1 and 2 may be omitted. At standstill the rotor resistances and reactances in the positive- and negative-sequence equivalent circuits are also equal. The positive- and negative-sequence torques are then equal and the motor will not start. The positive- and negative-sequence terminal potentials are also equal at starting. If torque is applied to the motor shaft so that it runs with a slip $s$ in what may be taken as the positive direction, the equivalent rotor resistance $r_{b1}/s$ in the positive-sequence diagram becomes greater than the corresponding resistance $r_{b2}/(2 - s)$ in the negative-sequence diagram and

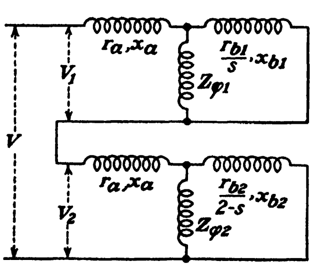


FIG. 107.

the torque in the positive direction is greater than the torque in the negative direction. If the net torque is great enough to overcome the retarding torque, the motor speeds up until it reaches a stable operating condition near synchronous speed. At the same time, $V_{a1}$ increases and $V_{a2}$ decreases, owing to the changes taking place in the impedance of the rotor winding to positive and to negative sequence.

It should be observed that, since breakdown occurs at a slip of about 6 to 10 per cent, the impedance of the motor to negative sequence is practically constant during stable operation. Furthermore, because of the relatively low value of the rotor impedance in the negative-sequence equivalent circuit, it is sufficiently accurate to disregard the magnetizing impedance $Z_{\phi 2}$ in this circuit. Thus, the equivalent circuit for the motor, when operating with one terminal free, is similar to the equivalent circuit used when the motor operates under balanced conditions, provided the resistance and leakage reactance of the stator are increased by about 200 per cent. When the constants of the motor and the slip at which it is running are known, its complete performance can be calculated by the general method just outlined. For example, if the representative motor is operating with one terminal free, but with normal line potential across the other two terminals, and the slip has the same value as before (page 301), 0.022, the positive-sequence stator current is

$$\frac{1}{(0.925 + j0.559) + (0.0533 + j0.188)} = 0.691 - j0.479$$

The operating power factor is 0.822. If the motor had been operating at the same slip but with balanced line potentials, the power factor would have been 0.856. The positive-sequence rotor current $I_{b1}$ is $(0.824 + j0.316)I_{a1}$ (page 302) and the positive-sequence torque is

$$T_1 = \frac{3p}{2\omega}0.63$$

This represents a reduction of 17.1 per cent from the torque with balanced applied potentials. Since the negative-sequence rotor current is $0.98I_{a2}$ and since $|I_{a2}| = |I_{a1}|$, the negative-sequence torque is

$$T_2 = -\frac{3p}{2\omega}0.020$$

The net torque is $\frac{3p}{2\omega}0.61$.

This is nearly 20 per cent less than the torque when operating under balanced conditions and at the same slip, 0.022. A more important fact, however, is that the motor will break down at a much smaller torque than when operating under balanced conditions. Since the negative-sequence torque is small compared with the positive-sequence torque, it is sufficiently accurate to assume that the motor will break down when the positive-sequence torque reaches its maximum value. Actually, the motor will break down for a slightly smaller slip than that corresponding to the maximum positive-sequence torque and the actual breakdown torque is slightly greater than the computed value. When the motor operates under balanced conditions, breakdown occurs when

$$s = \frac{r_{b1}}{\sqrt{r_{a1}^2 + (x_{a1} + x_{b1})^2}}$$

For the representative motor whose constants have been given, breakdown occurs at slip 0.113. Since the impedance of the motor to negative sequence is approximately

$$r_{a2} + \frac{r_{b2}}{2} + j(x_{a2} + x_{b2})$$

and since, during single-phase operation, this impedance is in effect added to the impedance of the stator in the positive-sequence equivalent circuit (Fig. 105), the positive sequence torque is maximum when the slip is

$$s = \frac{r_{b1}}{\sqrt{\left(r_{a1} + r_{a2} + \frac{r_{b2}}{2}\right)^2 + (x_{a1} + x_{b1} + x_{a2} + x_{b2})^2}}$$

For the representative motor this is a slip of 0.06.

When it is operating single phase at no load ($s = 0$ approximately) the positive sequence magnetizing current is approximately

$$\frac{V}{\sqrt{3}(x_{a1} + x_{a2} + x_{b2} + x_{\phi 1})}$$

We shall follow the method suggested on page 298 and subtract numerically from $V/\sqrt{3}$ the potential drop in the reactance $(x_{a1} + x_{a2} + x_{b2})$ due to the magnetizing current. If the potential $V/\sqrt{3}$ is taken as unity, this subtraction gives

$$V' = 0.90.$$

Note that, when the motor operates single-phase, the effect of the drop due to the magnetizing current is about three times as great as when the motor operates three-phase. The expression for the positive-sequence component $I_{b1}$ of the rotor current is approximately

$$I_{b1} = \frac{V'}{\sqrt{\left(r_{a1} + r_{a2} + \frac{r_{b1}}{s} + \frac{r_{b2}}{2 - s}\right)^2 + (x_{a1} + x_{a2} + x_{b1} + x_{b2})^2}}$$
$$= 1.41 \qquad \text{for} \qquad s = 0.06$$

The expression for the maximum value of the positive-sequence torque is

$$\frac{\frac{3p}{4\omega}(V')^2}{\left(r_{a1} + r_{a2} + \frac{r_{b2}}{2}\right) + \sqrt{\left(r_{a1} + r_{a2} + \frac{r_{b2}}{2}\right)^2 + (x_{a1} + x_{b1} + x_{a2} + x_{b2})^2}}$$
$$= \frac{3p}{2\omega}0.82$$

The negative-sequence component, $I_{b2}$, of the rotor current is approximately

$$I_{b2} = \frac{Z_{b1} + Z_\phi}{Z_\phi} I_{b1}$$

or

$$I_{b2} = 1.48$$

The negative-sequence torque is

$$\frac{3p}{2\omega} I_{b2}^2 \frac{r_{b2}}{2 - s}$$

or

$$T_2 = \frac{3p}{2\omega}0.068$$

The breakdown torque is $\frac{3p}{2\omega}0.75$ and is 0.40 times the breakdown torque when the motor operates with balanced 3-phase potentials applied to the stator.

**109. Induction Generator.**—We shall now consider the case in which the rotor moves in the same direction as the magnetic

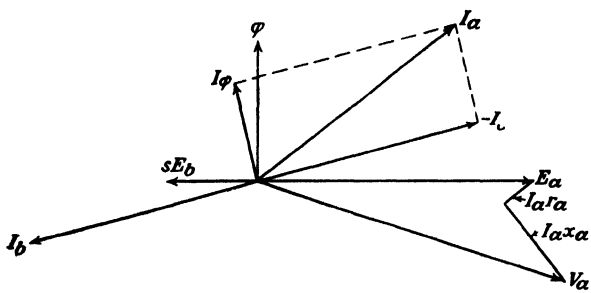



FIG. 108.

field but at a speed greater than that of the field, the slip $s$ being negative (see page 275). The vector diagram which represents the potential and current relations is shown in Fig. 108. As in Fig. 102, the potentials are taken as rises in potential in the direction of the current, but the slip $s$ is negative. The result is that the relative phase of the rotor currents is changed by nearly 180 degrees. The rotor current $I_b$ *appears* to *lead* the emf, $sE_b$, generated in the rotor by the air-gap flux, whereas when the rotor is running with a positive slip the rotor current *appears* to *lag* the emf generated in the rotor by the air-gap flux (Fig. 102). Since the electrical characteristics of the rotor winding are those of resistance and inductance, the current in the short-circuited rotor winding always *lags* the emf generated in the winding by the air-gap flux *for all values of the slip, both positive and negative.* Consider a cage winding rotating in the same direction as the magnetic field. In Fig. 109 the ordinates of the sine curve represent the resultant air-gap flux density which is rotating at synchronous speed in the positive direction (toward the right). The rotor bars are also moving in the same direction.

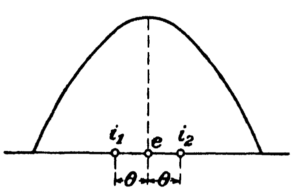



FIG. 109.

The emf generated in a rotor bar passes through its maximum value at the moment the bar is at the point of maximum flux density, as at *e*. If the speed of the bar is less than the speed of the field, the direction of this emf is away from the observer, whereas, if the speed of the bar is greater than the speed of the field, the direction of the emf is toward the observer. Owing to the presence of resistance and leakage inductance in bar and end ring the current in the bar, in either case, will reach its maximum value *after* the emf has passed through its maximum value. If the speed of the bar is less than the speed of the field, the bar will have slipped back $\theta$ degrees with respect to the field by the time the current in the bar reaches its maximum value and the bar will then occupy the relative position $i_1$. The angle $\theta$ is determined by the resistance and leakage inductance of the bar and end ring and the frequency of the current. If the angle between successive bars is also $\theta$, another bar will be at the point where the flux density is maximum and the generated emf in this bar will have its maximum value. Thus there will be simultaneously a bar at the point *e* in which the emf is maximum and a bar at the point $i_1$ in which the current is maximum. If it were possible to observe them, these points at which the emf and current are maximum would appear to flash by a *stationary* observer at synchronous speed in the positive direction and in this sense it might be said that the current appears to lag the generated emf. On the other hand, if the speed of the rotor bars is greater than the speed of the magnetic field, the bar will have moved forward $\theta$ degrees with respect to the field by the time the current in the bar reaches its maximum value and the bar will occupy the relative position $i_2$. Now there will be simultaneously a bar at the point *e* in which the emf is maximum and a bar at the point $i_2$ in which the current is maximum. To a *stationary* observer it would appear as though the current were *leading* the emf, since the point at which the current is maximum appears to be *ahead* of the point at which the emf is maximum. In Fig. 110 is drawn a portion of the vector

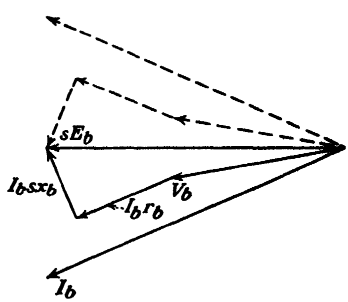



FIG. 110.

diagram which applies to a wound rotor across whose terminals there is a potential $V_b$. The *full* lines in this diagram show how the phase relations of the rotor potentials and currents appear to a stationary observer, while the *dotted* lines show the *actual* phase relations with respect to the emf $sE_b$ generated in the rotor by the air-gap flux. In any problem whose solution depends

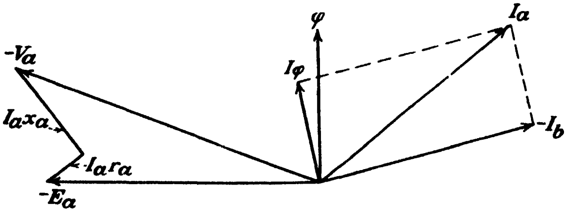



FIG. 111.

upon the actual phase relations of rotor potentials and rotor current, this difference in phase relations must be recognized. In Fig. 111 the potentials are drawn as *falls in potential* in the direction of the current and the stator current $I_a$ is assumed to consist of two components, a flux-producing one, $I_\phi$, and a so-called "load component" equal in magnitude and opposite in phase to the rotor current $I_b$. It is assumed that the rotor is short-circuited. The equivalent circuit which corresponds to the vector diagram Fig. 111 is in Fig. 112. In this equivalent circuit it is usual to indicate the negative quality of the slip by prefixing a minus sign so that $s$ itself is a positive number. It is most important to note that, while this equivalent circuit does not indicate the actual phase relation of the rotor current with respect to the rotor emf, it does indicate the correct relation between the current in the stator and the fall in potential across the stator terminals. This theory forms an essential part of the analysis of the induction machine whose rotor circuit is unsymmetrical.

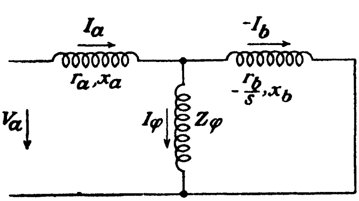



FIG. 112.

**110. Unsymmetrical Rotor Circuit.**—When an induction motor is operating with the stator connected to a symmetrical source of supply but with unbalanced impedances connected across

the terminals of the rotor, there will be produced in the rotor unbalanced currents of slip frequency, $sf$, which can be resolved into positive- and negative-sequence components. In the stator there will be two systems of balanced currents, one, the positive-sequence system, having a frequency $f$ equal to that of the supply and the other, the negative-sequence, having a frequency $(1 - 2s)f$.

Note that the negative-sequence system of stator currents is not negative sequence in the sense that the phase order of the currents is opposite to the phase order of the positive-sequence stator currents. The system is designated as "negative sequence" because the currents are due to the action of the negative-sequence currents in the rotor. When the slip $s$ is less than 0.5, the actual phase order of these stator currents is the same as the phase order of the positive-sequence stator currents, but for greater values of the slip their phase order is reversed. A similar use of the term "negative sequence" was made when unbalanced potentials were applied to the stator. In that case the positive-sequence rotor currents had a frequency $sf$, while the negative-sequence rotor currents had a frequency $(2 - s)f$. The negative-sequence rotor currents took their name from the fact that they were due to the effect of the negative-sequence stator currents. When the slip $s$ is positive, the actual phase orders of these two systems of rotor currents are opposite, but if the slip is negative, as it is when the machine operates as an induction generator, the actual phase orders of the two systems of rotor currents are the same.

When the rotor circuits are unsymmetrical, positive-sequence currents in the rotor produce a field which moves in the positive direction at slip speed with respect to the rotor, but, owing to the forward motion of the rotor, this field moves at synchronous speed with respect to the stator. This is the same condition that exists when the unsymmetry is on the stator side. All the reactions concerned with the currents which produce this field are analyzed by means of a positive-sequence vector diagram or a corresponding equivalent circuit applying to the case in which the speed of the rotor is less than the speed of the field.

The negative-sequence currents in the rotor produce a field which moves in the negative direction at slip speed with respect to the rotor, but, owing to the forward motion of the rotor,

this field moves in the positive direction with respect to the stator at $(1 - 2s)$ times synchronous speed. When the slip is 0.5, this field is stationary with respect to the stator and produces no reaction in it, but if $s$ becomes greater than 0.5, the field reverses its direction of motion with respect to the stator and moves in the negative direction with respect to both stator and rotor. All the reactions concerned with the currents which produce this field are analyzed by means of a negative-sequence vector diagram or a corresponding equivalent circuit applying to the case in which the speed of the rotor is greater than, and in the same direction as, the speed of the field. The relations between these positive- and negative-sequence equivalent circuits

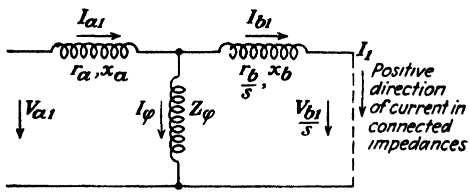



FIG. 113.—Note that Fig. 113 is like Fig. 105 except for the omission of the minus signs.

are determined by the unsymmetrical conditions existing on the rotor side.

The unbalanced impedances connected across the terminals of the rotor have the effect of producing both positive- and negative-sequence components in the terminal potentials of the rotor. Thus, the positive-sequence equivalent circuit of the motor should be shown with a terminal potential $V_{b1}/s$ across the rotor, where $V_{b1}$ is the actual positive-sequence component of the terminal potentials of the rotor and $s$ is the slip (see Fig. 113). If *equal* impedances were connected in $Y$ across the terminals of the rotor, $V_{b1}$ would be the potential drop through one of them. A convenient method of continuing the analysis is as follows:

It can be assumed that the negative-sequence potential across the rotor terminals, which arises from the unsymmetry in the connected impedances, acts like the potential applied to one of the windings of an induction machine whose other windings, in this case the stator, are short-circuited through a balanced

circuit. From this point of view the equivalent circuit used in analyzing the negative-sequence reactions in the motor is shown in Fig. 114. The potential applied to the rotor is $V_{b2}$ of frequency $sf$. The resistance of the rotor at the actual slip frequency is $r_b$ and its reactance is $sx_b$, according to the method established for expressing the reactance. Since the magnetizing impedance observed from the rotor is the ratio of the emf $sE_b$, generated by the air-gap flux, to the component current $I_\phi$, which produces the flux, the magnetizing impedance is $sZ_\phi$, where $Z_\phi$ is the magnetizing impedance observed on the stator side. If it is assumed that both stator and rotor are given a backward speed $(1 - s)$ times synchronous speed, the rotor will become stationary and the stator will have a speed $(1 - s)$

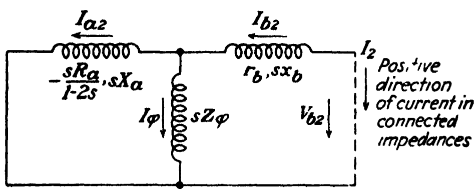



FIG. 114.

times synchronous speed in the same direction in which the negative-sequence field rotates. The slip of the stator with respect to this negative-sequence field is

$$\frac{s - (1 - s)}{s} \quad \text{or} \quad -\frac{(1 - 2s)}{s}$$

The negative-sequence stator current, of frequency $(1 - 2s)f$, must also flow in the supply network to which the stator is connected. The resistance $R_a$ and reactance $X_a$ must include, therefore, not only the stator constants but also the resistance and reactance per phase of the supply system viewed from the terminals of the stator. If the stator is directly connected to a supply system of very much greater kilovolt-ampere capacity than the motor, the resistance and reactance of the system will be negligibly small compared with the constants of the motor so that $R_a$ and $X_a$ are the resistance and leakage reactance of the stator winding alone. On the other hand, if the stator is connected to such a supply system through a feeder whose impedance has approximately the same value as the leakage

impedance of the stator, $R_a$ and $X_a$ must include the resistance and reactance of the feeder. Note that $R_a$ is the resistance of the circuit at frequency $(1 - 2s)f$ and that $X_a$ is $2\pi f$ times the inductance of the circuit at this same frequency. Hence, the actual reactance of the stator circuit at the actual stator frequency is $(1 - 2s)X_a$. The resistance that appears in the equivalent circuit, being the actual resistance divided by the slip, is $\dfrac{-sR_a}{1 - 2s}$ and the reactance, being the actual reactance divided by the slip, is $\dfrac{(1 - 2s)}{(1 - 2s)}sX_a$ or $sX_a$ as shown. Since, in the positive-sequence diagram (Fig. 113) the terminal potential of the rotor appears as $V_{b1}/s$, it is better to modify the negative-sequence diagram to conform with $V_{b1}/s$ by dividing the potential $V_{b2}$ and the circuit constants by $s$, as in Fig. 115. When the positive- and negative-sequence equivalent circuits have been established, the next step is to determine the relation that exists between the terminal potential of the rotor in the positive-sequence circuit and the corresponding potential in the negative-sequence circuit. If the zero-, positive-, and negative-sequence components of the three unbalanced impedances connected in $Y$ across the rotor are $Z_0'$, $Z_1'$, and $Z_2'$, the positive- and negative-sequence components of the terminal potential of the rotor are

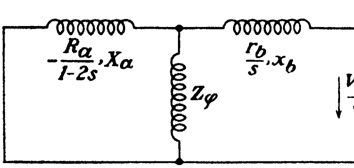


FIG. 115

$$V_{b1} = I_1Z_0' + I_2Z_2' \tag{342}$$
$$V_{b2} = I_2Z_0' + I_1Z_1' \tag{343}$$

On dividing equations (342) and (343) by the slip $s$,

$$\frac{V_{b1}}{s} = I_1\frac{Z_0'}{s} + I_2\frac{Z_2'}{s} \tag{344}$$
$$\frac{V_{b2}}{s} = I_2\frac{Z_0'}{s} + I_1\frac{Z_1'}{s} \tag{345}$$

Note that, if the impedances connected across the rotor terminals are of the form $r + jsx$, where $x$ is the reactance at the rated frequency $f$, the components, $Z_0'/s$, $Z_1'/s$, $Z_2'/s$ can be computed

from impedances of the form $\frac{r}{s} + jx$. In order to simplify the notation, replace $Z_0'/s$, $Z_1'/s$, $Z_2'/s$ by $Z_0$, $Z_1$, $Z_2$. It should be noted that the values of these impedances are given on the basis of a one-to-one ratio between the stator and rotor effective turns per phase. Thus the actual values of the impedances connected to the rotor must first be multiplied by the square of the ratio of transformation for the motor.

Note that the positive- and negative-sequence currents $I_1$ and $I_2$ are the components of the currents flowing in the impedances connected to the rotor and in the direction of the fall in potential across the rotor terminals. Note also that the direction of the positive-sequence rotor current in the positive-sequence equivalent circuit is the *same* as the direction of the positive-sequence current in the connected impedances, but that the direction of the negative-sequence rotor current in the negative-sequence equivalent circuit is *opposite* to the direction of the negative-sequence current in the connected impedances. That is, $I_1 = I_{b1}$, but $I_2 = -I_{b2}$.

If $Z_{m2}$ is the impedance of the motor to negative sequence as determined by the negative-sequence equivalent circuit of Fig. 115,

$$\frac{V_{b2}}{s} = I_{b2}Z_{m2}$$

and

$$I_{b2}Z_{m2} = -I_{b2}Z_0 + I_{b1}Z_1$$

or

$$I_{b2}(Z_{m2} + Z_0) = I_{b1}Z_1$$

If this relation between the positive- and negative-sequence rotor currents is substituted in equation (344),

$$\frac{V_{b1}}{s} = I_{b1}\left(Z_0 - \frac{Z_1Z_2}{Z_{m2} + Z_0}\right) \tag{346}$$

Since the ratio $V_{b1}/sI_{b1}$ can be looked upon as the value of the impedance which is connected across the rotor terminals in the positive-sequence equivalent circuit, the positive-sequence current can be computed from the circuit shown in Fig. 116. Now the complete performance of the motor can be determined by following the procedure that has already been outlined.

The power developed in the rotor due to the positive-sequence current depends upon the resistance of that portion of the equivalent circuit of Fig. 117 which lies to the right of the point 0. If the exciting impedance $Z_{\phi 2}$ is disregarded, this resistance is

$$\frac{2r_b}{s} - \frac{R_a}{(1 - 2s)}$$

If the negative-sequence exciting impedance is taken as $0 + jx$, or if the attendant core losses are neglected, this resistance is

$$\frac{2r_b}{s} - \frac{\dfrac{x^2 R_a}{1 - 2s}}{\left(\dfrac{R_a}{1 - 2s}\right)^2 + (X_a + x)^2} \tag{347}$$

This last expression needs to be used only when the slip is in the neighborhood of 0.5. This resistance may be negative when the slip is about 0.5 but somewhat less than this value. The range in the slip for which the resistance is negative depends upon the values of the resistances and reactances in the expression (347). When the resistance is negative, the positive-sequence torque is also negative, but otherwise the torque is positive. The expression for the average value of the positive-sequence torque developed by the motor (3-phase) is

$$T_1 = \frac{3p}{2\omega}|I_{b1}|^2\left[\frac{2r_b}{s} - \frac{\dfrac{x^2 R_a}{1 - 2s}}{\left(\dfrac{R_a}{1 - 2s}\right)^2 + (X_a + x)^2}\right]$$

Since the positive- and negative-sequence rotor currents are numerically equal, the expression for the average value of the total torque developed by the motor is

$$T_{av} = \frac{3p}{2\omega}|I_{b1}|^2\left[\frac{2r_b}{s} - \frac{2\dfrac{x^2 R_a}{1 - 2s}}{\left(\dfrac{R_a}{1 - 2s}\right)^2 + (X_a + x)^2}\right] \tag{348}$$

The positive-sequence component of the rotor current can be computed by the method of Art. 105 page 298. Following this procedure, let $V_1'$ equal the numerical difference of $V_1$ and the

potential drop in the stator due to the no-load current when the motor is operating with all three rotor terminals short-circuited. Then the positive-sequence rotor current is

$$|I_{b1}|^2 = |V_1'|^2 \div \left\{ \left[ r_a + \frac{2r_b}{s} - \frac{\dfrac{x^2 R_a}{1-2s}}{\left(\dfrac{R_a}{1-2s}\right)^2 + (X_a + x)^2} \right]^2 + \left[ x_a + 2x_b + \frac{x\left(\dfrac{R_a}{1-2s}\right)^2 + xX_a(X_a + x)}{\left(\dfrac{R_a}{1-2s}\right)^2 + (X_a + x)^2} \right]^2 \right\}$$

For very small values of slip, the following relation holds approximately:

$$|I_{b1}|^2 = |V_1'|^2 \div \left(\frac{2r_b}{s}\right)^2$$

and the average torque is

$$T_{av} = \frac{3p}{2\omega}|V_1'|^2 \frac{s}{2r_b} \text{ approximately}$$

Hence, for the same *small* values of slip, the torque, when but two of the rotor terminals are short-circuited, is one-half the torque when all three rotor terminals are short-circuited.

**112. Pulsating Torque Due to Effect of Single-phase Rotor Winding.**—The instantaneous torque developed by a motor having a single-phase rotor winding pulsates at twice slip frequency. This pulsating torque can be computed by the method discussed in Art. 104, page 287. Since the magnetic fields in the air gap rotate in opposite directions with respect to the rotor, the pulsating torque should be computed by means of equation (338), page 291. In this equation $E_1/\omega_1$ equals $sE_{b1}/s\omega$ and $E_2/\omega_2$ equals $sE_{b2}/s\omega$. Also, $I_1$ equals $I_{b1}$ and $I_2$ equals $I_{b2}$. Refer to page 299 for a comparison of the vector diagram and the equivalent circuit. In equation (338) the phase angle $\theta_1$ is the angle between the positive-sequence rotor current and *rise* in generated emf and equals the angle between $I_{b1}$ and $E_{b1}$ in Fig. 113, page 316. The phase angle $\theta_2$ is the angle between the negative-sequence rotor current and *rise* in generated emf and equals the *supplement* of the angle between $I_{b2}$ and the *fall* in emf $E_{b2}$ in Fig. 114,

page 317. Note that $E_{b1}$ and $E_{b2}$, which are not indicated in Figs. 113 and 114, are the *falls* in potential across the magnetizing branches of the equivalent circuits. The phase angles between $I_{b1}$ and $E_{b1}$ and between $I_{b2}$ and $E_{b2}$ are determined by the two impedances $Z_1$ and $Z_2$; where $Z_1$ equals the impedance

$$\left(\frac{2r_b}{s} + j2x_b\right)$$

plus the impedance $Z_2$ of the parallel branches $Z_{\phi 2}$ and

$$\left(-\frac{R_a}{1-2s} + jX_a\right)$$

[see Fig. 117]. Let $\alpha_1$ and $\alpha_2$ be the angles of the impedances $Z_1$ and $Z_2$. If the angle $\alpha_1$ is *positive*, $I_{b1}$ *lags* $E_{b1}$, and consequently $\theta_1 = -\alpha_1$, since both $\theta_1$ and $\theta_2$ are *positive* for *leading* currents. If the angle $\alpha_2$ is positive, $I_{b2}$ lags $E_{b2}$ and consequently

$$\theta_2 = \pi - \alpha_2.$$

Therefore, in equation (338), $-\cos(\theta_1 - \theta_2)$ equals $\cos(\alpha_1 - \alpha_2)$ and the magnitude of the pulsating torque equals the magnitude of the vector sum $\frac{3p}{2\omega}(I_{b2}I_{b1}Z_1 + I_{b1}I_{b2}Z_2)$, or

$$T_S = \frac{3p}{2\omega}|I_{b1}|^2 \cdot |Z_1 + Z_2|$$

If $Z_{\phi 2}$ is taken as $0 + jx$, the sum $Z_1 + Z_2$ is

$$|Z_1 + Z_2| = \left\{\left[\frac{2r_b}{s} - \frac{\dfrac{2x^2R_a}{1-2s}}{\left(\dfrac{R_a}{1-2s}\right)^2 + (X_a + x)^2}\right]^2 + \left[2x_b + 2\frac{x\left(\dfrac{R_a}{1-2s}\right)^2 + xX_a(X_a + x)}{\left(\dfrac{R_a}{1-2s}\right)^2 + (X_a + x)^2}\right]^2\right\}^{\frac{1}{2}}$$

This expression shows, by comparison with equation (348), that the *maximum* value of the *pulsating* torque is always somewhat greater than the *average* torque.

**113. Zero Sequence.**—A 3-phase induction motor can be operated under single-phase conditions by applying potential between one of the stator terminals and the neutral point of the windings as in Fig. 118. The current supplied to the motor will then consist of a positive-sequence, a negative-sequence,

and a zero-sequence component. The applied potential, $V$ will likewise consist of a positive-, a negative-, and a zero-sequence component. The relations between the positive- and negative-sequence components of potential and current have already been discussed. The ratio of the zero-sequence component of the applied potential to the zero-sequence component of the stator current is the impedance of the motor to zero sequence. The impedances to positive and negative sequence depend upon the slip. The impedance to zero sequence may or may not depend upon the slip according to the type of rotor winding. For the usual small values of slip the variation in the value of the impedance is very slight. The value of the impedance to zero sequence can be measured by connecting the three line terminals together (Fig. 119$a$) and applying potential between this common point and the neutral, $n$, of the stator. The impedance to zero sequence is the vector ratio of this applied potential to the current in *one* of the stator phases. During this test the motor should, in general, be run at approximately

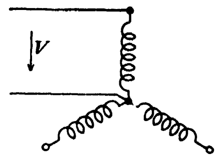



FIG. 118.

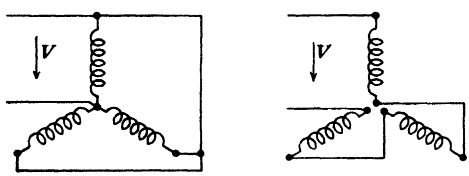



FIG. 119.

the same speed as that for which the calculations of performance are made subsequently. If it is possible to connect the three stator phases in series, as in Fig. 119$b$, the impedance to zero sequence is the vector ratio of one-third the applied potential $V$ to the current. When the motor is operated as in Fig. 118,

$$I_{a0} = I_{a1} = I_{a2} = \frac{I_{an}}{3}$$

$$V_{a0} + V_{a1} + V_{a2} = V_{an}$$

which can be written

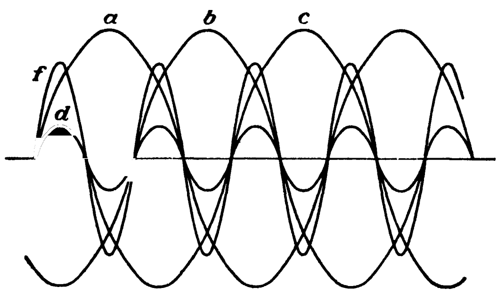
$$I_{a0}Z_0 + I_{a1}Z_1 + I_{a2}Z_2 = V$$

where $Z_0$, $Z_1$, $Z_2$ are the impedances per phase of the motor to zero, positive, and negative sequence. Since the component currents are equal,

$$I_0 = I_1 = I_2 = \frac{V}{Z_0 + Z_1 + Z_2}$$

The current $I_{an}$ the motor takes, being equal to the sum of its symmetrical components is

$$I_{an} = \frac{V}{\frac{1}{3}(Z_0 + Z_1 + Z_2)},$$




*The three fundamental components, a, b, c are spaced 120° apart, giving a resultant of zero. The three third-harmonic components, d, are in space-phase, giving a resultant, f, of 3 times each component.*

FIG. 120.

or it is the applied potential divided by the average of the impedances to zero, positive, and negative sequence.

The impedance to zero sequence can be determined by measurement as in Fig. 119, *a* or *b*, but it is of considerable interest to construct the equivalent circuit and to observe the relations that exist among its component parts and the impedances that appear in the positive- and negative-sequence equivalent circuits. There can be no fundamental component in the air-gap flux due to the zero-sequence components of the currents since these currents are equal at any instant in three equal windings displaced 120 degrees in space phase. This is also true of all the harmonic fluxes except the third and its multiples. The third-harmonic flux components, due to the individual phases, are all equal and are all in space phase. Thus there is a resultant third-harmonic flux distribution which is three times as great as

would be produced by one phase alone. These conditions are illustrated in Fig. 120. This third-harmonic flux distribution pulsates at fundamental frequency and is equivalent to two equal third-harmonic fields which rotate in opposite directions at one-third synchronous speed, or at one-third the speed of the fundamental flux produced by the positive-sequence components of the stator currents.

If the slip of the rotor with respect to this fundamental flux is $s$, the slip with respect to the positively rotating third-harmonic field is $(-2 + 3s)$ and the slip with respect to the negatively rotating third-harmonic field is $(4 - 3s)$. Thus, emfs having corresponding frequencies will be produced in the rotor winding. If the rotor winding consists of coils symmetrically grouped in three phases, the emfs generated in them by a third-harmonic field are in time-phase. If the rotor windings are connected in $Y$, these emfs will produce no difference in potential across their outer terminals and no currents will flow if these terminals are short-circuited. If, however, each individual rotor phase is short-circuited when $Y$ connected, or if the windings are connected in $\Delta$, currents equal in magnitude and in time-phase with each other will flow in the windings. They are due to the emfs produced by the two third-harmonic rotating fields. These zero-sequence rotor currents will produce only third-harmonic component fields which rotate synchronously with the corresponding fields produced by the zero-sequence currents in the stator winding. If the rotor winding consists of a symmetrical cage, the short-circuit currents flowing in the rotor bars will produce third-harmonic fields which rotate synchronously with the corresponding fields produced by the zero-sequence currents in the stator winding.

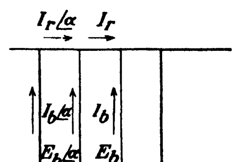



FIG. 121.

**114. Cage Winding.**—We shall now consider in detail the cage winding. Assume there are $n$ bars, each having a leakage impedance $Z_b''$ at the given rotor frequency, and assume that each end ring has a total leakage impedance $Z_r$. The relations between the bar and ring currents and the emf generated in each bar by the particular air-gap flux being considered are determined by solving the two Kirchhoff equations which apply to the cage winding (Fig. 121). Thus,

$$I_b/\underline{\alpha} + I_r/\underline{\alpha} = I_r$$

where $\alpha = \frac{p}{2}\frac{360°}{n}$; $n$ is the total number of rotor bars and $p$ is the number of poles of the flux distribution whose effect is being analyzed. In a 6-pole induction motor, $p$ is 6 for the fundamental flux distribution and is 18 for the third-harmonic distribution. Also,

$$(I_b/\underline{\alpha} - I_b)Z_b'' + 2I_r\frac{Z_r}{n} = E_b/\underline{\alpha} - E_b$$

where $E_b$ is the emf generated by the air-gap flux in the bar which carries the current $I_b$. The solution of these two equations for the current $I_b$ in the bar is

$$I_b = \frac{E_b}{Z_b'}$$

where the equivalent leakage impedance $Z_b'$ of the bar is

$$Z_b' = Z_b'' + \frac{Z_r}{2n\sin^2\frac{\alpha}{2}} \tag{349}$$

This equivalent leakage impedance of each bar must be referred to the stator. This is accomplished by the following procedure. First assume that the cage winding is replaced by a 3-phase winding arranged exactly like the stator winding. The phase impedance of this 3-phase rotor winding must be adjusted so that the rotor currents will produce a component flux distribution in the air gap indistinguishable from the distribution produced by the currents in the actual cage winding.

If a set of coils having one turn per pole are equally spaced around a uniform air gap, the flux distribution in the gap is rectangular if the reluctance of the iron portion of the magnetic circuit is negligible compared with the reluctance of the air gap. The flux density $B_0$ per ampere is $B_0 = 0.4\pi/g$ where $g$ is the length of the air gap in centimeters. The effect of the teeth and slots can be taken into account by increasing $g$, or by using an equivalent length of air gap. The maximum value of the fundamental and third-harmonic components of this rectangular distribution are

$$B_1 = \frac{4}{\pi}\frac{0.4\pi}{g} = \frac{1.6}{g}$$
$$B_3 = \frac{1}{3}\frac{4}{\pi}\frac{0.4\pi}{g} = \frac{0.53}{g}$$

Since the average value of a sinusoidal distribution is $2/\pi$ times the maximum value, the fundamental flux per pole is $A\frac{2}{\pi}\frac{1.6}{g}$ or $1.02\frac{A}{g}$; the third-harmonic flux per pole is $\frac{A}{3}\frac{2}{\pi}\frac{0.53}{g}$ or $0.113\frac{A}{g}$; where $A$ is the fundamental pole area in square centimeters. In a 6-pole motor $A$ is one-sixth the cylindrical surface of the rotor. If the coils carry a sinusoidal current of 1 ampere, both the fundamental and third-harmonic flux distributions pulsate and each may be replaced by two equal components rotating in opposite directions in the air gap. Since these rotating components are each one-half the maximum value of the pulsating field, the fundamental rotating components per ampere-turn are each $\frac{1.02}{\sqrt{2}}\frac{A}{g}$ or $0.72\frac{A}{g}$; the third-harmonic rotating components per ampere-turn are each $\frac{0.113}{\sqrt{2}}\frac{A}{g}$ or $0.080\frac{A}{g}$. For simplicity, assume that the windings are of the series type, all of the coils in any phase being connected in series. The following results can readily be modified so they will apply to a series-parallel type of winding. With positive-sequence current $I_{a1}$ in the stator, the positively rotating fundamental flux per pole $\phi_{a1}$ is

$$\phi_{a1} = 0.72\frac{A}{g}K_1\frac{N_a}{p}I_{a1}$$

where $N_a/p$ is the stator turns per pole and $K_1$ is the fundamental reduction factor for breadth and pitch. The negatively rotating fundamental flux is zero. There is no third-harmonic component produced by positive-sequence current. With zero-sequence current, $I_{a0}$, in the stator, the positively rotating third-harmonic flux per pole, $\phi_{a0}$, is

$$0.080\frac{A}{g}K_3\frac{N_a}{p}I_{a0}$$

where $K_3$ is the third-harmonic reduction factor for breadth and pitch; $p$ represents the number of poles for which the motor is wound, but is not the number of third-harmonic poles. The negatively rotating third-harmonic flux per pole also present, has the same value as the positively rotating third-harmonic flux per pole. There is no fundamental flux produced by the zero-sequence current.

The expression for the emf generated in a distributed, fractional pitch winding by a rotating sinusoidal field is $E = 4.44KNf\phi 10^{-8}$ volts. The emf, $E_{a1}$, generated in each phase of the stator by the positive-sequence fundamental flux is

$$E_{a1} = 4.44K_1\frac{N_a}{3}f\,0.72\frac{A}{g}K_1\frac{N_a}{p}I_{a1}10^{-8}$$

where $f$ is the rated frequency. If the iron losses due to this rotating field are disregarded, the ratio $E_{a1}/I_{a1}$ is the magnetizing reactance. In the usual induction motor, the iron losses are relatively so small that the phase displacement between the air-gap flux and the magnetizing current that is producing this flux is about 5 degrees. In such a case the difference between the magnitudes of the magnetizing impedance and the magnetizing reactance is less than 1 per cent. Thus, the positive-sequence magnetizing reactance is

$$X_{\phi 1} = 1.07\frac{A}{g}K_1^2\frac{N_a^2}{p}f10^{-8} \text{ ohms}$$

The negative-sequence magnetizing reactance has the same value. Since there are both positively rotating and negatively rotating third-harmonic fields produced by the zero-sequence current, the total magnetizing reactance for the third-harmonic field should be divided in two equal parts, one to be associated with the positively rotating third-harmonic field and one with the negatively rotating third-harmonic field. Since these third-harmonic fields have three times as many poles but rotate one-third as fast as the fundamental field, the frequency of the emfs they generate in the stator winding is the same as the frequency of the emfs generated by the fundamental field. Thus, the zero-sequence magnetizing reactance $X_{\phi 0}$, associated with *one* of the third harmonic component fields, is

$$X_{\phi 0} = 0.118\frac{A}{g}K_3^2\frac{N_a^2}{p}f10^{-8}$$

On comparing the zero-sequence and positive-sequence magnetizing reactances, it will be seen that

$$X_{\phi 0} = \frac{1}{9}\frac{K_3^2}{K_1^2}X_{\phi 1}$$

In a typical case where there are 12 slots per pole and the winding pitch is 10 slots

$$X_{\phi 0} = 0.027X_{\phi 1}$$

Since a representative value of $X_{\phi 1}$ is 2.8 on a per unit basis, the corresponding per unit value of $X_{\phi 0} = 0.076$, its value approximating that of the positive-sequence *leakage* reactance of either the stator or rotor winding.

It can be shown that the general relation between the current $I_b$ in one bar of a cage winding and the flux per pole $\phi$ produced by the currents in the entire cage winding is

$$\phi = A\frac{2}{\pi}\frac{1.6}{g}\frac{1}{\sqrt{2}}I_b\frac{N_b}{2p}$$

where the entire cage has $N_b$ bars; the current $I_b$ in each bar is produced by the rotation of the cage with respect to a flux distribution which has $p$ poles; the area of one pole is $A$, or $pA$ is the area of the cylindrical surface of the rotor. The flux per pole, $\phi_{b1}$, produced by the action of the rotor currents, $I_{b1}$, generated by the positive-sequence flux distribution, is

$$\phi_{b1} = 0.36\frac{A}{g}\frac{N_b}{p}I_{b1}$$

where $p$ is the number of poles for which the motor is wound and $A$ is the area of one pole. A like relation holds for the negative-sequence flux and current. The flux per pole $\phi_{b0}$ produced by the action of the rotor currents $I_{b0}$ generated by the positively rotating component of the zero-sequence third-harmonic flux distribution is

$$\phi_{b0} = 0.040\frac{A}{g}\frac{N_b}{p}I_{b0}$$

Here there are three times as many poles, each of which has one-third the area of the positive-sequence pole. The numerical coefficient has been changed so that $A$ and $p$ still represent the

same quantities as before. When these expressions are compared with the corresponding ones for the flux per pole produced by the stator currents, it should be noted that while $N_a$ is the total stator *turns* $N_b$ is the total rotor *bars* or half-turns.

We shall now assume that there is a positive-sequence fundamental field in the air gap for which the flux per pole is $\phi_1$ and that the slip of the rotor with respect to this field is $s$. The current $I_{b1}'$ in a rotor bar is

$$I_{b1}' = \frac{2.22sf\phi_1 10^{-8}}{Z_{b1}'}$$

where $f$ is the frequency of the positive-sequence currents in the stator. $Z_{b1}'$ is of the form $r_b' + js2\pi fL_b'$, where $r_b'$ and $L_b'$ are the equivalent resistance and equivalent leakage inductance of one rotor bar, including the effect of the end rings, at the frequency of the current in the bar. The flux per pole for the positive-sequence component field produced by these rotor currents is

$$0.80\frac{A}{g}\frac{N_b}{p}\frac{sf\phi_1}{Z_{b1}'}10^{-8}$$

If it is assumed that the cage winding is replaced by a winding arranged exactly like that on the stator, the current per phase in this winding will be

$$\frac{1.48K_1N_asf\phi_1}{Z}10^{-8}$$

where $Z$ is the leakage impedance per phase of this winding and is as yet undetermined. The positive-sequence flux per pole produced by the currents in the equivalent winding is

$$1.065\frac{A}{g}K_1^2\frac{N_a^2}{p}\frac{sf\phi_1}{Z}10^{-8}$$

If the flux per pole due to the currents in this equivalent winding is to be indistinguishable from the flux due to the currents in the actual cage winding

$$1.065\frac{A}{g}K_1^2\frac{N_a^2}{p}\frac{sf\phi_1}{Z}10^{-8} = 0.80\frac{A}{g}\frac{N_b}{p}\frac{sf\phi_1}{Z_{b1}'}10^{-8}$$

Hence the leakage impedance per phase of the equivalent winding is

$$Z = \frac{4}{3}K_1^2\frac{N_a^2}{N_b}Z_{b1}' \tag{350}$$

The impedance of the rotor in the positive-sequence equivalent circuit is

$$\frac{Z}{s} = Z_{b1} = \frac{4}{3}K_1^2\frac{N_a^2}{N_b}\left(\frac{r_{b1}'}{s} + j2\pi f L_{b1}'\right)$$

Assume there is a positively rotating third-harmonic field in the air gap due to zero-sequence currents in the stator, and let the flux per pole be $\phi_0$. The current $I_{b0}'$ in a rotor bar is

$$I_{b0}' = \frac{2.22(-2 + 3s)f\phi_0}{Z_{b0}'}10^{-8}$$

where $s$ is the slip with respect to the positive-sequence fundamental field and $Z_{b0}'$ is the rotor impedance per bar including the effect of the end rings at frequency $(-2 + 3s)f$. The flux per pole for the third-harmonic field produced by the bar currents, $I_{b0}'$, is

$$0.0888\frac{A}{g}\frac{N_b}{p}\frac{(-2 + 3s)}{Z_{b0}'}f\phi_0 10^{-8}$$

If it is assumed that the cage winding is replaced by an equivalent winding arranged exactly like that on the stator, the current per phase in this winding will be

$$1.48\frac{K_3N_a(-2 + 3s)f\phi_0}{Z}10^{-8}$$

where $Z$ is the undetermined impedance of this winding. The positively rotating third-harmonic flux per pole produced by the currents in the equivalent winding is

$$0.118\frac{A}{g}K_3^2\frac{N_a^2}{p}\frac{(-2 + 3s)}{Z}f\phi_0 10^{-8}$$

If the flux produced by this winding is to be indistinguishable from that produced by the actual cage winding

$$0.118\frac{A}{g}K_3^2\frac{N_a^2}{p}\frac{(-2 + 3s)}{Z}f\phi_0 10^{-8} = 0.0888\frac{A}{g}\frac{N_b}{p}\frac{(-2 + 3s)}{Z_{b0}'}f\phi_0 10^{-8}$$

Hence, the leakage impedance of the equivalent winding is

$$Z = \frac{4}{3}K_3^2\frac{N_a^2}{N_b}Z_{b0}' \tag{351}$$

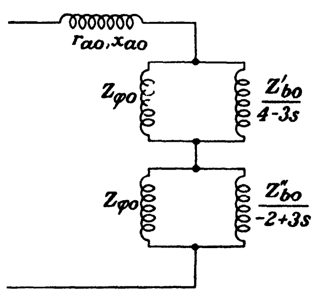
On comparing equations (350) and (351), it will be noted that the ratio of the multiplying coefficients which convert the impedance of a single rotor bar into the impedance of an equivalent 3-phase winding is $K_3^2/K_1^2$, where $K_1$ and $K_3$ are the total fundamental and the third-harmonic reduction factors for the stator winding. For a typical stator which has 12 slots per pole and a winding pitch of 10 slots $(K_3/K_1)^2 = 0.24$. It should be noted that at any given frequency $Z_{b0}'$ does not equal $Z_{b1}'$ exactly since the portion of the impedance due to the end ring is greater for fundamental flux than for third-harmonic flux.

The leakage reactance of the stator consists principally of slot leakage and coil-end leakage, both of which depend upon the winding pitch. With a winding-pitch unity, the currents in both coil sides lying in the same slot are always in time-phase, whether the currents be of positive, negative, or zero sequence, and the slot-leakage reactance is the same for all three sequences. If the winding pitch is two-thirds, the currents in the top and bottom coil sides in every slot are in different phases. For positive- and negative-sequence currents, the time-phase displacement of the currents in these coil-sides is 60 degrees, whereas for zero sequence the displacement is 180 degrees. For this reason the slot leakage reactance is much less for zero sequence than for positive or negative sequence. The coil-end leakage reactance for zero sequence is different from the leakage reactance for either positive or negative sequence for all values of the winding pitch on account of the mutual effect of one coil end upon another. There are well-known methods for calculating these leakage reactances from the dimensions of the slots and of the coils.[1]

The fall in potential across one stator phase equals the sum of the fall in potential due to the zero-sequence leakage impedance of the stator and the fall in potential due to the positively and

[1] ALGER, P. L., The Calculation of the Armature Reactance of Synchronous Machines, *Am. Inst. Elec. Eng.*, **47**, 493, 1928. Kilgore, L. A., Calculation of Synchronous Machines Constants, *Am. Inst. Elec. Eng.*, **50**, 1201, 1931. Also see footnote on p. 452.

negatively rotating third-harmonic air-gap fields produced by the zero-sequence currents. Thus the impedance of the motor to zero sequence consists in part of the leakage impedance of the stator to zero sequence and in part of the effect of the zero-sequence third-harmonic field in the air gap together with the reaction of the currents induced in the rotor by the action of this field. The net effect of the third-harmonic field is equivalent to a magnetizing impedance connected in parallel with the equivalent leakage impedance of the rotor (actual rotor leakage impedance divided by the slip). Since there are two zero-sequence third-harmonic fields in the air gap, there are two parallel combinations, one for the positively rotating component field and one for the negatively rotating component field. It follows that the equivalent circuit which represents the impedance of the motor to zero sequence is that shown in Fig. 122. We have outlined in some detail the method for obtaining the values of the constants of the branches in this equivalent circuit. It should be noted that if the rotor bars are deep, the rotor resistance to be used with the slip $(4 - 3s)$ may be considerably greater than that to be used with the slip $(-2 + 3s)$.




*$Z'_{bo}$ is the leakage impedance of the cage at frequency (4-3s) times fundamental frequency referred to the stator.*

FIG. 122.

**115. Effect of Saturation.**—The usual relation between the emf, $E_a$, generated by the air-gap flux and the current, $I_\phi$, which is necessary to produce this flux, is in Fig. 123. This is very closely the so-called "magnetization or saturation curve," or the open-circuit characteristic, of the machine. The principal difference between the two curves is that the ordinates in one (Fig. 123) are $E_a$ and in the other, open-circuit characteristic, are $V_a$. The potential $V_a$ is greater than $E_a$ by the leakage-impedance drop in the stator due to the exciting current. This drop is usually 2 or 3 per cent of $V_a$ and can be subtracted numerically from $V_a$ with little error. In the following analysis this small difference is properly ignored. Since the magnetizing impedance $|Z_\phi|$ equals $|E_a/I\phi|$, it is proportional to the

tangent of the angle between the horizontal axis and a line drawn from the origin to a point on the magnetization curve. As the saturation increases, the magnetizing impedance diminishes. When unbalanced potentials are applied to the stator, the magnitude of the air-gap flux varies as the flux rotates and consequently the saturation and the magnetizing impedance vary cyclically. The following approximate method is proposed for taking this variation into account. Assume that the motor is operating so that the positive- and negative-sequence com-

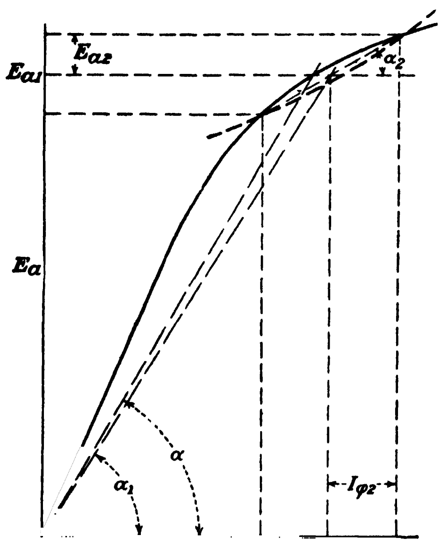



FIG. 123.

ponents of the *sinusoidal* emfs generated in the stator windings by the air-gap flux are $E_{a1}$ and $E_{a2}$ as indicated in Fig. 123. If the negative-sequence component is absent, the magnetizing impedance is proportional to tan $\alpha$, as shown. The minimum value of the air-gap flux is proportional to $(|E_{a1}| - |E_{a2}|)$ and the maximum value of the air-gap flux is proportional to $(|E_{a1}| + |E_{a2}|)$. Assume that the magnetizing currents are sinusoidal and have positive- and negative-sequence components $I_{\phi 1}$ and $I_{\phi 2}$. The corresponding minimum and maximum values of the magnetomotive force which produces the air-gap flux are proportional to $(|I_{\phi 1}| - |I_{\phi 2}|)$ and $(|I_{\phi 1}| + |I_{\phi 2}|)$.

These relations and the construction shown in Fig. 123 suggest at once that the positive-sequence magnetizing impedance be taken as proportional to tan $\alpha_1$ and that the negative-sequence magnetizing impedance be taken as proportional to tan $\alpha_2$. If this is done, it follows that the magnetizing impedance to negative sequence is less than the magnetizing impedance to positive sequence and that the superposition of the negative-sequence magnetization reduces the magnetizing impedance to positive sequence. The simplifying assumptions made in regard to the emfs and currents lead to certain other conclusions that it is well to consider. When the air-gap flux consists of oppositely rotating component fields, $\phi_1$ and $\phi_2$, the magnitude of the resultant field $\phi$ at any time $t$ is given by

$$\phi^2 = \phi_1^2 + \phi_2^2 + 2\phi_1\phi_2 \cos 2\omega t$$

where $t$ is measured from the moment the two component fields coincide. Similarly, the magnitude of the resultant magnetomotive force $F$, which is producing the air-gap flux $\phi$, is given by

$$F_1^2 + F_2^2 + 2F_1F_2 \cos 2\omega t$$

where $F_1$ and $F_2$ are the oppositely rotating components of $F$, and $t$ is measured from the time these components coincide. If it is assumed that the components of the flux coincide at the same moment that the components of the magnetomotive force coincide, the time angle $\omega t$ can be eliminated from the expressions for $\phi$ and $F$ and the relation between $\phi$ and $F$ can be established. When this is done,

$$\frac{\phi^2}{\phi_1\phi_2} - \frac{F^2}{F_1F_2} = \left(\frac{\phi_1}{\phi_2} + \frac{\phi_2}{\phi_1}\right) - \left(\frac{F_1}{F_2} + \frac{F_2}{F_1}\right)$$

This is the equation of an hyperbola whose axis is along the axis of $\phi$ and whose asymptote makes, with the axis of $F_1$, an angle whose tangent is $\sqrt{\phi_1\phi_2/F_1F_2}$. The slope of the asymptote is proportional to the geometric mean of the positive- and negative-sequence magnetizing impedances, in the same way that tan $\alpha_1$ is proportional to the positive-sequence magnetizing impedance. A portion of this hyperbola is shown as a dotted line in Fig. 123. The assumption that both the flux and the magnetomotive force consist of positive- and negative-sequence

components is incompatable with the magnetization curve obtained by test. Furthermore, since the ellipse whose radius represents the magnetomotive force is flatter than the ellipse whose radius represents the flux, and since the angular velocity of the radius is inversely proportional to the square of its length, the resultant flux does not coincide (in space position) with the resultant magnetomotive force except when both are passing through their minimum and maximum values.

When the correction for the effect of saturation is applied, a preliminary solution should be made in order to obtain approximate values of the positive- and negative-sequence generated emfs. The values of the positive- and negative-sequence magnetizing impedances can then be estimated and a corrected solution obtained. The magnitude of the correction depends upon the relative values of the impedances in the three branches of the equivalent circuit. In the usual induction motor, the magnetizing impedance is so large in comparison with the leakage impedances that any ordinary change in the saturation will produce a very small effect. In the synchronous machine, however, the air gap is usually much longer and the magnetizing impedances may be relatively no more than 10 or 20 per cent as great as in the induction machine. The change produced by saturation may then be great enough to warrant its consideration.

## CHAPTER VIII

# INDUCTION MACHINES

## UNSYMMETRICAL STATOR AND ROTOR WINDINGS

**116. Split-phase Motor.**—The torque developed by an induction motor is due to the effect of the *rotating* field in the air gap. The current in a single-phase stator winding produces a *pulsating* field which is incapable of developing torque, except when its oppositely rotating components become unbalanced by the effect of the currents induced in the rotor windings. This unbalancing effect occurs only when the rotor is turning. At standstill no torque is developed. If there is added to the stator a second winding displaced in space phase from the original single-phase winding and if in this second winding there flows a current which differs in time-phase from the current in the original winding, a rotating field is produced in the air gap and torque will be developed at standstill. If the two windings are connected in parallel to the same source of potential, it is necessary merely that the ratios of resistance to reactance be different in the two windings in order that the currents may differ in phase and a rotating field be developed. This difference in ratio of resistance to reactance is sometimes produced by connecting a condenser in series with one winding and sometimes by designing one winding so that its resistance is relatively much greater than that of the other winding.[1] The method of symmetrical components offers a convenient means for analyzing the performance of this type of motor.

**117. Three-phase Motor.**—Consider a 3-phase induction motor, having symmetrical stator and rotor windings and a condenser of reactance $x$ connected in series with one phase $a$ of the stator as in Fig. 124. In the following analysis it will be assumed that there is no loss in this condenser. Thus, the impedance of the condenser is assumed to have the form $0 - jx$. It will be found that this assumption permits writing simpler expressions for the desired results. In practice, the power

[1] See p. 350.

factor of the condenser may be as great as 0.20 and in such a case more accurate results will be obtained if an impedance $Z_c$ is substituted for $-jx$ *throughout* the analysis. Let us assume that the induction motor is connected through unbalanced impedances, $-jx$, 0, 0, to a source of unbalanced 3-phase potentials.[1] If the positive- and negative-sequence components of the potential $V_{an}$ applied to the series condenser and phase $a$ of the motor are $V_{a1}$ and $V_{a2}$ and the corresponding components of the current in phase $a$ are $I_{a1}$ and $I_{a2}$, we have, since the zero-, positive-, and negative-sequence components of the condenser reactance are each $-j\frac{x}{3}$,

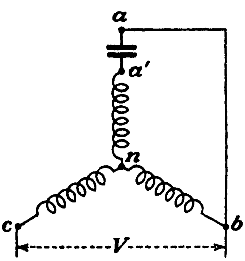



FIG. 124.

$$\left.\begin{aligned} V_{a1} &= I_{a1}\left(-j\frac{x}{3} + Z_{m1}\right) - j\frac{x}{3}I_{a2} \\ V_{a2} &= I_{a2}\left(-j\frac{x}{3} + Z_{m2}\right) - j\frac{x}{3}I_{a1} \end{aligned}\right\} \qquad (352)$$

The impedances of the motor to positive and to negative sequence are $Z_{m1}$ and $Z_{m2}$. The applied potentials are $V_{ab} = 0$, $V_{bc} = V$, and $V_{ca} = -V$. The method of Art. 8, page 53, shows that

$$V_{a1} = \frac{V}{3}\underline{/60}$$

$$V_{a2} = \frac{V}{3}\underline{/-60}$$

The vector diagram showing the symmetrical components of the applied potentials is in Fig. 125. The solution of equations (352) gives the following values for the positive- and negative-sequence components of the stator current in phase $a$:

$$\left.\begin{aligned} I_{a1} &= \frac{\frac{V}{3}\left(Z_{m2}\underline{/60} + \frac{x}{\sqrt{3}}\right)}{Z_{m1}Z_{m2} - j\frac{x}{3}(Z_{m1} + Z_{m2})}, \\ I_{a2} &= \frac{\frac{V}{3}\left(Z_{m1}\underline{/-60} - \frac{x}{\sqrt{3}}\right)}{Z_{m1}Z_{m2} - j\frac{x}{3}(Z_{m1} + Z_{m2})} \end{aligned}\right\} \qquad (353)$$

[1] The analysis can also be made on the assumption that an unbalanced circuit is connected in *parallel* with the motor.

Following the method outlined in Chap. VII, the positive- and negative-sequence components of the rotor currents can be determined, and the complete performance of the motor can be computed.

Since the single-phase current taken from the supply circuit is the current in the $c$ phase,

$$I_c = I_{a1}\underline{/120^\circ} + I_{a2}\underline{/-120^\circ}$$
$$= -V\frac{(Z_{m1} + Z_{m2} - jx)}{3Z_{m1}Z_{m2} - jx(Z_{m1} + Z_{m2})} \tag{354}$$

If the condenser is removed from its series position and placed directly across the supply potential $V$ and the terminal $a'$ of the motor is free, the current taken by the motor and condenser is

$$\left(\frac{V}{Z_{m1} + Z_{m2}} + \frac{V}{-jx}\right)$$

or

$$V\frac{(Z_{m1} + Z_{m2} - jx)}{-jx(Z_{m1} + Z_{m2})}$$

We shall now determine the reactance of the condenser for which the starting torque has its greatest possible value. Since the stator and rotor currents of each sequence are substantially equal at the time of starting, the starting torque is approximately

$$T_{start} = \frac{3p}{2\omega}(|\,I_{a1}\,|^2 - |\,I_{a2}\,|^2)r_b$$

where $r_b$ is the phase resistance of the rotor at rated frequency referred to the stator. Moreover, at starting, the impedances of the motor to positive and negative sequence are also equal and thus the starting torque can be written

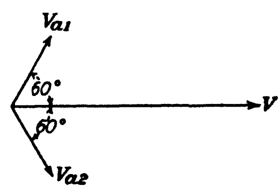



FIG. 125.

$$T_{start} = \frac{3p}{2\omega}\left|\frac{V}{3}\right|^2\left(\frac{\left|Z\underline{/60} + \frac{x}{\sqrt{3}}\right|^2 - \left|Z\underline{/-60} - \frac{x}{\sqrt{3}}\right|^2}{\left|Z^2 - j\frac{x}{3}2Z\right|^2}\right)r_b$$

where $Z_{m1} = Z_{m2} = Z$. This expression for the starting torque can be reduced readily to its simplest form by means of the principle that $|\,V\,|^2 = V \cdot \underline{V}$, where $\underline{V}$ is the conjugate of $V$.

The expression for the torque becomes

$$T_{\text{start}} = \frac{3p}{2\omega}\left|\frac{V}{3}\right|^2 \frac{\frac{2}{\sqrt{3}}Rxr_b}{(R^2 + X^2)\left[R^2 + \left(X - \frac{2x}{3}\right)^2\right]} \tag{355}$$

where the blocked or starting impedance $Z$ equals $R + jX$.

The blocked or starting impedance $Z$ can be measured by applying *balanced* 3-phase potentials to the stator while the rotor is at rest, when $Z$ is the vector ratio of the phase potential to the phase current.

The starting torque has its greatest value when the reactance of the condenser is 50 per cent greater than the starting impedance, or when $x = \frac{3}{2}\sqrt{R^2 + X^2}$. Assuming the potential rating of this condenser is $V/\sqrt{3}$, the kva rating of the condenser approximately equals the kva rating of the motor.

For this value of the reactance of the condenser, the starting torque is

$$\text{max. } T_{\text{start}} = \frac{3p}{2\omega}\frac{V^2}{6\sqrt{3}}\frac{Rr_b}{(R^2 + X^2)(\sqrt{R^2 + X^2} - X)}$$

If the equivalent reactance $X$ is more than three times the equivalent resistance $R$, the quantity $\sqrt{R^2 + X^2} - X$ is approximately $R^2/2X$ and the expression for the starting torque becomes

$$\text{max. } T_{\text{start}} = \frac{3p}{2\omega}\frac{V^2}{3}\frac{r_b}{R^2 + X^2}\frac{X}{\sqrt{3}R}$$

If the balanced line potentials $V$ are applied to the motor, the starting torque is

$$\frac{3p}{2\omega}\left|\frac{V}{\sqrt{3}}\right|^2 \frac{r_b}{R^2 + X^2}$$

The maximum torque developed at starting is $X/\sqrt{3}R$ times the torque developed when balanced line potentials of value $V$ are applied to the motor.

**118. Two-phase Motor.**—Consider the case in which the motor has 2-phase symmetrically arranged stator windings $a$ and $c$ with

a condenser of reactance $-jx$ connected in series with one phase, as in Fig. 126. Since 2-phase is a special case of 4-phase, the different sequences are indicated by the subscripts 0, 1, 2, and 3 (see Chap. III). Let $V_{c1}$ and $V_{c3}$ be the positive- and negative-sequence components of the potential applied in the $c$ phase to the motor and condenser, or $V_{c1} + V_{c3} = V$. Also, let $I_{c1}$ and $I_{c3}$ be the positive- and negative-sequence components of the current in the $c$ phase, and let $Z_{c0}$ and $Z_{c2}$ be the zero-order and second-order components of the impedance in this phase between the motor terminals and the source. The method of resolution described in Chap. III shows that

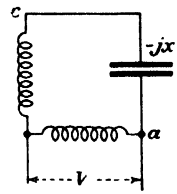


FIG. 126.

$$Z_{c0} = Z_{c2} = -j\frac{x}{2}$$

If $Z_{m1}$ and $Z_{m3}$ are the impedances of the motor to positive and negative sequence,

$$V_{c1} = I_{c1}\left(-j\frac{x}{2} + Z_{m1}\right) - j\frac{x}{2}I_{c3}$$

$$V_{c3} = I_{c3}\left(-j\frac{x}{2} + Z_{m3}\right) - j\frac{x}{2}I_{c1}$$

The method of resolution described in Chap. III shows that

$$V_{c1} = \frac{V}{2}(1 + j)$$

$$V_{c3} = \frac{V}{2}(1 - j)$$

Thus the solution for the positive- and negative-sequence components of the stator current in phase $c$ is

$$\left.\begin{aligned} I_{c1} &= \frac{V}{2}\frac{(1 + j)Z_{m3} + x}{Z_{m1}Z_{m3} - j\frac{x}{2}(Z_{m1} + Z_{m3})} \\ I_{c3} &= \frac{V}{2}\frac{(1 - j)Z_{m1} - x}{Z_{m1}Z_{m3} - j\frac{x}{2}(Z_{m1} + Z_{m3})} \end{aligned}\right\} \quad (356)$$

As in the 3-phase motor, the complete performance can now be computed.

The current in the other or $a$ phase of the stator winding is $-jI_{c1} + jI_{c3}$.

The total current taken from the source is

$$(1 - j)I_{c1} + (1 + j)I_{c3}$$

If the values of $I_{c1}$ and $I_{c3}$ are substituted in this expression, the single-phase current $I$ taken from the source is

$$I = V\frac{Z_{m1} + Z_{m3} - jx}{Z_{m1}Z_{m3} - j\frac{x}{2}(Z_{m1} + Z_{m3})} \tag{357}$$

If the condenser is removed from its series position and placed directly across the supply potential $V$, the current taken by the motor and the condenser is (see page 241)

$$\frac{2V}{Z_{m1} + Z_{m3}} + \frac{V}{-jx}$$

or

$$V\frac{Z_{m1} + Z_{m3} - j2x}{-jx(Z_{m1} + Z_{m3})}$$

Since at starting there is usually little difference between the stator and rotor currents, the torque developed at starting is substantially

$$T_{\text{start}} = \frac{2p}{2\omega}(|I_{c1}|^2 - |I_{c3}|^2)r_b$$

where $r_b$ is the phase resistance of the rotor at rated frequency referred to the stator. If the impedances of the motor to positive and negative sequence at starting are each equal to $Z$, the expression for the starting torque is

$$T_{\text{start}} = \frac{2p}{2\omega}\left|\frac{V}{2}\right|^2\left[\frac{|(1 + j)Z + x|^2 - |(1 - j)Z - x|^2}{\left|Z^2 - j\frac{x}{2}2Z\right|^2}\right]r_b$$

This reduces to

$$T_{\text{start}} = \frac{p}{\omega}V^2\frac{Rxr_b}{(R^2 + X^2)[R^2 + (X - x)^2]} \tag{358}$$

where the blocked or starting impedance $Z$ equals $R + jX$ and can be measured by applying balanced 2-phase potentials to the stator with the rotor at rest and is the vector ratio of the phase potential to the phase current. When a 2-phase motor is at rest and potential is applied to one phase of the stator, no potential will appear across the other phase of the stator, or there is no resultant mutual impedance between the phases of the stator when the rotor is at rest. Thus, it is not necessary to apply potential to *both* phases of the stator when measuring the starting impedance. When the rotor is turning and potential is applied to one phase of the stator, potential will also appear across the other phase. The resultant mutual impedance which now exists between the two stator phases is due wholly to the action of the currents generated in the closed *rotor* windings. There is no direct mutual impedance between the two stator phases since they are placed in space quadrature.

When the reactance of the condenser equals the magnitude of the starting impedance $Z$, the starting torque has its greatest value. If the starting impedance is approximately 0.20 per unit, this condenser would have a kilovolt ampere rating approximately five times the kilovolt ampere rating per phase of the motor.[1] Such a condenser is probably much larger than it would be economical to use. Also the starting current would be relatively great. The maximum value of the starting torque is

$$\text{max. } T_{\text{start}} = \frac{p}{\omega}\frac{V^2}{2}\frac{Rr_b}{(R^2 + X^2)(\sqrt{R^2 + X^2} - X)}$$

If the equivalent reactance $X$ is more than three times the equivalent resistance $R$, the quantity $\sqrt{R^2 + X^2} - X$ is approximately $R^2/2X$ and the expression for the maximum value of the starting torque becomes

$$\text{max. } T_{\text{start}} = \frac{p}{\omega} V^2 \frac{r_b}{(R^2 + X^2)} \frac{X}{R}$$

Thus the maximum torque developed at starting is approximately $X/R$ times the torque developed when balanced 2-phase potentials of the same value, $V$, are applied to the motor.

**119. Unsymmetrical Stator Windings.**—Heretofore, we have applied the method of symmetrical components to induction

[1] Assuming the rated potential of the condenser is $V$.

machines whose stator and rotor windings were arranged symmetrically. The unsymmetry that existed in the circuit lay outside the windings themselves. We shall now investigate the performance of a machine in which there is a certain unsymmetry in the arrangement of either the stator or rotor windings. The particular case we shall consider is a 2-phase winding, or one that is equivalent to such a winding, in which the phases have different number of turns but are spaced 90 degrees apart electrically around the air gap. Or, the windings of each phase, while different from each other, are symmetrical about axes in space quadrature. It is a characteristic of such a winding that, if the currents in the two phases are displaced in time-phase by 90 degrees and have such values that the *effective ampere-turns* of the two phases are *equal*, there will be produced in the air gap a rotating field of constant magnitude. When the windings have the same number of turns, the currents which produce a constant rotating field are equal and in time-quadrature and are then designated as positive or negative sequence, depending upon the direction in which the field rotates.

It is always advantageous, however, to resolve the currents in the windings into components such that each component will produce a constant or circular rotating field. When the ratio of the effective turns of the two stator windings is

$$\frac{K_c N_c}{K_a N_a} = k$$

the positive-sequence components of the stator currents should fulfill the relation $kI_{c1} = jI_{a1}$. Thus, in Fig. 126 the positive-sequence current in phase $c$, which is in series with the condenser, *leads* the positive-sequence current in phase $a$ by 90 degrees. Likewise the negative-sequence components of the stator currents should fulfill the relation $kI_{c3} = -jI_{a3}$. It will then be true that positive-sequence currents will produce a positively rotating field of constant magnitude, and negative-sequence currents will produce a negatively rotating field of constant magnitude. The resolution of the actual currents $I_c$ and $I_a$ into their positive- and negative-sequence components is accomplished as follows

$$I_c = I_{c1} + I_{c3}$$
$$I_a = -jkI_{c1} + jkI_{c3}$$

from which

$$\left.\begin{aligned} I_{c1} &= \frac{kI_c + jI_a}{2k} \\ I_{c3} &= \frac{kI_c - jI}{2k} \end{aligned}\right\} \tag{359}$$

If positive-sequence emfs are to be generated by a positively rotating field of constant magnitude, it must follow that

$$E_{c1} = jkE_{a1},$$

since these generated emfs are proportional to the effective turns in the two stator windings. Similarly,

$$E_{c3} = -jkE_{a3}$$

If $E_c$ and $E_a$ are the actual generated emfs due to an elliptical field in the air gap, the positive- and negative-sequence components of $E_c$ are

$$\left.\begin{aligned} E_{c1} &= \frac{E_c + jkE_a}{2} \\ E_{c3} &= \frac{E_c - jkE_a}{2} \end{aligned}\right\} \tag{360}$$

If the impedances of the two stator phases are so related that $Z_c = k^2 Z_a$, positive-sequence components of currents will produce only positive-sequence components of impedance potential drop. It follows that the zero-order and second-order components of the impedance $Z_c$ are (see page 108)

$$\left.\begin{aligned} Z_{c0} &= \frac{Z_c + k^2 Z_a}{2} \\ Z_{c2} &= \frac{Z_c - k^2 Z_a}{2} \end{aligned}\right\} \tag{361}$$

Further proof of these relations can be adduced by using equations (360) to determine the symmetrical components of $I_a Z_a$ and $I_c Z_c$ when $I_a$ and $I_c$ are expressed in terms of the symmetrical components of $I_c$. It will be found that the coefficient of $I_{c1}$ in $E_{c1}$ and of $I_{c3}$ in $E_{c3}$ is $Z_{c0}$, while the coefficient of $I_{c3}$ in $E_{c1}$ and of $I_{c1}$ in $E_{c3}$ is $Z_{c2}$, where $Z_{c0}$ and $Z_{c2}$ are defined by equations (361). If $V_{c1}$ and $V_{c3}$ are the positive- and negative-sequence

components of the impedance potential drop in phase $c$, it follows that

$$V_{c1} = I_{c1}Z_{c0} + I_{c3}Z_{c2}$$
$$V_{c3} = I_{c3}Z_{c0} + I_{c1}Z_{c2}$$

Consider an induction motor of this type whose stator windings are designed so that the ratio of their leakage impedances equals the square of the ratio of their effective turns. If only positive-sequence currents flow in the stator windings, the leakage impedance potential drop in the windings will be wholly positive-sequence. With only positive-sequence currents in the stator windings, the field in the air gap due to the action of both stator and rotor currents will be circular, or of constant magnitude, and will rotate positively. The emfs generated in the stator windings by this field will be wholly positive-sequence. Since the terminal potential of the stator equals the sum of this leakage impedance potential drop and the emf generated by the air-gap flux, it will be wholly positive sequence. Thus, if positive-sequence potentials are impressed on the motor, positive-sequence currents will flow in the stator windings. Note that the impedance of the motor to positive sequence is not the same for the two phases, but that the ratio of this impedance for phase $c$ to the corresponding impedance for phase $a$ is $k^2$.

If a condenser of reactance $-jx$ is connected in series with phase $c$ in order to produce starting torque when the motor is connected to a single-phase source of potential $V$, Fig. 126, the following equations may be written

$$\left.\begin{aligned} V_{c1} &= I_{c1}\left(-j\frac{x}{2} + Z_{mc1}\right) - j\frac{x}{2}I_{c3} \\ V_{c3} &= I_{c3}\left(-j\frac{x}{2} + Z_{mc3}\right) - j\frac{x}{2}I_{c1} \end{aligned}\right\} \qquad (362)$$

where $Z_{mc1}$ and $Z_{mc3}$ are the impedances of the motor to positive and negative sequence for phase $c$ and $-j\frac{x}{2}$ is both the zero-order and the second-order component of the impedance of the condenser [see equation (361)]. These equations (362) have the same form as those which apply to the motor having two equal

stator windings. The positive- and negative-sequence components of the applied potential are now

$$V_{c1} = \frac{V}{2}(1 + jk)$$
$$V_{c3} = \frac{V}{2}(1 - jk)$$

The positive- and negative-sequence components of the current in phase $c$ are

$$\left.\begin{aligned} I_{c1} &= \frac{V}{2}\,\frac{(1 + jk)Z_{mc3} + kx}{Z_{mc1}Z_{mc3} - j\frac{x}{2}(Z_{mc1} + Z_{mc3})} \\ I_{c3} &= \frac{V}{2}\,\frac{(1 - jk)Z_{mc1} - kx}{Z_{mc1}Z_{mc3} - j\frac{x}{2}(Z_{mc1} + Z_{mc3})} \end{aligned}\right\} \quad (363)$$

The winding, $c$, which has the condenser in series with it, is frequently called the *starting* winding. The other winding, $a$, which is connected directly to the source of power, is called the *main* winding.

Since motors of this type may be run under load conditions with the starting winding disconnected from the line, it is better to determine the symmetrical components of the current in the main winding, $a$, in terms of the motor impedances, $Z_{ma1}$ and $Z_{ma3}$, for this winding, to positive and negative sequence.

Since $I_{a1} = -jkI_{c1}$ and $I_{a3} = jkI_{c3}$, and since $Z_{mc1} = k^2Z_{ma1}$ and $Z_{mc3} = k^2Z_{ma3}$,

$$\left.\begin{aligned} I_{a1} &= \frac{V}{2}\,\frac{(k^2 - jk)Z_{ma3} - jx}{k^2Z_{ma1}Z_{ma3} - j\frac{x}{2}(Z_{ma1} + Z_{ma3})} \\ I_{a3} &= \frac{V}{2}\,\frac{(k^2 + jk)Z_{ma1} - jx}{k^2Z_{ma1}Z_{ma3} - j\frac{x}{2}(Z_{ma1} + Z_{ma3})} \end{aligned}\right\} \quad (364)$$

Now, the complete performance of the motor can be compared with that of a symmetrically wound motor for which $k$ is 1.

The expression for the starting torque is

$$T_{\text{start}} = \frac{p}{\omega}V^2\frac{r_b}{R^2 + X^2}\,\frac{kxR}{k^4R^2 + (k^2X - x)^2}$$

As before, $r_b$ is the resistance of the rotor referred to the main winding of the stator and $R + jX$ is the blocked impedance of the motor referred to the main winding.

When the reactance of the condenser equals $k^2$ times the starting impedance for the main winding, or when this reactance equals the starting impedance for the starting winding, the starting torque will have its maximum value,

$$\text{max. } T_{\text{start}} = \frac{p}{\omega} V^2 \frac{r_b}{R^2 + X^2} \frac{1}{k} \frac{X}{R}$$

Note that this maximum value of the starting torque varies inversely as the ratio of the effective turns in the starting winding to the effective turns in the main winding, and that the capacitance of the necessary condenser varies inversely as the square of the same ratio.

Consider the case in which the phase displacement of the current in the two stator windings necessary to produce the requisite starting torque is attained by designing the windings to have different ratios of resistance to leakage reactance. The starting winding and the main winding in parallel are connected directly to the source of power. The method of analysis follows that used in the preceding case, if it is assumed that the starting winding has a leakage impedance $k^2$ times the leakage impedance of the main winding and also that there is connected in series with the starting winding an impedance equal to the actual leakage impedance of the starting winding less the assumed value. If the actual leakage impedances of the starting winding and the main winding are $Z_c$ and $Z_a$, it is assumed that the starting winding has a leakage impedance $k^2Z_a$ and that there is connected in series with it an impedance $(Z_c - k^2Z_a)$. It is important to note that, since the unsymmetry is wholly in the stator, the difference $(Z_c - k^2Z_a)$ expressed in terms of stator leakage impedances also equals the difference $(Z_c - k^2Z_a)$, where $Z_c$ and $Z_a$ are the blocked impedances measured at the terminals of phase $c$ and of phase $a$. The analysis follows that used in the preceding case except that the impedance $(Z_c - k^2Z_a)$ replaces the reactance $-jx$ of the condenser. Thus, the positive- and negative-sequence components of the current in the main winding are

$$\left.\begin{aligned} I_{a1} &= \frac{V}{2}\,\frac{(k^2 - jk)Z_{ma3} + z}{k^2Z_{ma1}Z_{ma3} + \frac{z}{2}(Z_{ma1} + Z_{ma3})} \\ I_{a3} &= \frac{V}{2}\,\frac{(k^2 + jk)Z_{ma1} + z}{k^2Z_{ma1}Z_{ma3} + \frac{z}{2}(Z_{ma1} + Z_{ma3})} \end{aligned}\right\} \tag{365}$$

where $z = Z_c - k^2Z_a$. The expression for the starting torque is

$$T_{\text{start}} = \frac{p}{\omega}V^2\frac{r_b}{R^2 + X^2}\,\frac{k(Xr - Rx)}{(k^2R + r)^2(k^2X + x)^2}$$

where $r + jx = (Z_c - k^2Z_a)$ and $R + jX$ is the blocked impedance of the motor referred to the *main* winding $a$. Note, since it is also true that $r = R_c - k^2R$ and $x = X_c - k^2X$, where $R_c + jX_c$ is the blocked impedance of the motor referred to the *starting* winding, this expression for the starting torque can be written

$$T_{\text{start}} = \frac{p}{\omega}V^2\frac{r_b}{R^2 + X^2}\,\frac{k(XR_c - RX_c)}{R_c^2 + X_c^2}$$

This latter expression may prove more useful when designing the starting winding so that the motor will develop a specified starting torque.

**120. Unsymmetrical Cage Winding.**—Unsymmetry of the foregoing type may exist also in the rotor windings. An important case is that in which the rotor has a cage winding so arranged that it is symmetrical about two and only two sets of axes which are in quadrature electrically. For example, if both end rings of an induction motor are cut through at opposite points located one pole pitch apart, the cage will be symmetrical about each of two sets of quadrature axes one of which is located at the cuts in the end rings. A more important case, and one in which the same method of analysis can be used, is that of the damper or amortisseur winding often set in the rotor of a synchronous machine. The damper is usually arranged so that it is symmetrical both about axes through the centers of the poles and axes through the centers of the interpolar spaces. The difference between the analyses of these two cases is merely in details.

The following general method of analysis can be used to determine the behavior of an unsymmetrical cage winding of this

type. First, note that the effect of a circular field on the winding is exactly the same as the effect of two equal sinusoidally pulsating fields that are in both space quadrature and time quadrature. Each of these component fields pulsates at a frequency determined by the relative angular velocities of the rotating field and the rotor. If the axes of these two fields are taken so that they coincide with the axes of symmetry of the cage winding, the distribution of the currents induced in the winding by each of these component fields will be symmetrical about the same axes. Since the cage is symmetrical about each axis, the current distribution in the cage will consist of two component distributions each of which is *symmetrical* about one of the two quadrature axes. Thus there is no direct reaction between these two component current distributions in the cage. Owing to this symmetry in the distribution of the currents, there is little difficulty in computing their values. Next, determine the magnitude of the component flux that each of these current distributions will produce in the air gap. Then a symmetrical 2-phase winding, having an unbalancing impedance connected in series with one phase, can be found which will produce the same component fields as the actual cage winding produces, when both windings are acted upon by the same resultant circular field. The axis of each phase of the 2-phase winding coincides with one of the axes of symmetry of the cage. This 2-phase winding can be replaced by a symmetrical 3-phase winding having unbalancing impedances connected in series with it. The problem is now ready to be solved by the method pursued in Art. 110, p. 314, where the stator and rotor windings are both symmetrical but the rotor has unsymmetrical impedances connected in series with it.

There are several methods by which the distribution of current in the cage winding can be determined. In general, it is simpler to assume that the rotor bars are uniformly spaced and sufficiently close together to permit calculus to be employed. With a finite number of bars, it is necessary to solve a number of simultaneous linear equations having complex coefficients. It is doubtful if the increased accuracy attained by using this latter method rather than the calculus is warranted in any case. In Fig. 127 there is shown a portion of a cage winding whose end rings are cut at points a polar pitch apart, leaving the winding

symmetrical about axes through $d$ and $q$ but about no others. Assume that there is a circular field in the air gap which may be resolved into two sinusoidally distributed pulsating component fields having their points of zero density at $d$-$d$ and at $q$-$q$. Hence these component fields are in space quadrature electrically. The problem of finding the distribution of current in the cage, due to each of these component fields, resembles in some respects the problem of determining the distribution of current in a long transmission line. The leakage impedance of the end rings corresponds to the series impedance of the line conductors, and the leakage impedance of the rotor bars corresponds to the charging impedance between the line conductors. However, in the cage winding, owing to the air-gap flux, there is a generated emf which is not present in the transmission line.

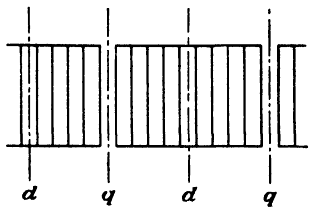


FIG. 127.

First, consider the field whose maximum density lies in the $d$ axis. Cutting the end rings at $q$ has no effect upon the distribution of the currents induced in the rotor bars by the action of this field. As in Chap. VII, let $Z_b''$ be the leakage impedance of one bar, $Z_r$ the leakage impedance of one complete end ring, $p$ the number of poles, and $N_b$ the total number of rotor bars. The bar admittance per electrical radian is $N_b/\pi p Z_b''$, and the ring impedance per electrical radian is $2Z_r/\pi p$. The instantaneous current density per radian in the bars at a point $x$ radians from the origin $q$ is of the general form $\sqrt{2}|\,C\,|\sin(\omega t + \theta)$, where $|\,C\,|$ and $\theta$ are functions of $x$. The rms value of this current density can be represented by a vector $C$ whose magnitude and time-phase are determined by $x$. The equation this rms vector current density $C$ must satisfy is

$$\frac{C\,dx}{N_b\,dx/\pi p Z_b''} + \int_x^{\frac{\pi}{2}} \int_0^x C \frac{2Z_r}{\pi p}\,dx\,dx = 2.22 f \phi 10^{-8} \int_x^{\frac{\pi}{2}} \sin x\,dx \quad (366)$$

where $x = 0$ at the point $q$, $\phi$ is the maximum value of the pulsating flux per pole, and $f$ is the frequency of its pulsation with respect to the cage winding. The sum of the terms on the left of the equality sign is the impedance potential drop around a

loop in which an emf equal to the term on the right of the equality sign is generated by the air-gap flux. Note that there is no potential between the points on the two end rings which lie in the $d$ axis. If some of the rotor bars are missing about the point $x = 0$, as they usually are in the damper winding of a synchronous machine, the lower limit of 0 in the definite integral is replaced by a value of $\alpha$, where $2\alpha/\pi$ is the portion of the space per pole in which bars are missing. On differentiating twice equation (366) reduces to

$$\frac{\pi p Z_b''}{N_b}\frac{d^2C}{dx^2} - \frac{2Z_r}{\pi p}C = -2.22f\phi 10^{-8}\cos x \tag{367}$$

The general solution for the vector current density per radian is

$$C = \frac{2.22f\phi 10^{-8}}{\dfrac{\pi p Z_b''}{N_b} + \dfrac{2Z_r}{\pi p}}\cos x + A\epsilon^{Dx} + B\epsilon^{-Dx} \tag{368}$$

where

$$D = \frac{1}{\pi p}\sqrt{\frac{2Z_rN_b}{Z_b''}}$$

For the conditions which are imposed by equation (366), the constants $A$ and $B$ are both zero. Thus the magnitude of the rms current density is proportional to $\cos x$, *but the phase* of the current density is *independent* of $x$. Since the current per bar $I_b$, is $\frac{\pi p}{N_b}C$,

$$I_b = \frac{2.22f\phi 10^{-8}}{Z_b'' + \dfrac{2N_b}{\pi^2p^2}Z_r}\cos x$$

The bar currents are all in time-phase but their magnitudes differ sinusoidally around the rotor.

The equivalent leakage impedance, $Z_b'$, per bar is

$$Z_b' = Z_b'' + \frac{2N_b}{\pi^2p^2}Z_r \tag{369}$$

This checks with the value given in Art. 114, page 327, if the angle $\alpha$ between successive bars becomes very small. The rms vector current density, $C$, in the bars is

$$\frac{N_b}{\pi p}\frac{2.22f\phi 10^{-8}}{Z_b'}\cos x$$

The resulting component of the ampere-turns per pair of poles acting along the $d$ axis, due to this current distribution, is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} C \cos x \, dx$$

The limits of integration tacitly assume that there are no missing bars about the point $x = 0$. The integration shows that the effective value of the ampere-turns per pair of poles acting in the direction of the $d$ axis, due to the currents in the cage winding, is

$$\frac{N_b}{2p} \frac{2.22 f\phi 10^{-8}}{Z_b'}$$

The resulting component along the $q$ axis, which is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} C \sin x \, dx,$$

is zero.

If the cage winding in the $d$ axis is to be replaced by a winding having one turn per pole and a leakage impedance $Z$ per turn, the effective value of the ampere-turns per pair of poles, due to the current in the single turn is

$$\frac{2 \times 4.44 f\phi 10^{-8}}{Z}$$

If the leakage impedance $Z$ is adjusted so that

$$Z = 4 \times \frac{2p}{N_b} Z_b'$$

the ampere-turns due to the cage and the replacing winding are identical. When all the turns of the latter winding are connected in series, the leakage impedance of all the turns of this phase of the winding is

$$\frac{8p^2}{N_b} Z_b' \tag{370}$$

This is the leakage impedance of one phase of a 2-phase winding, having one turn per pole per phase, and equivalent to the cage winding.

We shall now consider the field whose maximum density lies in the $q$ axis. The distribution of the currents induced in the rotor bars by the action of this component field is seriously affected by cutting the end rings. The equation to be satisfied by the effective vector current per radian in the bars is

$$\frac{(C_0 - C)\,dx}{\dfrac{N_b}{\pi p}\,\dfrac{dx}{Z_b''}} + \int_0^x \int_0^x C\frac{2Z_r}{\pi p}\,dx\,dx = 2.22f\phi 10^{-8} \int_0^x \sin x\,dx \tag{371}$$

where $x$ is now zero at the point $d$.

The sum of the terms on the left of the equality sign is the impedance potential drop around the closed loop in which an emf equal to the term on the right of the equality sign is generated by the air-gap flux. Here $C_0$ is the current density at the point $x = 0$. Owing to the break in the end rings at $q$, the total current flowing in the bars in one direction between the points $x = 0$ and $x = \pi/2$ is zero. That is, $\int_0^{\frac{\pi}{2}} C\,dx = 0$. If the end rings are complete, but some of the bars are missing about the $q$ axis, the solution for the vector current density is found by means of equation (366), if in that equation the upper limit, $\pi/2$, in the definite integral on the left of the equality sign is changed to $\frac{\pi}{2} - \alpha$ and if a term $I_\alpha \frac{2Z_r\alpha}{\pi p}$ is added on the left of the equality sign. The current $I_\alpha$ equals $\int_0^{\frac{\pi}{2}-\alpha} C\,dx$.

The general solution for equation (371) has the same form as for equation (368). Now, however, each of the coefficients $A$ and $B$ equals

$$-\frac{2.22f\phi 10^{-8}}{\dfrac{\pi p Z_b''}{N_b} + \dfrac{2Z_r}{\pi p}}\,\frac{D}{\epsilon^{\frac{D\pi}{2}} - \epsilon^{-\frac{D\pi}{2}}}$$

where

$$D = \frac{1}{\pi p}\sqrt{\frac{2Z_r N_b}{Z_b''}}$$

In order to simplify the analysis, substitute for the exact value of the vector current density the approximate value

$$C = \frac{N_b}{\pi p}\frac{2.22f\phi 10^{-8}}{Z_b'}\left(\cos x - \frac{2}{\pi}\right) \tag{372}$$

While this approximate value does not satisfy equation (371) exactly, it does satisfy the condition $\int_0^{\frac{\pi}{2}} C\,dx = 0$. Again the currents in the bars are in time-phase with one another. The effective value of the ampere-turns per pair of poles acting in the direction of the $q$ axis, due to this rms vector current density, is for this case $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} C \cos x\,dx$, which equals

$$\frac{N_b}{2p}\frac{2.22f\phi 10^{-8}}{Z_b'}\frac{\pi^2 - 8}{\pi^2}$$

This again assumes there are no missing rotor bars. Thus, the leakage impedance of one phase of a 2-phase winding, having one turn per pole per phase, and equivalent to the cage winding in this axis, is

$$\frac{8p^2}{N_b}Z_b'\frac{\pi^2}{\pi^2 - 8} \tag{373}$$

On comparing this impedance with the impedance given in (370), it will be noted that, when the end rings are cut in the manner described, the equivalent leakage impedance of the cage winding is approximately 5.3 times as great in one of the two quadrature axes as in the other.

Assume the leakage impedance of one of the two phases to be the same as that of the other phase, but while the terminals of one phase are short-circuited directly, the other phase has an impedance

$$\frac{8p^2}{N_b}Z_b'\frac{8}{\pi^2 - 8}$$

connected in series with it before it is short-circuited. Thus the unsymmetrical cage winding is reduced to an equivalent symmetrical 2-phase winding with an additional impedance in series with *one* of the *phases*.

**121. Equivalent Two-phase and Three-phase Windings.**—Assume that all the turns in each phase are connected in series. Let $N_a$ be the total turns in the 3-phase winding and $N_c$ the total turns in the 2-phase winding. Both of the windings are symmetrical. Let $K_a$ and $K_c$ be the total fundamental reduction factors for pitch and breadth for the two windings. These windings are equivalent if the short-circuit currents induced in them by the action of any elliptical field produce equal ampere-turns per pole. If the ampere-turns per pole are to be the same for each winding for both positive and negative sequence,

$$\frac{K_a N_a I_{a1}}{p} = \frac{K_c N_c I_{c1}}{p}$$

$$\frac{K_a N_a I_{a2}}{p} = \frac{K_c N_c I_{c3}}{p}$$

where $I_{a1}$ and $I_{c1}$ are the positive-sequence components of the phase currents in the two windings and $I_{a2}$ and $I_{c3}$ are the negative-sequence components of the phase currents in the two windings. Let $Z_{am1}$ and $Z_{am2}$ be the leakage impedances per phase of the 3-phase winding to positive and negative sequence and let $Z_{a0}$, $Z_{a1}$, $Z_{a2}$ be the symmetrical components of the three impedances connected in series with the 3-phase winding and through which the winding is short-circuited. Also let $Z_{cm1}$ and $Z_{cm3}$ be the leakage impedances per phase of the 2-phase winding to positive and negative sequence and let $Z_{c0}$ and $Z_{c2}$ be the symmetrical components of the impedances through which the phases are short-circuited.

If $\phi_1$ and $\phi_2$ are the values of the positive- and negative-sequence component fields in the air gap and $f_1$ and $f_2$ are the frequencies of the variation of these fields with respect to the windings being considered,

$$I_{a1}(Z_{am1} + Z_{a0}) + I_{a2}Z_{a2} = 4.44K_a\frac{N_a}{3}f_1\phi_1 10^{-8}$$

$$I_{a2}(Z_{am2} + Z_{a0}) + I_{a1}Z_{a1} = 4.44K_a\frac{N_a}{3}f_2\phi_2 10^{-8}$$

$$I_{c1}(Z_{cm1} + Z_{c0}) + I_{c3}Z_{c2} = 4.44K_c\frac{N_c}{2}f_1\phi_1 10^{-8}$$

$$I_{c3}(Z_{cm3} + Z_{c0}) + I_{c1}Z_{c2} = 4.44K_c\frac{N_c}{2}f_2\phi_2 10^{-8}$$

Since the windings must also be equivalent without the external impedance which is in series with them, these equations show that in general

$$\begin{aligned} Z_{am1} &= KZ_{cm1} \\ Z_{am2} &= KZ_{cm3} \\ Z_{a0} &= KZ_{c0} \\ Z_{a1} &= Z_{a2} = KZ_{c2} \end{aligned}$$

where the reducing ratio $K$ equals

$$\frac{2}{3}\left(\frac{K_aN_a}{K_cN_c}\right)^2 \tag{374}$$

When the 2-phase winding has one turn per pole per phase, this ratio $K$ equals $\frac{(K_aN_a)^2}{6p^2}$. Thus the leakage impedance $Z_b$ per phase of the 3-phase winding, equivalent to the cage winding, is

$$Z_b = \frac{4}{3}\frac{(K_aN_a)^2}{N_b}Z_b'$$

Since impedance is connected in series with only one phase of the equivalent 2-phase winding, the zero-order and second-order components of the added impedance are each

$$\frac{8p^2}{N_b}Z_b'\frac{4}{\pi^2 - 8}$$

Thus the symmetrical components of the external impedance connected to the equivalent 3-phase winding are each

$$\frac{4}{3}\frac{K_a^2N_a^2}{N_b}Z_b'\frac{4}{\pi^2 - 8}$$

or [see equation (350), page 333]

$$Z_{a0} = Z_{a1} = Z_{a2} = \frac{4}{\pi^2 - 8}Z_b$$

Note that, since $Z_b$ is the rotor impedance, it takes the form $\frac{r_b}{s} + jx_b$ in the equivalent circuit of the motor. The theory of the motor whose rotor winding is closed through unsymmetrical impedances is discussed in Art. 110, page 314. The

impedance connected across the rotor terminals in the positive-sequence equivalent circuit is $Z_0 - \frac{Z_1 Z_2}{Z_{m2} + Z_0}$ [see equation 346). In the present case, since the symmetrical components $Z_0$, $Z_1$, $Z_2$, of the impedances connected to the rotor terminals are each equal to $\frac{4}{\pi^2 - 8} Z_b$. The impedance $Z_0 - \frac{Z_1 Z_2}{Z_{m2} + Z_0}$ reduces

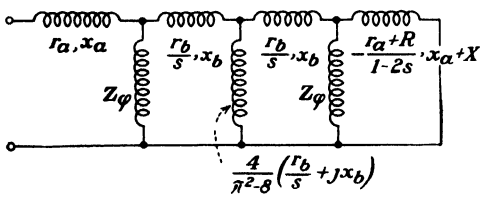



FIG. 128.

to $\frac{Z_{m2} Z_0}{Z_{m2} + Z_0}$, or equals the parallel impedance of $\frac{4}{\pi^2 - 8} Z_b$ and the impedance of the motor to negative sequence as viewed from the rotor. The equivalent circuit representing this condition is given in Fig. 128. Figure 128 is identical with Fig. 117, page 320, except for the added impedance $\frac{4}{\pi^2 - 8} Z_b$. This is

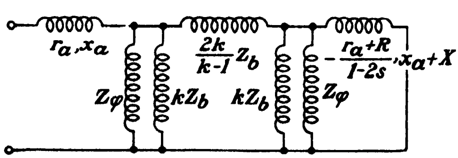



FIG. 129.

the general form of the equivalent circuit representing a motor having a cage winding symmetrical about each of two quadrature axes. The equivalent circuit of Fig. 128 can be considerably simplified. Replace the central $T$ of Fig. 128 by an equivalent $\pi$ as in Fig. 129. If $\frac{r_b}{s} + jx_b = Z_b$ and $\frac{\pi^2}{\pi^2 - 8} = k$, the impedances of the branches of this equivalent $\pi$ are

$$kZ_b, \qquad kZ_b, \qquad \frac{2kZ_b}{k - 1}.$$

Combine the parallel branches and replace the resulting $\pi$ by an equivalent $T$ as in Fig. 130. The impedances of the branches of this equivalent $T$ are

$$\frac{Z_bZ_\phi}{Z_b + Z_\phi}, \qquad \frac{Z_bZ_\phi}{Z_b + Z_\phi}, \qquad \frac{k-1}{2}\,\frac{Z_bZ_\phi^2}{(Z_b + Z_\phi)(kZ_b + Z_\phi)}.$$

Note that the impedance $r_a + jx_a + \frac{Z_bZ_\varphi}{Z_b + Z_\varphi}$, Fig. 130, is the equivalent impedance of the motor to positive sequence *before* the end rings are cut. It is readily shown that the impedance

$$\frac{k-1}{2}\,\frac{Z_bZ_\phi^2}{(Z_b + Z_\phi)(kZ_b)}$$

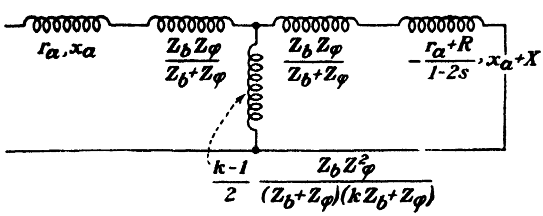



FIG. 130.

of the pillar of the $T$ is

$$\frac{1}{2}\left(\frac{kZ_bZ_\phi}{kZ_b + Z_\phi} - \frac{Z_bZ_\phi}{Z_b + Z_\phi}\right)$$

Hence, this impedance is one-half the difference of the equivalent impedance of the motor to positive sequence when the *rotor* impedance is $kZ_b$ and the equivalent impedance of the motor to positive sequence when the *rotor* impedance is $Z_b$. Note that $Z_b$ is the rotor impedance for the $d$ axis and that $kZ_b$ is the rotor impedance for the $q$ axis.

It is interesting to compare the analysis and equivalent circuit, Fig. 130, for the induction motor with the corresponding analysis and negative-sequence equivalent circuit for the salient pole synchronous machine, Art. 143, page 435.

**122. Unsymmetrical Capacitor Motor.**—It has been suggested that there may be an advantage in arranging the stator windings of a 2-phase capacitor motor with a space phase differing from 90 degrees. Consider such a motor in which the angle between the stator windings is $\alpha$, where $\alpha$ may be greater or less than $\pi/2$, as indicated in Fig. 131. The main winding is $a$ and the condenser is connected in series with the auxiliary or starting winding $c$. Assume the effective turns per pole for the $a$ and

$c$ phases as $A$ and $C$, and disregard as usual the space harmonics in the air-gap flux density. The flux density in the air gap due to the stator currents $i_a$ and $i_c$ at a point $x$ radians from one side of the $a$ phase is

$$\beta_x = k[Ai_a \sin x + Ci_c \sin(x + \alpha)]$$

There are two ways in which the Method of Symmetrical Components can be applied to the analysis of this air-gap density. First, the currents, or better, the ampere-turns, can be resolved by the Fortescue formula into positive- and negative-sequence components. However, if this is done, positive-sequence

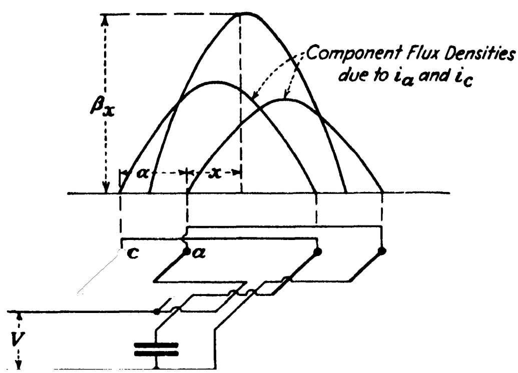



FIG. 131.

ampere-turns will not produce a positively rotating field of constant magnitude, since the stator windings are not symmetrically arranged. Since the field is not constant, it will consist of both positively and negatively rotating components. That is, positive-sequence ampere-turns will produce both positively and negatively rotating fields. This is also true of negative-sequence ampere-turns. Hence, there will be a resultant positively rotating field which is due in part to the positive-sequence ampere-turns and in part to the negative-sequence ampere-turns. There is also a negatively rotating field due to *both* the positive- and the negative-sequence ampere-turns. The simplicity that was obtained with a symmetrical winding, $\alpha = \pi/2$, has been lost. It is better to follow the spirit rather than the letter of the Fortescue method of resolution, and to divide the stator ampere-turns into such components that one

component will produce only a positively rotating field of *constant magnitude* and the other only a negatively rotating field of *constant magnitude.* It has been shown[1] that this second method of resolution is possible. If the angle between the effective turns of the two phases is $\alpha$, and if the effective ampere-turns in the two phases are equal in magnitude and differ in time-phase by $(\pi - \alpha)$, a single rotating field of constant magnitude will be produced. Assume the stator currents are

$$i_a = \sqrt{2}|\,I_a\,|\sin \omega t$$
$$i_c = \sqrt{2}|\,I_c\,|\sin(\omega t + \pi - \alpha)$$

The air-gap flux density due to these currents is

$$\beta_x = \sqrt{2}k[A|\,I_a\,|\sin x \sin \omega t + C|\,I_c\,|\sin(x + \alpha)\sin(\omega t + \pi - \alpha)]$$

If the effective ampere turns $A|\,I_a\,|$ and $C|\,I_c\,|$ are equal, the expression for the flux density reduces to

$$\beta_x = \frac{1}{\sqrt{2}}k2A|\,I_a\,|\sin \alpha \cos\left[x - \omega t - \left(\frac{\pi}{2} - \alpha\right)\right] \qquad (375)$$

This expression represents a positively rotating field of constant magnitude. The negatively rotating field has been entirely eliminated. With a symmetrical winding, the angle $\alpha$ is $\pi/2$, the currents in the two phases are equal and differ in time phase by $\pi/2$, and the expression for the flux density becomes

$$\beta_x = \frac{1}{\sqrt{2}}k2A|\,I_a\,|\cos(x - \omega t)$$

Note that the effect of the unsymmetry in the arrangement of the winding is to *reduce* the magnitude of the flux density to a value that is proportional to $\sin \alpha$ and to *advance* it in *space phase* by an angle $\left(\frac{\pi}{2} - \alpha\right)$. Since the magnetic field is rotating in the positive direction, the effect of the advance in space phase is to advance the time of maximum linkage with the armature winding *a*. That is, the maximum linkage with phase *a* does not occur at the time the current in phase *a* is maximum, as when both windings and currents are symmetrical.

[1] Chap. VII, p. 271.

Therefore, the vectors which represent the current and the air-gap flux should not be shown in phase as they are when the windings and currents are symmetrical, but the vector which represents the flux should lead by $\left(\frac{\pi}{2} - \alpha\right)$ the vector which represents the current in phase *a*.

Furthermore, it is important to note that, since the field moves synchronously with the current and since the current in

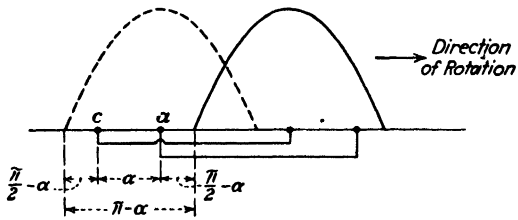



FIG. 132.

the *c* phase passes through its maximum value $(\pi - \alpha)$ radians before the current in the *a* phase reaches its maximum value, the maximum linkage with the *c* phase occurs $\left(\frac{\pi}{2} - \alpha\right)$ radians after the current in the *c* phase passes through its maximum value. This is illustrated in Fig. 132, which shows the *a* and *c*

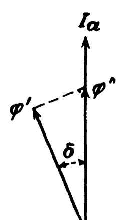



FIG. 133.

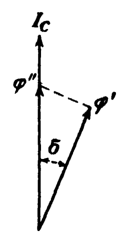



FIG. 134.

windings and the magnetic field at the moment that $i_a$ is maximum (full line) and also at the moment $i_c$ is maximum (dotted). Therefore care must be taken to specify which phase is being considered. Thus, in Fig. 133, if the current $I_a$ is one of two symmetrical currents which flow in a *symmetrical* winding, the air-gap flux due to the two symmetrical currents is represented by a vector $\phi''$ drawn in phase with the current. On the other hand, if the winding is unsymmetrical, the angle between the

effective turns of the two phases being $\alpha$, the air-gap flux due to the current $I_a$ in combination with the current $\frac{A}{C}I_a/\underline{\pi - \alpha}$ in the other phase $c$ should be represented by a vector $\phi'$ such that $\phi' = \phi''(\cos \delta)/\underline{\delta}$, where $\delta = \frac{\pi}{2} - \alpha$ is the displacement of the $c$ winding from its normal symmetrical position. The vector relation of the current and the flux linkage with the $c$ phase is shown in Fig. 134. Therefore, so far as the rotor is concerned it is as if the stator winding were symmetrical and the currents in it were symmetrical and in the $a$ phase equal to $I_a(\cos \delta)/\underline{\delta}$. Insofar as the magnetic effect produced in the air gap is concerned, the current $I_a$ in the unsymmetrical motor is equivalent both in magnitude and in time-phase to a current $I_a(\cos \delta)/\underline{\delta}$ in a symmetrical motor.

When the current in the $a$ phase leads the current in the $c$ phase by the phase angle $(\pi - \alpha)$ and the magnitudes of the ampere-turns in the two phases are equal, the expression for the air-gap flux density is

$$\beta_x = -\frac{1}{\sqrt{2}}k2A|\,I_a\,|\sin\alpha\cos\left[x + \omega t - \left(\frac{\pi}{2} - \alpha\right)\right] \qquad (376)$$

This shows that, owing to the unsymmetry in the winding, the flux density is reduced in magnitude and *advanced* in space phase. However, since this magnetic field rotates in the *negative* direction, the effect of the advance in space phase is to *retard* the time of its maximum linkage with the armature winding $a$. Thus, for the negatively rotating field, the proper phase relation between the current and the flux for phase $a$ is that shown in Fig. 135. The nomenclature is the same as in Fig. 133. That is, so far as the magnetic effect produced in the air gap is concerned, the current $I_a$ in the unsymmetrical motor is now equivalent both in magnitude and in time-phase to a current

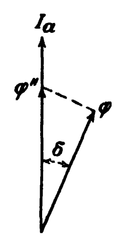



FIG. 135.

$$I_a(\cos \delta)/\underline{-\delta}$$

in a symmetrical motor. It should be noted that the effect of eddy-current losses and hysteresis losses in retarding the

flux with respect to the current has been neglected. When this retarding effect is neglected, the flux is in phase with the magnetomotive force which produces it. Thus, it is generally true that the vector $\phi'$ indicates the correct phase relation of the stator magnetomotive force with respect to the stator current when the stator windings are unsymmetrical. Therefore, if the currents in the $a$ and $c$ phases are divided into positive- and negative-sequence components in accordance with the principles that have just been established, positive-sequence currents will produce only a positively rotating field and negative-sequence currents will produce only a negatively rotating field.

The vector relations between the phase currents and their symmetrical components are

$$I_a = I_{a1} + I_{a3}$$
$$I_c = \frac{A}{C}I_{a1}\underline{/\pi - \alpha} + \frac{A}{C}I_{a3}\underline{/-\pi + \alpha}$$

All the equations that follow are written in terms of the current and emf in the main winding $a$. The solution for the positive- and negative-sequence components gives

$$\left.\begin{aligned} I_{a1} &= \frac{1}{2(\cos\delta)\underline{/\delta}}\left(I_a + \frac{C}{A}I_c\underline{/-\alpha}\right) \\ I_{a3} &= \frac{1}{2(\cos\delta)\underline{/\delta}}\left(I_a + \frac{C}{A}I_c\underline{/\alpha}\right) \end{aligned}\right\} \qquad (377)$$

where

$$\delta = \frac{\pi}{2} - \alpha$$

Positive-sequence currents produce a positively-rotating circular field. The effect of this field upon a symmetrical rotor, of either the wound or cage type, is exactly the same as if the field were produced by symmetrical currents in a symmetrical winding. This is also true of negative-sequence currents. Thus, in these respects the motor with an unsymmetrical stator winding is exactly like a motor with a symmetrical stator winding in which symmetrical currents flow. The general vector relations between the stator and rotor magnetomotive forces for either positive- or negative-sequence conditions are shown in Fig. 136. In the positive-sequence diagram, the angle between $E$ and $M_b$ is

about 10 or 15 degrees, while in the negative-sequence diagram this angle is about 75 degrees.

Therefore when positive-sequence currents flow in the stator windings, the *resultant* magnetic field in the air gap due to both stator and rotor currents, the rotor winding being symmetrical, has a constant magnitude and rotates in the positive direction, or it is a positively rotating circular field. When negative-sequence currents flow in the stator windings, there will be a resultant negatively rotating circular field in the air gap.

The emfs generated in the two stator windings by the positive-sequence field in the air gap are proportional to the effective

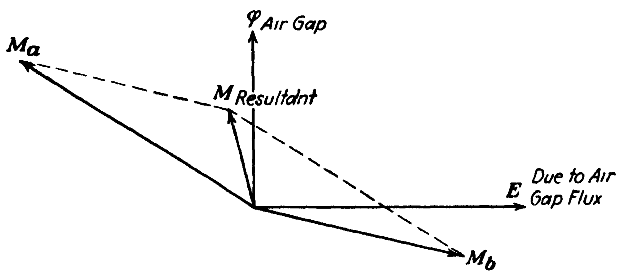



FIG. 136.—The stator and rotor magnetomotive forces are $M_a$ and $M_b$.

turns in the windings and are displaced in time-phase by an angle equal to the space phase displacement $\alpha$ between the windings. Designate the emfs that are generated by the positively rotating field as the positive-sequence emfs, and designate the emfs generated by the negatively rotating field as the negative-sequence emfs. Thus, if $E_a$ and $E_c$ are the actual generated emfs in the stator windings when both component fields are present in the air gap,

$$E_a = E_{a1} + E_{a3}$$
$$E_c = \frac{C}{A}E_{a1}\underline{/\alpha} + \frac{C}{A}E_{a3}\underline{/-\alpha}$$

The solution of these equations for the positive- and negative-sequence components of the emfs gives

$$\left.\begin{aligned} E_{a1} &= \frac{1}{2(\cos\delta)\underline{/-\delta}}\left(E_a - \frac{A}{C}E_c\underline{/\alpha}\right) \\ E_{a3} &= \frac{1}{2(\cos\delta)\underline{/\delta}}\left(E_a - \frac{A}{C}E_c\underline{/-\alpha}\right) \end{aligned}\right\} \tag{378}$$

where

$$\delta = \frac{\pi}{2} - \alpha$$

It should be noted that the formulas for resolving the currents and emfs are not alike as they are when the windings are symmetrical. Having determined the manner in which the currents and emfs are to be resolved into components, the method of resolving the self- and mutual impedances is fixed. Thus, if $Z_a$ and $Z_c$ are self-impedances in the $a$ and $c$ phases, the falls in potential in these phases due to these impedances are $I_aZ_a$ and $I_cZ_c$. If $I_aZ_a$ and $I_cZ_c$ are resolved into their positive- and negative-sequence components by means of equations (378) and at the same time the currents $I_a$ and $I_c$ are expressed in terms of the positive- and negative-sequence components of $I_a$,

$$\left.\begin{aligned} E_{a1} &= \frac{1}{(\cos\delta)\underline{/-\delta}}\left[I_{a1}\frac{(Z_a + N^2Z_c)}{2} + I_{a3}\frac{(Z_a + N^2Z_c\underline{/2\alpha})}{2}\right] \\ E_{a3} &= \frac{1}{(\cos\delta)\underline{/\delta}}\left[I_{a3}\frac{(Z_a + N^2Z_c)}{2} + I_{a1}\frac{(Z_a + N^2Z_c\underline{/-2\alpha})}{2}\right] \end{aligned}\right\} \quad (379)$$

where $N = A/C$. Note that $Z_a$ and $Z_c$ may include the leakage self-impedances of the two windings and any other impedances connected in series with the windings. For a symmetrical winding, $\alpha$ is $\pi/2$ and $\delta$ is zero, and the equations reduce to those for a symmetrical winding. For the moment, let us disregard the factors $(\cos\delta)\underline{/-\delta}$ and $(\cos\delta)\underline{/\delta}$. The coefficient

$$\left(\frac{Z_a + N^2Z_c}{2}\right)$$

of $I_{a1}$ in $E_{a1}$ is the same as the corresponding coefficient of $I_{a3}$ in $E_{a3}$. Following the nomenclature used in the analysis of 4-phase circuits (Chap. III),

$$\frac{Z_a + N^2Z_c}{2} = Z_{a0} \quad (380)$$

This is the zero-sequence component of the self-impedances.

In the Fortescue analysis of 4-phase circuits, the coefficients of $I_{a3}$ in $E_{a1}$ and of $I_{a1}$ in $E_{a3}$ are the same and are designated

as $Z_2$. The corresponding coefficients in equations (379) differ and must be distinguished from each other. Thus,

$$\left.\begin{aligned} \frac{Z_a + N^2 Z_c\underline{/2\alpha}}{2} &= Z_{a23} \\ \frac{Z_a + N^2 Z_c\underline{/-2\alpha}}{2} &= Z_{a21} \end{aligned}\right\} \qquad (381)$$

These are the second-order components of the self-impedances.

If there is mutual reactance between the $a$ and $c$ phases, due to flux which does not link the rotor windings, the positive- and negative-sequence components of the mutually induced emfs are

$$\left.\begin{aligned} E_{a1} &= \frac{1}{(\cos\delta)\underline{/-\delta}}(-I_{a1}jNx\cos\alpha - I_{a3}jNx\underline{/\alpha}) \\ E_{a3} &= \frac{1}{(\cos\delta)\underline{/\delta}}(-I_{a1}jNx\underline{/-\alpha} - I_{a3}jNx\cos\alpha) \end{aligned}\right\} \qquad (382)$$

It should be observed that the mutual reactances between the phases is positive if $\alpha$ is less than $\pi/2$ and is negative if $\alpha$ is greater than $\pi/2$. When the two windings are 90 degrees apart, there is no mutual reactance between them. The product $x \cos \alpha$ is always positive.

The emf generated by the air-gap flux in the stator winding of a symmetrically wound motor can be expressed as the product of the stator current and the parallel combination of the magnetizing impedance and the equivalent leakage impedance of the rotor. Since the equivalent positive-sequence current in the symmetrically wound motor is $I_{a1}(\cos\delta)\underline{/\delta}$, the fall in potential in the $a$ phase of the unsymmetrical motor due to the positive-sequence component of the air-gap flux is

$$\frac{Z_\phi Z_b}{Z_\phi + Z_b} I_{a1}(\cos\delta)/\delta$$

where $Z_\phi$ and $Z_b$ are the magnetizing impedance and the equivalent leakage impedance of the rotor per phase, both referred to phase $a$ of the stator. They are the impedances that would be computed or measured if the stator windings were symmetrically arranged and each had the same number of effective turns as phase $a$ of the unsymmetrical winding. As usual,

the rotor equivalent leakage impedance is $\frac{r_b}{s} + jx_b$ for positive-sequence and is $\frac{r_b}{2 - s} + jx_b$ for negative-sequence. As an abbreviation, write $Z_{r1}$ and $Z_{r3}$ for the positive- and negative-sequence parallel combinations of the magnetizing impedance and the rotor equivalent leakage impedances. As in the previous cases the motor is operated with a condenser connected in series with phase $c$. The positive- and negative-sequence components of the applied potential $V$ are

$$\left.\begin{aligned} V_1 &= \frac{V}{2(\cos\delta)\underline{/-\delta}}(1 - N\underline{/\alpha}) \\ V_3 &= \frac{V}{2(\cos\delta)\underline{/\delta}}(1 - N\underline{/-\alpha}) \end{aligned}\right\} \tag{383}$$

where $N = A/C$, as before.

We can now write the relations between the positive- and negative-sequence components of the applied potential and the corresponding components of the resulting stator currents. On equating the positive-sequence component of the applied potential to the sum of the positive-sequence components of the fall in potential in the stator, due to the stator self- and mutual leakage impedance, to the condenser that is connected in series with the $c$ phase, and also due to the resultant air-gap flux,

$$\frac{V(1 - N\underline{/\alpha})}{2(\cos\delta)\underline{/-\delta}} = \frac{1}{(\cos\delta)\underline{/-\delta}}(I_{a1}Z_0 + I_{a3}Z_{23}) + I_{a1}(\cos\delta)\underline{/\delta}\, Z_{r1} \tag{384}$$

The corresponding negative-sequence equation is

$$\frac{V(1 - N\underline{/-\alpha})}{2(\cos\delta)\underline{/\delta}} = \frac{1}{(\cos\delta)\underline{/\delta}}(I_{a3}Z_0 + I_{a1}Z_{21}) + I_{a3}(\cos\delta)\underline{/-\delta}\, Z_{r3} \tag{385}$$

where

$$Z_0 = \frac{Z_a + N^2Z_c}{2} - jNx\cos\alpha$$

$$Z_{23} = \frac{Z_a + N^2Z_c\underline{/2\alpha}}{2} - jNx\underline{/\alpha}$$

$$Z_{21} = \frac{Z_a + N^2Z_c\underline{/-2\alpha}}{2} - jNx\underline{/-\alpha}$$

Note that $Z_a$ is the self-leakage impedance of the $a$ phase, $x$ is the mutual-leakage reactance between phases $a$ and $c$, and $Z_c$ is the sum of the self-leakage impedance of the $c$ phase and the impedance of the condenser connected in series with the $c$ phase.

The solution of equations (384) and (385) for the positive- and negative-sequence currents gives

$$\left.\begin{aligned} I_{a1} &= \frac{V}{2}\,\frac{(1 - N/\underline{\alpha})(Z_{r3}\cos^2\delta + Z_0) - (1 - N/\underline{-\alpha})Z_{23}}{(Z_{r1}\cos^2\delta + Z_0)(Z_{r3}\cos^2\delta + Z_0) - Z_{21}Z_{23}} \\ I_{a3} &= \frac{V}{2}\,\frac{(1 - N/\underline{-\alpha})(Z_{r1}\cos^2\delta + Z_0) - (1 - N/\underline{\alpha})Z_{21}}{(Z_{r1}\cos^2\delta + Z_0)(Z_{r3}\cos^2\delta + Z_0) - Z_{21}Z_{23}} \end{aligned}\right\} \quad (386)$$

The actual current in the main winding $a$ is the sum of its symmetrical components and is

$$I_a = I_{a1} + I_{a3}$$

The actual current in the starting winding $c$ is

$$I_c = N[-\cos\alpha(I_{a1} + I_{a3}) + j\sin\alpha(I_{a1} - I_{a3})]$$

The total current supplied to the motor is the sum of these currents

$$(1 - N\cos\alpha)(I_{a1} + I_{a3}) + jN\sin\alpha(I_{a1} - I_{a3})$$

If the applied potential is taken as $|V| + j0$, the total vector volt-ampere input to the motor is

$$P + jQ = V[(1 - N\cos\alpha)(I_{a1} + I_{a3}) + jN\sin\alpha(I_{a1} - I_{a3})]$$

Thus the power input and the power factor are established.

The total copper loss in the stator is

$$|I_a|^2 r_a + |I_c|^2 r_c$$

where $r_a$ and $r_c$ are the effective resistances of the stator windings $a$ and $c$. The approximate value of the stator core loss can be determined from the values of the positive- and negative-sequence components of the emf generated in the stator by the air-gap flux. Art. 102, page 283. These component emfs are

$$I_{a1}(\cos\delta)/\underline{\delta}\, Z_{r1}$$
$$I_{a3}(\cos\delta)/\underline{-\delta}\, Z_{r3}.$$

The positive- and negative-sequence components of the rotor currents are

$$I_{b1} = \frac{Z_\phi}{Z_\phi + Z_{b1}} I_{a1}(\cos \delta)\underline{/\delta}$$

$$I_{b3} = \frac{Z_\phi}{Z_\phi + Z_{b3}} I_{a3}(\cos \delta)\underline{/-\delta}$$

The torque developed is

$$0.738\frac{p}{\omega}\left(|I_{b1}|^2\frac{r_b}{s} - |I_{b3}|^2\frac{r_b}{2-s}\right) \text{ lb. ft.}$$

If the rotor conductors are deep enough, it will be necessary to correct the rotor resistance $r_b$ for "skin effect." The rotor copper loss is

$$|I_{b1}|^2 r_b + |I_{b3}|^2 r_b$$

The vector volt-ampere input to the $a$ phase due to the positive-sequence potential and current is $\underline{V}_{a1}I_{a1}$, and that due to the $c$ phase is $\underline{V}_{c1}I_{c1}$, or $\frac{1}{N}\underline{V}_{a1}\underline{/-\alpha}NI_{a1}\underline{/\pi - \alpha}$. Thus the total vector volt-ampere input, $2\underline{V}_{a1}I_{a1}(\cos \delta)\underline{/\delta}$, is not divided equally between the two phases, as it is with a symmetrical motor. Similarly, it can be shown that the total vector volt-ampere input due to the negative-sequence components of potential and current is $2\underline{V}_{a3}I_{a3}(\cos \delta)\underline{/-\delta}$.

Although the analysis of the unsymmetrical motor follows the same procedure as the analysis of the symmetrically-wound motor, the unsymmetry has introduced rotating factors that add materially to the numerical computation.

The preceding analysis can be applied to the "shaded-pole" induction motor if one is willing to disregard all space harmonics in the distribution of the air-gap flux and to assume that the air gap is uniform. The angle $\alpha$ is measured from the center of the main winding $a$ to the center of the shading winding $c$. The shading winding may be short-circuited directly or closed through a condenser. In either case, the potential $V_c$ applied to it is zero. On referring to equations (383), it will be observed that the positive- and negative-sequence components of the potential applied to the main winding $a$ are

$$V_{a1} = \frac{V}{2(\cos \delta)\underline{/-\delta}}$$

$$V_{a3} = \frac{V}{2(\cos \delta)\underline{/\delta}}$$

The positive- and negative-sequence components of the current in the main winding $a$ can then be determined together with the copper losses in the stator and rotor windings. The current supplied to the motor is $I_{a1} + I_{a3}$ and the vector volt-ampere input is $V(I_{a1} + I_{a3})$.

## References

PUCHSTEIN, A. F., and LLOYD, T. C., Capacitor Motors with Windings Not in Quadrature, *Am. Inst. Elec. Eng.*, **54**, 1235, 1935.

LYON, W. V., and KINGSLEY, C., JR., Analysis of Unsymmetrical Machines, *Am. Inst. Elec. Eng.*, June, 1936.

# CHAPTER IX

## SYNCHRONOUS MACHINES

The operating characteristics of the synchronous machines connected to an electric power system play an important part in determining the behavior of the system as a whole. This has long been recognized and considerable effort has been expended in analyzing the performance of synchronous machines both under steady state and under transient conditions. The progress that has been made is due largely to the application of improved methods of analysis and to the aid of electrical or mechanical devices such as the network analyzer and the differential analyzer. When the synchronous machine is operated under unsymmetrical conditions, the Method of Symmetrical Components has proved an invaluable aid in predicting its behavior. In this chapter, we shall apply this method of analysis to determine the steady state performance of a synchronous machine. The Method of Symmetrical Components can be extended and applied to analysis of the machine under transient conditions.[1]

When the performance of a machine is being analyzed, it is necessary to make certain assumptions in regard to the arrangement and the electric or magnetic characteristics of its component parts. It is generally true that the more nearly the assumed conditions agree with the actual conditions, the more nearly will the computed performance agree with the actual performance. However, as the agreement between the assumed and actual conditions becomes closer the analysis becomes increasingly difficult. Considerable experience and skill are required to choose assumptions which on the one hand will make the analysis as accurate as possible and which on the other hand will lead to no great complexity in the solution of the problem. It is essential that the assumptions upon which the

[1] Bekku, Sadatoshi, Sudden Short-circuit of Alternator, *Researches of the Electrotechnical Laboratory*, Tokyo, No. 203, June, 1927.

analysis is based be clearly set forth. Otherwise, it is not possible to estimate the worth of the analysis or to understand its limitations. The ultimate value of any analysis is measured by the accuracy with which it will predict the actual performance of the machine.

**123. Ideal Synchronous Machine.**—In the first approach to a difficult problem, it is usually desirable to assume conditions such that the principal features of the analysis will be set forth clearly and will not be encumbered with relatively less important details. To this end, we shall first consider an "ideal" synchronous machine with the following specifications.

The field and armature iron are finely laminated and the air gap is uniform.

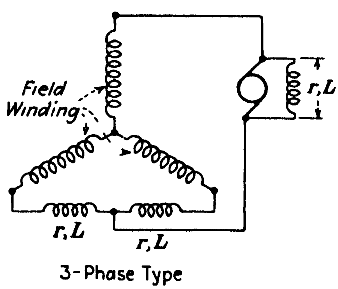


FIG. 137.

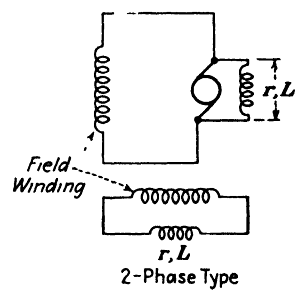


FIG. 138.

The armature has a symmetrical 3-phase winding.

The field has a symmetrical 3-phase or a symmetrical 2-phase winding.

The windings are arranged so that the air-gap flux is distributed sinusoidally.

The machine operates in an unsaturated condition.

Direct current is supplied to the field as in Fig. 137 or Fig. 138. The resistance $r$ and inductance $L$ of each of the impedances are the same as the measured values of resistance and inductance between the terminals of the exciter. Thus, in the 3-phase machine there are three equal impedances connected in $Y$ across the terminals of the field winding, and in the 2-phase machine each phase of the field winding is closed through an impedance, the two impedances having the same value. The reason for this peculiar type of field excitation will be set forth presently.

Although these specifications do not apply to synchronous machines as ordinarily built, they have been tacitly assumed by all who have analyzed the behavior of synchronous machines operating in the steady state under unbalanced conditions. In fact, we shall see that, unless these specifications are fulfilled, the Method of Symmetrical Components cannot be used strictly when a synchronous machine is connected to a power network unsymmetrical in any respect. After having analyzed the performance of the "ideal" machine, we shall consider the effect of the more important differences existing between the "ideal" and the actual machine. Thus, we shall discuss the effect of field windings of the ordinary type, of solid rotors when the air gap is uniform, of salient poles when the air gap is not uniform, of damper windings, and of saturation. Although this discussion may not arrive at definite quantitative results, it should point the way to a better understanding of the general problem and to more accurate methods for computing and measuring the essential constants of the machines.

**Operation of "Ideal" Synchronous Machine with Unsymmetrical Terminal Potentials.**—When the terminal potentials of the "ideal" synchronous machine are unsymmetrical, they can be replaced by their positive- and negative-sequence components. The phase order of the positive-sequence components is taken to be the same as that of the emfs generated on open circuit by the rotation of the direct-current field. By the principle of superposition, each of the resulting currents flowing in the field and armature windings consists of two components.

**124. Positive Sequence.**—The first of the two components in both field and armature is that due to the effect of the direct potential applied to the field and of the positive-sequence potentials at the terminals of the armature. The direct current in the field has its Ohm's law value. In accordance with the commonly accepted theory of an unsaturated, uniform air-gap, synchronous machine, the corresponding positive-sequence current in the armature is $\frac{E_1 - V_1}{Z_1}$, where $E_1$ is the positive-sequence emf generated in one phase of the armature on open circuit, $V_1$ is the positive-sequence terminal potential of one phase, and $Z_1$ is the positive-sequence or synchronous impedance of the armature winding per phase, each of these quantities being

expressed in its vector form. The open-circuit emf $E_1$ is subsequently referred to as the excitation emf, since it is due to the field excitation. The excitation emf must not be confused with the component emf generated by the *resultant* air-gap flux. The excitation emf and the air-gap emf are equal only on open circuit. Although the same symbols are sometimes used to represent these two different emfs, it is always specified which emf is indicated if the meaning is not otherwise clear. As defined and measured, the synchronous impedance is the ratio of the open-circuit phase potential to the short-circuit phase current for the same value of the field current. The open-circuit characteristic (occ) and the short-circuit characteristic (scc) are shown in Fig. 139. If one is to neglect consistently the effects of saturation, the synchronous impedance should be taken as the ratio $ef/if$ and not the ratio $EF/IF$. Furthermore, in the solution of problems where it is necessary to assign a value to the excitation emf corresponding to some given field current, this emf should be taken from the air-gap line and not from the open-circuit characteristic. That is, the excitation emf corresponding to the field current $F$ is $E_F$ and not $E$. This procedure is theoretically correct if the emf generated by the resultant air-gap flux does not exceed the value it has where the open-circuit characteristic departs from the air-gap line, at the point $e$ in Fig. 139. It should be noted it is here assumed that the only important effect of saturation is to modify the relation between the emf generated in the armature winding by the air-gap flux *and* the resultant magnetomotive force of the *field* and *armature* currents which produces this flux. Fortunately, in many short-circuit studies of synchronous machines that are connected to power networks, the emf $E_a$ generated by the air-gap flux is less than the critical value indicated by the point $e$ of Fig. 139. If the air-gap emf $E_a$ exceeds this critical value, it may be desirable to modify the foregoing theory in

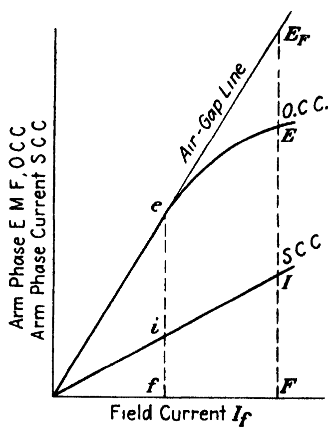



FIG. 139.

order to include the effects of saturation. Several methods have been suggested for accomplishing this result, one of which is described in Art. 150, page 463.

Since the armature resistance is relatively small, the synchronous impedance is substantially equal to the synchronous reactance. The effective armature resistance can be measured indirectly under short-circuit conditions in several ways. It may be considerably greater than the true or ohmic resistance of the armature winding.[1] The synchronous reactance equals the ratio of the total emf per phase generated in the armature to the current per phase when the machine is running in synchronism, with balanced currents in the armature winding but with no current in the field winding. The flux which generates this emf is divided broadly into two parts; that which links the armature winding but does not link the field winding, the so-called "armature leakage flux," and that which links both armature and field windings, commonly designated as the "armature-reaction flux." Thus, the synchronous reactance can be divided into two parts corresponding to these two component fluxes. The first part is commonly known as the "armature leakage reactance." To the second we shall give the less common name of "magnetizing reactance," since it corresponds in all respects to the magnetizing reactance of an induction motor.

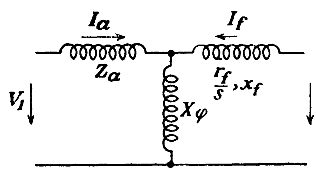


FIG 140

Consider first the general case of the synchronous machine in which the speed of the rotor differs from synchronism by slip $s$. Thus, the "ideal" synchronous machine can be represented by an equivalent circuit similar to that used to represent an induction motor when potentials are applied to both stator and rotor windings of the motor. In Fig. 140, $Z_a$ is the leakage impedance and $X_\phi$ is the magnetizing reactance of the armature per phase. The resistance at slip frequency and the leakage reactance at stator frequency of the field winding, both referred to the stator are $r_f$ and $x_f$ per phase. The actual terminal potential, $V_1$, of the stator has a frequency $f$, while the actual

[1] FIELD, A. B., Eddy Currents in Large Slot-wound Conductors, *Am. Inst. Elec. Eng.*, **24**, 761, 1905. W. V. LYON, Heat Losses in the Conductors of Alternating-current Machines, *Am. Inst. Elec. Eng.*, **40**, 1361, 1921.

terminal potential, $V_f$, of the rotor referred to the stator has a frequency $sf$. However, in the equivalent circuit, the terminal potential of the rotor takes a value $V_f/s$ and has a frequency the same as the frequency $f$ of the potential applied to the stator. Also, the frequency of the rotor current in the equivalent circuit is the same as the frequency $f$ the stator potential and current. The general expressions for the field and armature currents as deduced from the equivalent circuit are

$$I_f = \frac{V_f/s}{\dfrac{r_f}{s} + jx_f + \dfrac{jX_\phi Z_a}{jX_\phi + Z_a}} - \frac{V_1}{Z_a + \dfrac{jX_\phi\left(\dfrac{r_f}{s} + jx_f\right)}{jX_\phi + \dfrac{r_f}{s} + jx_f}} \frac{jX_\phi}{jX_\phi + \dfrac{r_f}{s} + jx_f}$$

$$I_a = \frac{V_1}{Z_a + \dfrac{jX_\phi\left(\dfrac{r_f}{s} + jx_f\right)}{jX_\phi + \dfrac{r_f}{s} + jx_f}} - \frac{V_f/s}{\dfrac{r_f}{s} + jx_f + \dfrac{jX_\phi Z_a}{jX_\phi + Z_a}} \frac{jX_\phi}{jX_\phi + Z_a}$$

If the machine is running in synchronism, the slip, $s$, has a limiting value zero. It will be observed that the limiting value of the rotor current, $I_f$, is $V_f/r_f$ and the current $I_a$ in the stator is

$$I_a = \frac{V_1}{Z_a + jX_\varphi} - \frac{jX_\varphi I_f}{Z_a + jX_\varphi}$$

The first term is due to the potential applied to the stator and the second term is due to the potential applied to the rotor. Note that, since $X_\phi$ is the mutual magnetizing reactance, $jX_\phi I_f$ is the open-circuit emf, $E_1$, generated in the armature winding by the field flux, and $(V_1 - E_1)/Z_1$, is the armature current where $Z_1$ is synchronous impedance or the impedance to positive sequence. This is the same value of current obtained by the usual method of analysis. The apparent difference in sign is explained when it is noted that in the equivalent circuit both $V_1$ and $E_1$ are taken as falls in potential while in the synchronous generator theory these quantities are always taken as rises in potential in the direction of the current. It is believed that the method of treating the synchronous machine as a special case of the induction machine, having potential applied to both stator

and rotor windings, has distinct advantages. It correlates the theory applying to the two machines, not only for balanced operation just considered, but also when there are negative-sequence potentials at the terminals of the stator winding.

**125. Negative Sequence.**—The second or negative-sequence components of the currents in the armature and field windings are due only to the effect of the negative-sequence potentials at the armature terminals. The negative-sequence currents in the armature winding produce a circular field in the air gap. This field rotates at synchronous speed but in a direction opposite to that in which the field poles are moving. Thus this negative-sequence field generates an emf of double frequency in the field winding. Although it is not entirely in accord with the facts, we shall assume that the exciter, which is still connected to the field terminals of the synchronous machine, generates no emf of this double frequency. Since the field winding itself is symmetrical and is closed through a balanced circuit (see Figs. 137 and 138), it behaves in exactly the same way as the symmetrical rotor winding of an induction motor while running at synchronous speed in the negative direction, that is, with slip 2.

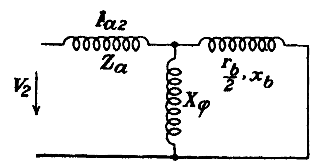



FIG. 141.—The impedance $Z_b$ of the rotor circuit equals $\frac{r_b}{2} + jx_b$.

The double-frequency currents induced in the symmetrical field winding produce a circular field in the air gap. This field rotates synchronously with the inducing field. Thus, since the *resulting* field in the air gap is circular, the negative-sequence equivalent circuit of this synchronous machine is exactly like the negative-sequence equivalent circuit of an induction motor for a slip 2. The equivalent circuit is in Fig. 141. If the rotor circuits are arranged as in Fig. 137, the impedance $Z_b$, of the rotor circuit in Fig. 141 is one-half the sum of the leakage impedance, $Z_b'$, of the rotor winding per Y phase and the impedance, $Z_e$, of the exciter, both taken at twice rated frequency and referred to the stator. Since the effect of frequency upon resistance is usually considered while the effect upon inductance is ignored, it is customary to indicate the impedance of the rotor branch of the equivalent circuit as $\frac{r_b}{2} + jx_b$ where $r_b$ is the resist-

ance at double rated frequency and $x_b$ is the leakage reactance at rated frequency.

If the rotor circuits are arranged as in Fig. 138, the same procedure should be followed except that the 2-phase rotor circuits should first be replaced by their equivalent 3-phase circuits. The method for accomplishing this result is developed in Art. 121, page 358. Hereafter it will be assumed that the impedance assigned to the rotor branch of the negative-sequence equivalent circuit includes the impedance of the exciter.

The impedance of the machine to negative sequence is the vector ratio of the negative-sequence terminal potential per phase to the negative-sequence current per phase. The obvious method of measuring this impedance is to apply negative-sequence potentials to the armature and at the same time to rotate the field structure at synchronous speed in the positive direction. The exciter should be connected to the field terminals but should supply the winding with no direct current, since direct current in the field winding will generate positive-sequence emfs in the armature winding. The resistance and reactance per phase to negative sequence are then measured in the usual manner with wattmeters, ammeter, and voltmeter. If $W$ is the total power and $V$ and $I$ are the line potential and line current, the resistance $R_2$ and reactance $X_2$ are

$$R_2 = \frac{W}{3I^2} \qquad X_2 = \sqrt{\frac{V^2}{3I^2} - \frac{W^2}{9I^4}}$$

The driving torque applied to the shaft during this test equals the sum of the torque due to friction and windage and the torque developed by the action of the negative-sequence field with the currents the field induces in the rotor winding. As in the induction motor, this latter torque is proportional to the electrical losses in the rotor divided by the slip. These electrical losses include both the copper loss and the core loss in the rotor due to the negative-sequence field. The slip of the rotor with respect to this field is 2. That is, the mechanical input to the shaft supplies not only the friction and windage losses but also supplies *one-half* the electrical losses in the rotor. Therefore, the negative-sequence electrical input to the stator supplies the copper loss and core loss in the stator and one-half the electrical losses in the rotor. The total electrical losses due to the nega-

tive-sequence current equal the total electrical *and* mechanical power supplied to the machine less the friction and windage losses. This net power per phase divided by the square of the negative-sequence current may be defined as the negative-sequence loss constant. It is greater than the resistance to negative sequence since the resistance is determined only by the electrical power supplied to the machine. Thus, the resistance to negative sequence is useful as a part of the impedance to negative sequence in determining the negative-sequence current, but is not useful in determining the losses due to the negative-sequence current. In the general unbalanced case in which there are both positive- and negative-sequence currents it is well to note, that the losses produced by the negative-sequence current do not depend strictly upon this current alone. The core losses in both stator and rotor are determined by the *resultant* magnetic field and thus the total loss does not equal the sum of the losses that would be produced by the individual action of the positive- and negative-sequence components of the resultant field. That is, the negative-sequence loss constant depends to some extent upon the positive-sequence component of the resultant field, particularly if the negative-sequence component is relatively large and the machine is operating in or near the saturated condition.

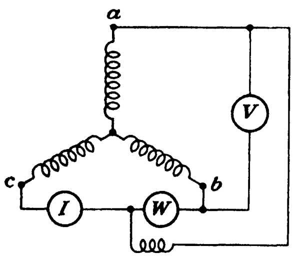


FIG 142

**126. Line-to-line Short Circuit.**—The impedance to negative sequence can be measured also as in Fig. 142 when the machine is operating as a generator with *two* of the line terminals short-circuited. As shown in Art. 10, page 55, the negative-sequence component $V_2$ of the phase potential $V_{an}$ is one-third the voltmeter reading, or $V_2 = \frac{1}{3}V$. It is also shown in Art. 13 that the negative-sequence component $I_2$ of the phase current $I_{an}$ is $-j\frac{1}{\sqrt{3}}I$, where $I$ is the vector value of the short-circuit current. The impedance to negative sequence is thus $j\frac{1}{\sqrt{3}}\frac{V}{I}$. If $W$ is the reading of the wattmeter, the reactance $X_2$ and the resistance $R_2$ to negative sequence are

$$X_2 = \frac{1}{\sqrt{3}} \frac{W}{|I|^2}$$

$$R_2 = \frac{1}{\sqrt{3}} \sqrt{\frac{|V|^2}{|I|^2} - \frac{W^2}{|I|^4}}$$

The reactance is determined by direct measurement but the *resistance* is computed by taking the square root of the difference of the squares of two numbers that are nearly equal. If the reactance is approximately five times the resistance, an error of 0.5 per cent in *one* of the meter readings produces an error of approximately 12 per cent in the computed value of the resistance. If, however, a reliable reactive meter is substituted for the wattmeter, the resistance to negative sequence, $R_2$, is directly determined and is

$$R_2 = \frac{1}{\sqrt{3}} \frac{Q}{|I|^2},$$

where $Q$ is the reading of the reactive meter.

The resistance component of the impedance to negative sequence can be determined also by measuring the power input, $P_S$, to the shaft of the generator during the single-phase short-circuit test. This power input equals the sum of the copper losses due to the positive- and negative-sequence currents in the stator ($a$) and to the negative-sequence current in the rotor ($b$), the friction and windage loss, and the core loss, which is commonly assumed to depend upon the rotating field in the air gap. That is,

$$P_S = 3I_{a1}^2 r_a + 3I_{a2}^2 r_a + 3I_{b2}^2 r_b + F + W + P_c \qquad (387)$$

Of the total power input, $P_S$, to the shaft, the part, $(F + W)$, is used in turning the rotor against friction and windage. Another part, $P_{ca1}$, is used in turning the rotor against the action of the eddy current and hysteresis losses produced in the stator by the rotation of the direct-current magnetic field. The greater part of the remainder, $(P - F - W - P_{ca1})$, is converted into positive-sequence electric power,[1] $3E_1I_1 \cos \angle \frac{E_1}{I_1}$, by the

[1] That is, power due to positive-sequence components of potential and current.

rotation of the direct-current field. A part, $3I_{a1}^2 r_a$, of this generated positive-sequence power heats the armature conductors. The remainder, $3V_1I_1 \cos \angle_{I_1}^{V_1}$, appears at the terminals of the armature where it is converted into an equal amount of negative-sequence power, $3V_2I_2 \cos \angle_{I_2}^{V_2}$, and is returned to the armature. A part, $3I_{a2}^2 r_a$, of this negative-sequence input to the armature heats the armature conductors and another part, $P_{ca2}$, supplies the eddy-current and hysteresis losses produced in the armature by the negative-sequence field. The remainder of the negative-sequence input is trans-

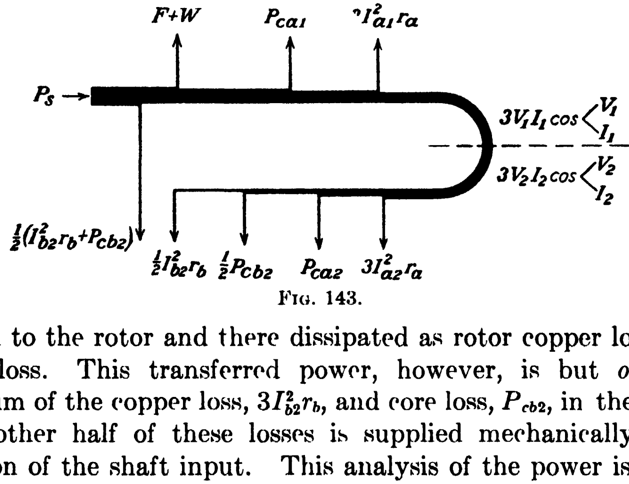



FIG. 143.

ferred to the rotor and there dissipated as rotor copper loss and core loss. This transferred power, however, is but *one-half* the sum of the copper loss, $3I_{b2}^2 r_b$, and core loss, $P_{cb2}$, in the rotor. The other half of these losses is supplied mechanically by a portion of the shaft input. This analysis of the power is somewhat idealized in that it is assumed that hysteresis loss can be resolved into component losses. The analysis can be represented graphically as in Fig. 143. The resistance component, $R_2$, of the impedance to negative sequence is

$$R_2 = \frac{V_2 I_2}{I_2^2} \cos \angle_{I_2}^{V_2}$$

$$R_2 = \frac{1}{3I_{a2}^2}\left(3I_{a2}^2 r_a + P_{ca2} + \frac{3I_{b2}^2 r_b}{2} + \frac{P_{cb2}}{2}\right) \qquad (388a)$$

The negative-sequence equivalent circuit, Fig. 141, shows that the negative-sequence rotor current, $I_{b2}$, is approximately

$\frac{X_\phi}{X_\phi + x_b} I_{a2}$. Thus, if the core losses due to the negative-sequence field are neglected, the negative-sequence resistance, $R_2$, is

$$R_2 = r_a + \frac{X_\phi^2}{(X_\phi + x_b)^2} \frac{r_b}{2} \tag{388b}$$

The negative-sequence loss constant, $K_2$, equals the *total* power due to the negative-sequence current and the negative-sequence field divided by $3I_{a2}^2$. Thus,

$$K_2 = \frac{1}{3I_{a2}^2}(3I_{a2}^2 r_a + P_{ca2} + 3I_{b2}^2 r_b + P_{cb2}) \tag{389}$$

The core losses in both these expressions (388*a*) and (389) are of minor importance in comparison with the copper losses. If the core losses vary as the square of the flux densities to which they are due, which is approximately true, both the resistance $R_2$ and the "loss constant" $K_2$ are constant.

Since the positive- and negative-sequence currents are each $1/\sqrt{3}$ of the indicated short-circuit current $I$, inspection of equations (388*a*) and (389) and of Fig. 143 shows that

$$R_2 = \frac{P_S - F - W - P_{ca1} + P_{ca2}}{2I^2} \tag{390}$$

$$K_2 = \frac{P_S - F - W - P_{ca1} - I^2 r_a}{I^2} \tag{391}$$

If the entire core loss $P_c$ is negligible, the resistance to negative sequence is

$$R_2 = \frac{P_s - F - W}{2I^2} \tag{392}$$

Since the core losses in equation (390) may not be negligible, we shall estimate their values. If $V_1$ and $V_2$ represent the positive- and negative-sequence components of the rise in terminal potential across the idle phase and $I_1$ and $I_2$ are the corresponding components of the current in this phase

$$V_1 = V_2 \qquad \text{and} \qquad I_1 = -I_2$$

The positive-sequence emf, $E_1$, generated by the air-gap flux is

$$E_1 = V_1 + I_1 z_a$$

where $z_a$ is the leakage impedance of the stator. Likewise the negative-sequence emf, $E_2$, generated by the air-gap flux is

$$E_2 = V_2 + I_2 z_a$$

The negative-sequence *fall* in terminal potential, $-V_2$, equals $I_2 Z_2$, where $Z_2$ is the impedance of the alternator to negative sequence. Thus

$$E_1 = I_1(Z_2 + z_a)$$
$$E_2 = I_1(Z_2 - z_a)$$

If the per unit values of $Z_2$ and $z_a$ are as great as 0.30 and 0.15, the positive-sequence rotating field is 45 per cent of its normal value when the positive-sequence current has its rated value. At the same time the negative-sequence rotating field is one-third as great as the positive-sequence field. The problem of the core loss produced by a field of this type is discussed in some detail in Chap. VII, pages 277–286. For simplicity, let us assume here that the resulting core loss equals the sum of the losses occasioned by the individual action of the positive- and negative-sequence component fields. Assume that the core loss varies as $f^{1.3}\beta^{1.7}$. For the assumed values of the impedances, the negative-sequence loss, $P_{ca2}$, in the stator is approximately $0.15^{1.7}$ or is 0.04 of the normal no-load core loss at normal excitation. Assuming that the losses in the stator and rotor are equal for the same frequency and flux density, the total negative-sequence core loss, $P_{cb2}$, in the rotor is $2^{1.3} \times 0.15^{1.7}$, or is 0.10 of the normal no-load core loss. If one-half this loss is supplied by the driving torque, the total negative-sequence core loss supplied electrically is 0.09 of the normal no-load core loss. The core loss due to the positive-sequence field is $0.45^{1.7}$ or 0.26 of normal no-load core loss. Thus the total core loss during this single-phase test is 0.40 of the normal no-load core loss. The estimated value of the core-loss correction term $(-P_{ca1} + P_{ca2})$ in the expression (390) for the resistance to negative sequence, $R_2$, is $(-0.26 + 0.04)$, or $(-0.22)$ of the normal no-load core loss. Since the normal no-load core loss is frequently twice the friction and windage, this core loss correction may be as much as 0.40 of the friction and windage loss, and in such a case should not be neglected.

**127. Zero Sequence.**—Since synchronous machines are operated frequently with their neutrals grounded, the armature

currents may contain zero-sequence components, particularly when line-to-ground faults occur on the network to which they are connected. The impedance, $Z_0$, of the machine to zero sequence is the vector ratio of the zero-sequence phase potential, $V_0$, to the zero-sequence phase current, $I_0$. If the current in *each* of the three phases of the armature produces only a sinusoidal component flux distribution in the air-gap, zero-sequence currents in the armature will produce no resulting air-gap flux, since the three component distributions are instantaneously equal and are spaced 120 degrees apart. For this reason the impedance of the "ideal" machine to zero sequence equals the leakage impedance of the armature to zero sequence. This is usually much smaller than the leakage impedance to positive or to negative sequence. If the armature coils have a pitch of unity, the currents in the top and bottom coil sides of every slot are the same. The slot leakage reactance is then the same for zero, positive, and negative sequence. For a fractional pitch winding the currents in the top and bottom coil sides of some slots are displaced 60 degrees in time-phase for positive and for negative sequence, but are displaced 180 degrees in time-phase for zero sequence. Thus, in the case of a fractional pitch winding the slot leakage reactance is less for zero sequence than for either positive or negative sequence.

The impedance to zero sequence can be measured by connecting the three armature terminals together and applying an emf of rated frequency between them and the neutral point. Since in the "ideal" machine no currents are induced in the rotor circuits by the zero-sequence armature currents, it is theoretically unnecessary to drive the rotor at synchronous speed in this test. However, this is not true of machines as actually constructed (see Arts. 135, 139, 140, 148, 149). The impedance to zero sequence is *three times* the vector ratio of the applied potential to the total current.

If it is possible to connect the three phases of the armature winding in series, the measured impedance to zero sequence is the vector ratio of one-third the applied potential to the current.

**128. Calculation of the Unsymmetrical Operation of Three-phase Synchronous Machines.**—It is important to note that current of one sequence does not affect the terminal potential of a different sequence. Furthermore, by the rotation of the

direct-current field, only a positive-sequence emf, $E_1$, is generated in the armature windings. The following fundamental equations apply to one phase of the "ideal" synchronous machine when it is operating under unbalanced conditions.

$$\left.\begin{aligned} E_1 - I_1Z_1 &= V_1 \\ 0 - I_2Z_2 &= V_2 \\ 0 - I_0Z_0 &= V_0 \end{aligned}\right\} \qquad (393)$$

where the excitation emf $E_1$, and the components $V_0$, $V_1$, and $V_2$ of the terminal potential are the *rises* in potential in the direction in which the component currents $I_0$, $I_1$, and $I_2$ are taken. Also $Z_1$, $Z_2$, and $Z_0$ are the impedances of the machine to positive, to negative, and to zero sequence.

These equations have a much broader significance than is at first apparent. In Chap. X it is shown that their meaning can be extended to form the basis for the solution of the conditions arising when *unsymmetrical* faults occur on any *symmetrical* power network. The relations among the terminal potentials $V_0$, $V_1$, and $V_2$ are determined by the character of the network to which the machine is connected. These equations are also used commonly in analyzing the performance of actual machines. If proper values are given to the sequence impedances $Z_0$, $Z_1$, $Z_2$ satisfactory results often will be obtained. The difficulty lies in computing or measuring the proper values of the sequence impedances. This is discussed subsequently. When analyzing the performance of a salient-pole synchronous machine *not* operating at an inductive power factor lower than about 0.2, it is best to use the Blondel or "double-reaction" method. The first of the three equations (393) is then modified and becomes

$$E_1 - I_dZ_d - I_qZ_q = V_1$$

where $I_d$ and $I_q$ are the direct and quadrature components of the positive-sequence current $I_1$, and $Z_d$ and $Z_q$ are the values of the positive-sequence impedance or synchronous impedance in the direct and quadrature axes. When the power-factor is lower than 0.2 and is inductive, the results obtained by the double reaction method or by assuming that the machine has "ideal" characteristics are substantially the same. We shall

now apply this method of analysis to some typical cases of unsymmetrical operation.

**I. Line-to-line Short Circuit.**—If the terminals $b$ and $c$ are short-circuited, the line current in phase $a$ is zero and the terminal potential $V_{bc}$ is also zero. Thus,

$$I_{a1} + I_{a2} = 0$$
$$V_{a1} = V_{a2} \quad \text{Compare Art. 10, page 55.}$$

If these values are substituted in the first and second of the fundamental equations, $E_{a1}$ being the excitation emf for phase $a$,

$$I_{a1} = \frac{E_{a1}}{Z_1 + Z_2}$$

It is interesting to note that the *positive-sequence* current which flows when the machine is subjected to this *unsymmetrical* short circuit is the same as the current which would be supplied to a *symmetrical* load having an impedance equal to the impedance of the machine to negative sequence.

Since $I_b = -j\sqrt{3}I_{a1}$ and $E_{bc} = -j\sqrt{3}E_{a1}$,

$$I_{bc} = \frac{E_{bc}}{Z_1 + Z_2}$$

That is, the short-circuit armature current equals the open-circuit armature emf between terminals divided by the sum of the impedance to positive sequence, or synchronous impedance, and the impedance to negative sequence. With a symmetrical 3-phase short circuit, the current per phase is $E_{bc}/\sqrt{3}Z_1$. This is smaller than the line-to-line short-circuit current, since $Z_2$ is ordinarily less than $0.73Z_1$. The positive-sequence potential, $V_{a1}$, across the open phase, being $E_{a1} - I_{a1}Z_1$ is $\dfrac{Z_2}{Z_1 + Z_2}E_{a1}$. The negative-sequence potential $V_{a2}$ across the open phase is also $\dfrac{Z_2}{Z_1 + Z_2}E_{a1}$. Thus, the actual potential across the open phase is

$$\frac{2Z_2}{Z_1 + Z_2}E_{a1}$$

The field current consists of a direct component and a double frequency component due to the action of the negative-sequence current in the armature.

**II. Line-to-neutral Short Circuit.**—If phase $a$ is short-circuited, its terminal potential is zero and there is no current in either phase $b$ or phase $c$. That is,

$$V_{a0} + V_{a1} + V_{a2} = 0$$
$$I_{a0} = I_{a1} = I_{a2} = \frac{I_a}{3}$$

where $I_a$ is the short-circuit current in phase $a$. If these relations are substituted in the fundamental equations (393),

$$I_{a1} = \frac{E_{a1}}{Z_0 + Z_1 + Z_2}$$

In this case, the positive-sequence current which flows when the machine is subjected to a line-to-neutral short circuit is the same as the current which would be supplied to a symmetrical load having an impedance equal to the sum of the impedances of the machine to negative sequence and to zero sequence. Since the actual current $I_a$ in the short-circuited phase is three times the positive-sequence component,

$$I_a = \frac{E_{a1}}{\frac{1}{3}(Z_0 + Z_1 + Z_2)}$$

That is, the short-circuit armature current equals the open-circuit emf per phase divided by the average value of the three sequence impedances. Note that this current is greater than the line-to-line short-circuit current for the same field current. The positive-sequence component, $V_{a1}$, of the terminal potential of the phase which is short-circuited is

$$\frac{Z_0 + Z_2}{Z_0 + Z_1 + Z_2} E_{a1}$$

The negative-sequence component of this same potential is

$$\frac{-Z_2}{Z_0 + Z_1 + Z_2} E_{a1}$$

while the zero-sequence component is

$$\frac{-Z_0}{Z_0 + Z_1 + Z_2} E_{a1}$$

The terminal potential of phase $b$ is

$$V_{a0} + V_{a1}\underline{/-120^\circ} + V_{a2}\underline{/120^\circ}$$

or

$$\sqrt{3}\left(\frac{Z_0\underline{/-30^\circ} + Z_2\underline{/30^\circ}}{Z_0 + Z_1 + Z_2}\right)E_{b1}$$

The field current consists of a direct component and a double-frequency component due to the action of the negative-sequence current in the armature. The zero-sequence armature current induces no current in the field circuits of the "ideal" machine.

**III. Double Line-to-neutral Short Circuit.**—If phases $b$ and $c$ are both short-circuited and there is no current in phase $a$,

$$V_{a0} = V_{a1} = V_{a2} = \tfrac{1}{3}V_a$$
$$I_{a0} + I_{a1} + I_{a2} = 0$$

If these relations are substituted in the fundamental equations (393),

$$\frac{E_{a1}}{Z_1} = \frac{V_a}{3}\left(\frac{1}{Z_0} + \frac{1}{Z_1} + \frac{1}{Z_2}\right)$$

If this relation is substituted in the general equation (393) for the positive-sequence component of the current,

$$I_{a1} = \frac{1}{Z_1}\left(E_{a1} - \frac{E_{a1}}{Z_1}\,\frac{\frac{1}{Z_1}}{\frac{1}{Z_0} + \frac{1}{Z_1} + \frac{1}{Z_2}}\right)$$

which reduces to

$$I_{a1} = \frac{E_{a1}}{Z_1 + \dfrac{Z_0Z_2}{Z_0 + Z_2}}$$

That is, the positive-sequence current which flows when the machine is subjected to a double line-to-neutral short circuit is the same as the current which would be supplied to a symmetrical load having an impedance equal to that of the parallel combination of the impedances of the machine to negative sequence and to zero sequence. The relation between $E_{a1}$ and $V_{a1}$ can also be written in terms of the sequence admittances. If $Y_0$, $Y_1$,

and $Y_2$ are the admittances per phase of the machine to zero, positive, and negative sequence,

$$V_{a0} = V_{a1} = V_{a2} = \frac{Y_1}{Y_0 + Y_1 + Y_2} E_{a1}$$

On substituting this value of the sequence components of the terminal phase potential in the three fundamental equations (393), the positive-, negative-, and zero-sequence components of the current are

$$I_{a1} = E_{a1} Y_1 \frac{(Y_0 + Y_2)}{Y_0 + Y_1 + Y_2}$$
$$I_{a2} = \frac{-E_{a1} Y_1 Y_2}{Y_0 + Y_1 + Y_2}$$
$$I_{a0} = \frac{-E_{a1} Y_1 Y_0}{Y_0 + Y_1 + Y_2}$$

The current in phase $b$ is

$$\begin{aligned} I_b &= I_{a0} + I_{a1}/\underline{-120^\circ} + I_{a2}/\underline{120^\circ} \\ &= \frac{\sqrt{3} E_{a1} Y_1}{Y_0 + Y_1 + Y_2} (Y_0/\underline{-150^\circ} + Y_2/\underline{-90^\circ}) \\ &= \frac{\sqrt{3} E_{b1} Y_1}{Y_0 + Y_1 + Y_2} (Y_0/\underline{-30^\circ} + Y_2/\underline{30^\circ}) \end{aligned}$$

The current in phase $c$ is

$$\begin{aligned} I_c &= I_{a0} + I_{a1}/\underline{120^\circ} + I_{a2}/\underline{-120^\circ} \\ &= \frac{\sqrt{3} E_{a1} Y_1}{Y_0 + Y_1 + Y_2} (Y_0/\underline{150^\circ} + Y_2/\underline{90^\circ}) \\ &= \frac{\sqrt{3} E_{c1} Y_1}{Y_0 + Y_1 + Y_2} (Y_0/\underline{30^\circ} + Y_2/\underline{-30^\circ}) \end{aligned}$$

This shows that the currents in phases $b$ and $c$ are approximately equal in magnitude if the angles of $Y_0$ and $Y_2$ are approximately the same. For example, if $Z_0 = 0 + j0.1$ per unit,

$Z_1 = 0 + j0.8$ per unit,

$Z_2 = 0 + j0.3$ per unit,

$$I_{a1} = -j1.14 E_{a1} \qquad I_{a2} = +j0.29 E_{a1} \qquad I_{a0} = +j0.86 E_{a1}$$
$$I_b = (-0.49 - j1.72) E_{b1} \qquad I_c = (0.49 - j1.72) E_{c1}$$

With a *symmetrical* 3-phase short circuit, the current is

$$I_a = 0 - j1.25E_a.$$

**IV. Line-to-neutral Short Circuit on Phase *a* and Line-to-line Short Circuit on *bc*.**—Since $V_a = 0$ and $V_{bc} = 0$,

$$V_{a0} + V_{a1} + V_{a2} = 0$$
$$V_{a1} = V_{a2}$$

Since the sum of the currents flowing *out* from the terminals *b* and *c* is zero,

$$(I_{a0} + I_{a1}\underline{/-120^\circ} + I_{a2}\underline{/120^\circ}) + (I_{b0} + I_{a1}\underline{/120^\circ} + I_{a2}\underline{/-120^\circ}) = 0$$

or

$$2I_{a0} - I_{a1} - I_{a2} = 0$$

The solution of these three relations, in conjunction with the three fundamental equations (393), gives the following values of the sequence currents:

$$I_{a1} = \frac{E_{a1}Y_1(4Y_0 + Y_2)}{4Y_0 + Y_1 + Y_2}$$
$$I_{a2} = -\frac{E_{a1}Y_1Y_2}{4Y_0 + Y_1 + Y_2}$$
$$I_{a0} = \frac{2E_{a1}Y_1Y_0}{4Y_0 + Y_1 + Y_2}$$

Again $E_{a1}$ is the open-circuit generated emf per phase.

The positive-sequence component of the current can be written also as

$$I_{a1} = \frac{E_{a1}}{Z_1 + \dfrac{Z_0Z_2}{Z_0 + 4Z_2}}$$

That is, the positive-sequence component of the current which flows when the machine is subjected to this unsymmetrical short circuit is the same as the current which would be supplied to a symmetrical load having an impedance $Z_0Z_2/(Z_0 + 4Z_2)$ per phase. The current in the *a* phase is $3I_{a0}$ and that in the

$b$ phase is $\frac{(2Y_0 + Y_2)}{4Y_0 + Y_1 + Y_2} Y_1 E_{bc}$, where $E_{bc}$ is the open-circuit terminal emf across $bc$. For example, if $Z_0 = 0 + j0.1$

$$Z_1 = 0 + j0.8 \qquad Z_2 = 0 + j0.3 \text{ per unit,}$$
$$I_{a1} = 0 - j1.22E_{a1}$$
$$I_{a2} = 0 + j0.094E_{a1}$$
$$I_{a0} = 0 - j0.56E_{a1}$$
$$I_a = 0 - j1.68E_{a1}$$
$$I_b = 0 - j0.66E_{bc}$$

**V. Line-to-ground Fault on Several Synchronous Machines in Parallel, the Neutrals of Some Not Being Grounded.**—In addition to the fundamental equations (393) for the $A$ and $B$ machines (see Fig. 144), there are the relations between the sequence

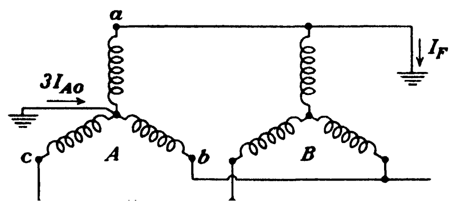



FIG. 144

components of the potential and current at the fault. If the impedance of the fault is assumed to be zero, the terminal potential of the $a$ phase of generator $A$ is zero and

$$V_{a0} + V_{a1} + V_{a2} = 0 \tag{394}$$

If the machines are delivering no other current except to the fault, the positive-, negative-, and zero-sequence components of the total current in the $a$ phase are each equal to one-third the fault current $I_F$. Thus, for the $a$ phase

$$I_{A1} + I_{B1} = I_{A2} + I_{B2} = I_{A0} = \tfrac{1}{3}I_F \tag{395}$$

There is no zero-sequence component in the current delivered by the $B$ machine since its neutral is not grounded. If the values of the sequence components of the currents in each machine as determined by the fundamental equations are substituted in equation (395), there results, on equating the

negative- and zero-sequence components to the positive-sequence components

$$-V_2(Y_{A2} + Y_{B2}) = (E_{A1} - V_1)Y_{A1} + (E_{B1} - V_1)Y_{B1}$$
$$-V_0Y_{A0} = (E_{A1} - V_1)Y_{A1} + (E_{B1} - V_1)Y_{B1}$$

where $Y_{A1}$, $Y_{B1}$, $Y_{A2}$, $Y_{B2}$, and $Y_{A0}$ are the admittances of the $A$ and $B$ machines to positive, negative, and zero sequence. If the values of $V_2$ and $V_0$ as determined by these equations are substituted in equation (394), there results

$$V_1\left[1 + \left(\frac{1}{Y_{A2} + Y_{B2}} + Z_{A0}\right)(Y_{A1} + Y_{B1})\right] = \left(\frac{1}{Y_{A2} + Y_{B2}} + Z_{A0}\right)(E_{A1}Y_{A1} + E_{B1}Y_{B1}) \quad (396)$$

The positive-sequence component of the line-to-neutral potential, $V_1$, at the fault can thus be determined in terms of the excitation emfs of the two generators and their sequence constants. The positive-sequence component of the current delivered by each generator can then be determined by means of the first of the fundamental equations (393). Observe that the second fundamental equation shows that the ratio of the negative-sequence components of the two machine currents is inversely as the impedances of the machines to negative sequence. The first of the fundamental equations shows that the positive sequence component of the fault current $I_{F1}$ or $(I_{A1} + I_{B1})$ is

$$I_{A1} + I_{B1} = (E_{A1}Y_{A1} + E_{B1}Y_{B1}) - V_1(Y_{A1} + Y_{B1})$$

If this relation is substituted in equation (396),

$$V_1 = \left(\frac{Z_{A2}Z_{B2}}{Z_{A2} + Z_{B2}} + Z_{A0}\right)I_{F1}$$

That is the positive-sequence component of the fault current equals the positive sequence component of the phase potential at the fault divided by the sum of the impedance of the $A$ machine to zero sequence and the parallel combination of the impedances of the $A$ and $B$ machines to negative sequence. Thus if the fault were replaced by a symmetrical load having this impedance per phase, the symmetrical current supplied to this

load would equal the positive-sequence component of the fault current.

**Illustration.**—In Fig. 144 assume that the grounded machine, $A$, has a rating equal to one-half that of the ungrounded machine, $B$. Let us choose as the unit of current the rated current of $A$. If the per unit values of the impedance to positive and to negative sequence of $B$ are $Z_1$ and $Z_2$ with the rated current of $B$ as the base, their per unit values with the rated current of $A$ as the base are $Z_1/2$ and $Z_2/2$. With this change of base the full-load current of $B$ has a value of 2 units.

Assume that

$$Z_{A0} = 0 + j0.1 \qquad Z_{A1} = 0 + j0.8 \qquad Z_{A2} = 0 + j0.3$$

per unit and, on the same base, assume that

$$Z_{B1} = 0 + j0.5 \qquad Z_{B2} = 0 + j0.2$$

Also assume that $E_A = 1\underline{/0°}$ and $E_B = 1\underline{/10°}$. That is, both excitation emfs have their normal values but the emf of $B$ leads that of $A$ by 10 degrees. When these values are substituted in the preceding equations, the computed per unit values of the sequence components of the potential and current for the phase on which the fault occurs are

$$\begin{aligned}
V_1 &= 0.413 + j0.0445 & & \\
I_{A1} &= -0.0556 - j0.734 & I_{B1} &= 0.258 - j1.14 \\
P_{A1} &= -0.0556 & P_{B1} &= 0.0553 \\
I_{A2} &= 0.081 - j0.748 & I_{B2} &= 0.121 - j1.12 \\
I_{A0} &= 0.202 - j1.87 & I_{B0} &= 0 \\
I_A &= 0.227 - j3.35 & I_B &= 0.379 - j2.26
\end{aligned}$$

$$\text{Fault current } I_F = 0.61 - j5.61$$

The current in the $a$ phase of the $B$ machine is $(0.19 - j1.13)$ times its own rated current. It is interesting to note that, were the $B$ machine not connected in parallel with the $A$ machine, the fault current would be but 2.50 per unit. This is less than the current that $A$ delivers to the fault when the machines are connected in parallel and is only 44 per cent of the total fault current delivered by both machines.

**VI. Synchronous Machine Supplying an Unbalanced Static Load.**—As a special case, consider a single-phase load of imped-

ance $Z$ connected across the armature terminals $bc$. There is then no current in phase $a$ and for this phase

$$I_1 + I_2 = 0$$

There are several methods of establishing the relations between the symmetrical components of the terminal potentials of the generator and the components of the currents supplied to the load. In the general case, either equations (55), page 82, which express these relations in terms of the symmetrical components of the load impedances, or equations (56), page 85, which express the relations in terms of the symmetrical components of the load admittances, can be used.

For a single-phase load, however, there is a somewhat simpler procedure. The general relation between the terminal potential $V_{bc}$ and the symmetrical components $V_1$ and $V_2$ of the phase potential $V_{an}$ is (see Art. 10, page 55)

$$V_{bc} = -j\sqrt{3}(V_1 - V_2)$$

In the case of a single-phase load, the current $I_{bc}$ in the loaded phase is

$$I_{bc} = -j\sqrt{3}I_1$$

where $I_1$ is the positive-sequence component of the current in the open phase. If the impedance of the load is $Z$, it follows that

$$-j\sqrt{3}I_1Z = -j\sqrt{3}(V_1 - V_2)$$

That is

$$I_1 = -I_2 = \frac{V_1 - V_2}{Z}$$

If these values of $I_1$ and $I_2$ are substituted in the fundamental equations (393), the solution for $I_1$ is

$$I_1 = \frac{E_1}{Z + Z_1 + Z_2} \tag{397}$$

The positive-sequence component of the current supplied to the single-phase load is the same as the current that would be supplied to a balanced $Y$-connected load having an impedance $(Z + Z_2)$ per phase. The power losses in the generator, however,

would be quite different for these two loads. The current supplied to the load is $\frac{E_{bc}}{Z + Z_1 + Z_2}$ where $E_{bc} = -j\sqrt{3}E_1$, is the open-circuit generated emf across the terminals *bc*. The terminal potential $V_{bc}$ of the generator is

$$V_{bc} = E_{bc}\frac{Z}{Z + Z_1 + Z_2}$$

from which it follows that

$$E_1 = j\frac{Z + Z_1 + Z_2}{Z}\frac{V_{bc}}{\sqrt{3}} \tag{398}$$

This determines the value of the excitation emf, $E_1$, necessary to establish a given terminal potential $V_{bc}$. Note that, while $Z$ is the impedance of the single-phase load, $Z_1$ and $Z_2$ are the impedances of the generator per phase to positive sequence and to negative sequence.

**Losses.**—The copper loss in the armature due to the positive-sequence current, $I_1$, is $3I_1^2r_a$, where $r_a$ is the resistance of the armature per phase at rated frequency. The corresponding loss in the field is $I_f^2r_f$, where $I_f$ is the direct exciting current in the field winding and $r_f$ is the resistance of the entire field winding at zero frequency. In the case of the "ideal" machine with excitation supplied as in Fig. 137, $r_f$ is 1.5 times the resistance at zero frequency of *one phase* of the field winding. As previously noted, the field current, $I_f$, is determined by the value of $E_1$ on the air-gap line if the saturation of the magnetic circuit is neglected. The effect of saturation is discussed in Art. 150, page 463.

The copper loss in the armature due to the negative-sequence current $I_2$ is $3I_2^2r_a$, where $r_a$ is again the resistance of the armature per phase at rated frequency. The corresponding loss in the field is $3I_{b2}^2r_b$, where $r_b$ is the resistance at twice rated frequency of one phase of the field winding referred to the armature, and $I_{b2}$ is the negative-sequence current in the field winding referred to the armature. The current $I_{b2}$ differs from $I_2$ by the exciting component of the negative-sequence current. It should be noted that the total copper loss in both armature and field windings, due to the negative-sequence component of the current, does *not* equal the square of the negative-sequence

current multiplied by the resistance of the machine to negative sequence, in as much as one-half the loss in the field winding is supplied mechanically and not electrically (see Art. 126, page 382).

The rotating magnetic fields that produce the core losses are usually measured in terms of the emfs they generate in the armature winding. The positive-sequence emf, $E_{a1}$, per phase due to the positively rotating *air-gap* field, is

$$E_{a1} = V_1 + I_1 z_a$$

where $z_a$ is the *leakage impedance* of the armature per phase. If the values of $V_1$, $I_1$, and $E_1$, as determined by equations (393), (397), and (398), are substituted in this equation,

$$E_{a1} = j\frac{Z + z_a + Z_2}{Z}\frac{V_{bc}}{\sqrt{3}}$$

The negative-sequence emf, $E_{a2}$, per phase due to the negatively rotating *air-gap* field, is

$$E_{a2} = V_2 + I_2 z_a$$

which reduces to

$$E_{a2} = j\frac{Z_2 - z_a}{Z}\frac{V_{bc}}{\sqrt{3}}$$

Thus, the ratio of the negatively and positively rotating *air-gap* fields is $\frac{Z_2 - z_a}{Z + Z_2 + z_a}$. Note that the numerator of this fraction is the parallel combination of the magnetizing impedance and the leakage impedance assigned to the rotor in the negative-sequence equivalent circuit. If the resistance and leakage reactance of the rotor circuits are small, this impedance is small and there will be relatively little negatively rotating field to produce core losses. In practice, it is often necessary to employ an amortisseur or damper to reduce the impedance of the rotor circuits to such a value that the negatively rotating field is so small it will not produce undue core losses in the rotor. In some of the early designs of single-phase generators, this effect was not appreciated, with the result that the generators became dangerously hot when operated under load. The defect was remedied by the addition of an amortisseur to the rotor.

The resulting field in the air gap is elliptical with respect to the stator, and the core losses produced in the stator can be calculated by the method proposed in Art. 102, page 283. If the negative-sequence component of this field is relatively small, it may be satisfactory to assume that the core loss in the stator is the sum of the losses produced by the *individual* action of the positive-sequence and negative-sequence components of the resultant elliptical field. With respect to the rotor, the air-gap flux consists of a constant part—the positive-sequence component—and a part that rotates at twice synchronous speed—the negative-sequence component. The core loss in the rotor is due wholly to the negative-sequence component but the magnitude of this core loss is influenced by the positive-sequence component of the rotating field (see Art. 100, page 279). As an approximation, the total copper and core losses due to the negative-sequence current can be determined, as suggested in Art. 126, page 382, by multiplying the square of the negative-sequence current by a negative-sequence loss constant. The core loss produced by the positively rotating field is then determined in the usual manner from the no-load core loss as if the generator were operating under balanced conditions. It should be pointed out, however, that the relation between the losses under a *balanced* load and the measured no-load losses is understood rather imperfectly and merits further investigation.

**VII. Synchronous Machine Supplying a Single-phase Load in Parallel with an Induction Motor.**—The arrangement of the generator and load is illustrated diagrammatically in Fig. 145. The impedances of the generator per phase to positive and negative sequence are $Z_{a1}$ and $Z_{a2}$. The corresponding impedances of the motor are $Z_{b1}$ and $Z_{b2}$. The impedance of the static load is $Z_c$. Since the positive-sequence current in the generator equals the sum of the positive-sequence currents supplied to the motor and load and since this is also true of the negative-sequence currents,

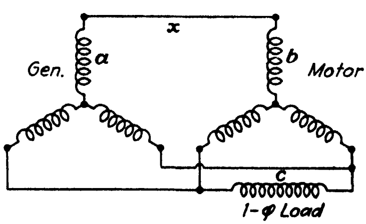


FIG. 145.

$$I_{a1} = I_{b1} + I_{c1}$$
$$I_{a2} = I_{b2} + I_{c2}$$

These are the components of the generator, motor and static-load currents in the line marked with an $x$. Of the current $I_c$ it should be noted that $I_{c1} + I_{c2} = 0$. Similarly, the generated excitation emf $E_{a1}$ and the terminal potentials $V_1$ and $V_2$ are the component potentials from the line marked $x$ to the neutral. Thus,

$$E_{a1} - I_{a1}Z_{a1} = V_1$$
$$0 - I_{a2}Z_{a2} = V_2$$

For the motor,

$$V_1 = I_{b1}Z_{b1}$$
$$V_2 = I_{b2}Z_{b2}$$

For the static load,

$$I_{c1} = Y_c(V_1 - V_2)$$
$$I_{c2} = Y_c(V_2 - V_1) \text{ [see equation (64), page 86],}$$

where $Y_c$ is the admittance of the static load. If the foregoing relations are substituted in equations (393),

$$E_{a1} - Z_{a1}\left[\frac{V_1}{Z_{b1}} + Y_c(V_1 - V_2)\right] = V_1 \qquad (399)$$

$$0 - Z_{a2}\left[\frac{V_2}{Z_{b2}} + Y_c(V_2 - V_1)\right] = V_2 \qquad (400)$$

From these equations (399) and (400) it follows that

$$\frac{V_2}{V_1} = \frac{Z_{a2}Z_{b2}}{Z_{a2}Z_{b2} + Z_c(Z_{a2} + Z_{b2})}$$

Note that this ratio of the negative- to the positive-sequence component of the terminal potential does not depend upon the impedance $Z_{b1}$ of the motor to positive sequence and thus is independent of the load on the motor (see Art. 88, page 243). The excitation emf of the generator is

$$E_{a1} = V_1\left(1 + Z_{a1}\frac{Z_{b1} + Z_c}{Z_{b1}Z_c} + \frac{Z_{a1}Z_{a2}/Z_c^2}{1 + Z_{a2}\dfrac{Z_{b2} + Z_c}{Z_{b2}Z_c}}\right)$$

If

$$Z_{a1} = 0.02 + j0.60$$
$$Z_{a2} = 0.05 + j0.20$$
$$Z_{b1} = 1.8 + j0.90$$
$$Z_{b2} = 0.1 + j0.40$$
$$Z_c = 3.5 + j0.0$$
$$\frac{V_2}{V_1} = 0.011 + j0.037$$
$$\frac{|V_2|}{V_1} = 0.039$$

Also

$$E_{a1} = V_1(1.14 + j0.44)$$

The excitation emf $E_{a1}$ is 21 per cent greater than the positive-sequence component of the terminal potential. Further substitution of the numerical values of the constants shows that, if the positive-sequence component of the terminal potential is assumed to have its rated value, 1.0 per unit, the positive- and negative-sequence components of the current supplied by the generator are

$$I_{a1} = \frac{0.14 + j0.44}{0.02 + j0.6}$$
$$= 0.74 - j0.21$$
$$I_{a2} = -\frac{0.011 + j0.037}{0 + j0.25}$$
$$= -0.19 + j0.008$$

Thus, the current in the marked phase ($x$) of the generator is

$$I_a = 0.55 - j0.20$$

That is, $I_a$ is 59 per cent of its rated value. In the most heavily loaded phase ($c$) of the generator the current is

$$I_{a1}\underline{/+120^\circ} + I_{a2}\underline{/-120^\circ}$$

or $I_c = -0.086 - j0.91$, which is 91 per cent of the rated full-load current of the generator.

**VIII. A Single-phase Load of Impedance *Z* Connected across Terminals *b* and *c* of a Salient-pole Generator.**—The necessary fundamental equations applying to phase $a$ are now

$$E_1 - I_d Z_d - I_q Z_q = V_1 \tag{401}$$

$$0 - I_2 Z_2 = V_2 \tag{402}$$

where $I_d$ is the component of the positive-sequence current in quadrature with the excitation emf $E_1$, and $I_q$ is the component that is in phase with $E_1$, $Z_d$ and $Z_q$ are the so-called "direct-axis" and "quadrature-axis synchronous impedances." That is $I_d Z_d$ is the synchronous-impedance drop in the armature due to the component of the armature current in quadrature with the excitation emf, and $I_q Z_q$ is the synchronous-impedance drop due to the component of the armature current in phase with the excitation emf. In the usual salient-pole machine, the direct-axis impedance $Z_d$ is nearly twice as great as the quadrature-axis impedance $Z_q$, since the armature magnetomotive force produces about twice as much flux when it acts along the polar or direct axis as when it acts along the interpolar or quadrature axis. The second equation (402) assumes that in spite of the nonuniformity in the magnetic circuit, the negative-sequence terminal potential is proportional to the negative-sequence current, just as it is in the case of the "ideal" machine. The positive-sequence component of the generated excitation emf in phase $a$ is $E_1$. There is no negative-sequence excitation emf, since the armature windings are assumed to be symmetrically arranged. The positive- and negative-sequence components of the terminal-potential rise in this phase ($a$) are $V_1$ and $V_2$. The positive direction both for phase potentials and for currents is taken as *toward* the neutral. For a detailed discussion of this problem see Art. 132, page 420.

Since there is no current in phase $a$, $I_{a1} + I_{a2} = 0$. It follows that $I_b = -j\sqrt{3}I_{a1}$. The fall in potential $-V_{cb} = I_b Z$ Thus,

$$-V_{cb} = -j\sqrt{3}I_{a1}Z$$

or

$$V_{bc} = -j\sqrt{3}I_{a1}Z$$

But in general,

$$V_{bc} = -j\sqrt{3}(V_{a1} - V_{a2})$$

Therefore

$$V_{a1} - V_{a2} = I_{a1}Z$$

If these relations are substituted in the fundamental equations (401), (402), noting that $I_{a1} = I_d + I_q$,

$$E_1 - I_d(Z_d + Z_2 + Z) - I_q(Z_q + Z_2 + Z) = 0 \qquad (403)$$

A method for solving this equation is illustrated in the following example. Let

$$Z_d = 0 + j0.8 \qquad Z_2 = 0 + j0.3$$
$$Z_q = 0 + j0.45 \qquad Z = \sqrt{3}(0.8 + j0.6)$$

per unit. Thus, the impedance $Z$ will take unit, or rated, current when the *terminal* line potential has its rated value. Represent the excitation emf $E_1$ by a vector along the axis of reals, or let $E_1 = 1.50 + j0$, for example, where the rated phase potential is 1.0 unit. Since the direct-axis component $I_d$ of the current is in quadrature with $E_1$, it is of the form $0 - jB$. The quadrature-axis component $I_q$ of the current is in phase with $E_1$ and is of the form $A + j0$. That is, equation (403) can be written

$$1.50 + j0 - (0 - jB)(1.39 + j2.14) - (A + j0)(1.39 + j1.79) = 0$$

On equating the reals and imaginaries,

$$1.50 - 1.39A - 2.14B = 0$$
$$1.79A - 1.39B = 0$$

The solution of these two equations gives

$$A = 0.36$$
$$B = 0.47$$

The positive-sequence current is $I_{a1} = 0.36 - j0.47$. The load current is $-j\sqrt{3}I_{a1}$, or $-0.63 - j0.81$, which is approximately 3 per cent greater than full-load current.

**129. Synchronous Phase-balancer.**—If a synchronous motor is connected through a negative-sequence generator to a power circuit, it is possible both to maintain balanced potentials at the point of connection and to raise the power factor of the circuit. Thus, in Fig. 146, $G$ is a negative-sequence generator connected in series with a synchronous motor $M$ to a circuit at a point where it is desired to maintain balanced potentials. Figure 146

indicates that, whereas the phase sequence of the excitation emfs generated in the windings of the motor is the same as the phase sequence of the corresponding line-to-neutral potentials, the phase sequence of the excitation emfs in the generator is opposite to that of the corresponding line-to-neutral potentials. Thus, the excitation emfs of the motor form a positive sequence system and those of the generator form a negative-sequence system. It then follows that currents which are of positive-sequence phase order with respect to the motor are of negative-sequence phase order with respect to the generator. Therefore the following equations may be written

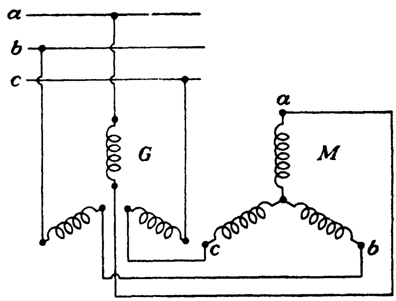



FIG. 146.

$$V_1 = I_1(Z_{G2} + Z_{M1}) - E_M$$
$$V_2 = 0 = I_2(Z_{G1} + Z_{M2}) - E_G$$

where $V_1$ is the fall in line-to-neutral potential, $I_1$ and $I_2$ are the positive- and negative-sequence currents with respect to the synchronous motor, $Z_{M1}$ and $Z_{M2}$ are the impedances of the motor to positive and to negative sequence, $Z_{G1}$ and $Z_{G2}$ are the corresponding constants of the generator, that is, $Z_{M1}$ and $Z_{G1}$ are the synchronous impedances of the motor and generator, and $E_M$ and $E_G$ are the excitation emfs of the motor and generator. The function of the negative-sequence generator is to unbalance the terminal potentials of the synchronous motor so that it will take a negative-sequence current that is exactly equal in magnitude but opposite in phase to the negative-sequence current demanded by the load. It may then be said that the negative-sequence current flows between the motor and the load and is thus withdrawn from the transmission circuit. The currents and the line potentials of the transmission circuit will then be balanced. This condition is similar to that which exists when the leading current taken by a synchronous or static condenser equals the lagging current required by an adjacent inductive load. The second of the preceding equations

shows that the necessary excitation emf of the negative-sequence generator is

$$E_G = I_2(Z_{G1} + Z_{M2})$$

or the excitation emf is proportional to the negative-sequence current that is to be neutralized. Since the time-phase of this negative-sequence current depends upon the time-phase of the line-to-neutral potential $V$, some means must be provided for adjusting the relative time-phase of the excitation emf of the negative-sequence generator with respect to the line-to-neutral potential. For example, if the rotors of the synchronous motor and the generator are rigidly coupled, the necessary phase adjustment can be obtained if the armature of the generator is supported in bearings so that it can be rotated at least 180 degrees electrically. Again, if the air gap of the generator is uniform and the field is supplied with a symmetrical 2-phase winding, the magnetic axis of the field can be rotated the required amount by adjusting the relative amount of current in each of the two field windings. It is then possible to change the relative time-phase of the excitation emf without moving the armature.

Since the line potentials are assumed to be balanced, the total power taken by the synchronous motor and generator is due wholly to the positive-sequence component of the current. This power is entirely expended in rotation, core, and copper losses in the two machines. Thus the power factor of the positive-sequence current with respect to the line-to-neutral potential may be in the neighborhood of 0.10 if there is no shaft load on the motor. At this low power factor the excitation emf of the synchronous motor is approximately

$$|E_M| = |V_1| + |I_1| \cdot |X_{G2} + X_{M1}|$$

The reactive volt-ampere input of the inductive load which is neutralized by the action of the synchronous condenser is approximately $3V_1I_1$.

**130. Effect of Single Field Winding.**—The air gap is uniform and all the other specifications relating to the "ideal" machine, except those relating to the field winding, are unchanged. From a theoretical point of view the advantage of the symmetrical type of field excitation must now be manifest, for, if the excitation is of the usual single-phase type, the double-frequency currents

induced in the field winding by the negative-sequence field will produce *two* oppositely rotating fields in the air gap instead of a single field. One of these fields rotates synchronously with the inducing field produced by the negative-sequence currents in the stator, while the other rotates in the positive direction and thus generates a positive-sequence triple-frequency emf in the stator. This action is the same as that which takes place in an induction motor having a single-phase rotor winding. It is discussed in Art. 111, page 320. The equivalent circuit for the induction motor in Fig. 117 is given for slip $s$. Since the slip of the rotor of the synchronous machine is 2 with respect to the negative-sequence field produced by the stator currents, the negative-sequence equivalent circuit of the synchronous machine is that shown in Fig. 147, if the circuit to which the

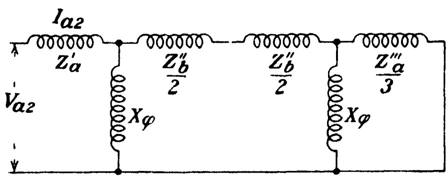



FIG. 147.

stator is connected is symmetrical. It should be noted that the leakage impedance $Z_a'$ of the stator per phase is computed at system frequency $f$, while the leakage impedance $Z_b''$ of the rotor per phase is computed for frequency $2f$. However, both magnetizing reactances $jX_\phi$ are computed for frequency $f$. The triple-frequency currents generated in the stator winding must flow in all parts of the power network to which the synchronous machine is connected. Thus, the impedance $Z_a'''$, which should be computed at frequency $3f$, must include not only the leakage impedance of the stator but also the resistance and reactance per phase of the power network as viewed from the terminals of the stator winding. The impedance of the machine to negative sequence is the ratio of $V_2$ to $I_{a2}$, or the impedance of this synchronous machine to negative sequence depends not only upon the constants of the machine itself but also upon those of the circuit to which it is connected. If this circuit is unsymmetrical, due, for example, to faults or to unbalanced loads, the resulting *unsymmetrical* triple-frequency currents that flow in the stator

winding will produce an additional field which rotates at triple speed in the air gap but in a direction opposite to that in which the field poles rotate. This rotating field will produce currents of four times rated frequency in the field winding. The multiplication of frequency will continue as already shown in Art. 99, page 276. Under these conditions the method of symmetrical components cannot be applied rigorously except under the assumptions noted in Art. 144, page 442. Further experimental work is needed to determine what adjustments should be made in the machine constants so there will be a reasonable agreement between the actual conditions and the conditions as computed on the basis of a symmetrical field winding. In this connection it should be noted that machines are usually constructed with conducting paths on the field poles in addition to the field winding and currents are induced in these paths by the negative-sequence component of the air-gap flux. The general effect of these conducting paths may make the rotor appear as if it carried a *symmetrical* field winding closed through external circuits that are not greatly unbalanced. The eddy currents set up in the solid cylindrical rotor of a steam-turbine-driven synchronous generator produce this result. The same effect is also produced by the currents induced in the amortisseur or damper winding of a salient-pole synchronous machine.

**131. Solid Cylindrical Rotor.**—There are several features in the construction of a solid cylindrical rotor that have an important bearing upon the operation of the alternator under unbalanced conditions. Of these the more important are: the single field winding, the metallic retaining wedges which hold the winding in place in the slots, the solid teeth between the slots, and the main body of the solid rotor itself. We shall not attempt to determine the joint effect these features have upon the operation of the alternator, but will consider the different items separately. The discussion is limited to the effects produced by the negatively rotating fundamental flux due to the negative-sequence currents in the armature.

**Single Field Winding.**—The effect of the single field winding is discussed in Art. 130.

**Wedges.**—The metallic retaining wedges act somewhat like the bars of a cage winding. There being no end rings, the electric circuit through the bars is closed through the solid teeth

of the rotor. The wedges are not continuous throughout the length of the slot but consist of several sections in electrical contact with each other only through the teeth. If the negatively rotating field due to the negative-sequence armature currents is sufficiently great, as it may be during a sudden single-phase short circuit, the induced currents in the wedges may develop enough heat actually to melt portions of the wedges and teeth. The induced currents in the wedges can be reduced by reducing the lengths of individual sections of the wedges.

**Teeth.**—As an approximation, the solid teeth can be treated as transformer laminations in determining the distribution of the flux and the induced eddy currents. Assume that the sides of the teeth are parallel and that the flux density is parallel to the sides. Owing to the eddy currents induced in the teeth by the negatively rotating field, the flux density is least at the center of a tooth and greatest at the sides. Assume that the flux density in any tooth is uniform in any plane parallel to the sides of the teeth and that the direction of the flux density is parallel to a radius through the center of the rotor and the tooth. If $\beta$ is the instantaneous flux density at a point $x$ cm. from the center of the tooth and the *permeability*, $\mu$, *is assumed constant,*

$$4\pi\mu\frac{\partial\beta}{\partial t} = \rho\frac{\partial^2\beta}{\partial x^2} \tag{404}$$

where $\rho$ is the resistivity of the rotor teeth. If the flux density varies sinusoidally with time, equation (404) can be written in vector form

$$j4\pi\omega\mu B = \rho\frac{d^2B}{dx^2}$$

where $B$ is the *vector* representing the sinusoidally varying flux density and $\omega$ is $2\pi$ times the frequency of the variation. The magnitude of $B$ is the maximum instantaneous value of the flux density. If the distribution of the flux is symmetrical about the center line of the tooth, the solution for the flux density vector, $B$, is

$$B = B_0 \cosh \alpha x$$

where $B_0$ is the flux density at the center of the tooth and

$$\alpha = 2\pi\sqrt{\frac{2f\mu}{\rho}}\underline{/45^\circ}$$

The flux density, $B_{t/2}$, at the side of the tooth is $B_0 \cosh \alpha t/2$, where $t$ is the width of a tooth. The ratio of the flux density at the side of the tooth to the flux density at the center of the tooth is $\cosh \alpha t/2$. For example, if $f = 120$ cycles, $\mu = 500$, $\rho = 15{,}000$ abohms per centimeter cube, and $t = 4$ cm., the ratio of the flux densities is

$$\cosh 2 \times 2\pi\sqrt{\frac{240 \times 500}{15{,}000}}\underline{/45^\circ} = \cosh(8\pi + j8\pi) =$$
$$0.5\epsilon^{8\pi}(\cos 8\pi + j \sin 8\pi) \text{ approximately}$$

That is, the flux density at the surface of a tooth is $0.5\epsilon^{8\pi}$ times, or $4 \times 10^{10}$ times the flux density at the center of a tooth. The flux is almost entirely confined to a very thin "skin" at the surface of the tooth. The maximum flux, $\phi$, per tooth per centimeter of axial length is

$$\phi = \int_{-\frac{t}{2}}^{\frac{t}{2}} B\,dx$$

where $t$ is the width of a tooth.

$$\phi = \frac{2B_0}{\alpha} \sinh \frac{\alpha t}{2}$$

The average flux density, $B_{av}$, for the tooth is

$$B_{av} = \frac{2B_0}{\alpha t} \sinh \frac{\alpha t}{2}$$

Hence, the flux density $B_{t/2}$ at the side of the tooth is

$$B_{t/2} = B_{av}\frac{\alpha t}{2} \coth \frac{\alpha t}{2}$$

For the given numerical values of $\alpha$ and $t$, $\coth \alpha t/2$ is approximately unity and

$$B_{t/2} = B_{av} 8\sqrt{2}\pi\underline{/45^\circ} \text{ approximately}$$

The fall in magnetic potential per centimeter length of tooth is $\frac{1}{\mu}B_{t/2}$ or $\frac{B_{av}}{\mu}8\sqrt{2}\pi\underline{/45^\circ}$. If the average density, $B_{av}$, in the teeth is twice the maximum air-gap density and the teeth are five times as long as the air gap, the fall in magnetic potential

for the teeth is $\frac{2 \cdot 5 \cdot 8\sqrt{2}\pi}{500}$ times or approximately 0.71 times the fall in magnetic potential for the air gap. Also, there is a time-phase displacement of 45 degrees between these two component magnetic potentials.

The distribution of the eddy currents in the teeth is determined by an equation similar to equation (404), if the *permeability is assumed constant*, or

$$4\pi\mu\frac{\partial c}{\partial t} = \rho\frac{\partial^2 c}{\partial x^2}$$

where $c$ is the instantaneous eddy-current density at a point $x$ cm. from the center of the teeth. If the flux density, and hence the current density varies sinusoidally with time,

$$j4\pi\omega\mu C = \rho\frac{d^2C}{dx^2}$$

where $C$ is the vector representing the instantaneous current density.

Since the eddy-current density is zero at the center of the tooth and is positive for positive values of $x$ and negative for negative values of $x$, the solution for the *vector*, $C$, representing the eddy-current density is

$$C = C_0 \sinh \alpha x$$

Note that the *magnitude* of the *vector* $C$ is the *maximum instantaneous value* of the eddy-current density. Hence, the power loss at a point $x$ cm. from the center of a tooth is $\frac{\rho}{2}|C|^2$ per centimeter cube.

The average heat loss per centimeter cube due to the eddy currents is

$$\frac{2}{t}\int_0^{\frac{t}{2}} \frac{\rho|C|^2}{2}\,dx$$

where $\rho$ is the resistivity in abohms per centimeter cube.

Since $\alpha = 2\pi\sqrt{\frac{2\mu f}{\rho}}\underline{/45^\circ}$,

$$\alpha = 2\pi\sqrt{\frac{\mu f}{\rho}}(1 + j1) = m(1 + j1)$$

Thus,

$$\begin{aligned} |C|^2 &= |C_0|^2 |\sinh mx(1+j1)|^2 \\ &= |C_0|^2(\sinh^2 mx \cos^2 mx + \cosh^2 mx \sin^2 mx) \\ &= |C_0|^2(\cosh^2 mx - \cos^2 mx) \\ &= \tfrac{1}{2}|C_0|^2(\cosh 2mx - \cos 2mx) \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\rho}{t}\int_0^{\frac{t}{2}} |C|^2\,dx &= \frac{\rho}{2t}\int_0^{\frac{t}{2}} |C_0|^2(\cosh 2mx - \cos 2mx)\,dx \\ &= \frac{\rho}{4mt}|C_0|^2(\sinh mt - \sin mt) \end{aligned}$$

The general differential relation between the eddy current density and the flux density for the particular case we are considering is

$$4\pi\mu C = -\frac{dB}{dx}$$

It follows that

$$C_o = -j\frac{\pi f t}{\rho}\,\frac{B_{av}}{\sinh\frac{\alpha t}{2}}$$

Thus,

$$|C_o|^2 = \frac{\pi^2 f^2 t^2}{\rho^2}\,\frac{2B_{av}^2}{\cosh mt - \cos mt}$$

Therefore, the average eddy-current loss per centimeter cube is

$$\frac{\pi f^{\frac{3}{2}} t}{4\sqrt{\mu\rho}} B_{av}^2 \frac{\sinh mt - \sin mt}{\cosh mt - \cos mt}$$

For large values of $mt$, the loss is approximately

$$\frac{\pi f^{\frac{3}{2}} t}{4\sqrt{\mu\rho}} B_{av}^2 10^{-7} \text{ watts per centimeter cube}$$

Note that $B_{av}$ is the *maximum instantaneous* flux per tooth divided by the cross-sectional area of the tooth in square centimeters.

**Solid Rotor.**—We shall now consider the distribution of flux and eddy currents in the main body of the solid rotor, due

solely to the presence of a negative-sequence field in the air gap. Since it will be found that both the flux and eddy currents are almost entirely confined to a thin "skin" near the surface of the rotor, we shall assume that this surface is a plane rather than a cylinder. This is the same assumption made in Chap. VII when determining the flux distribution in the laminated stator or rotor of an induction motor. Choose coordinate axes so that the $Y$ axis is perpendicular to the surface of the rotor and the $X$ and $Z$ axes lie in the surface of the rotor, the $Z$ axis being

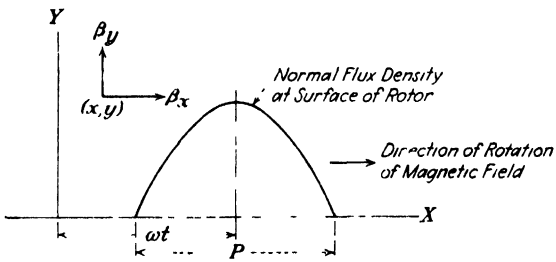


Fig. 148.

parallel to the shaft, as in Fig. 148. Assume that the flux density does not vary in a direction parallel to the $Z$ axis and that there is no component in the eddy currents parallel to either the $X$ or $Y$ axis. The error due to these assumptions decreases as the ratio of the shaft length to the pole pitch increases. The differential equations that the flux density must satisfy are

$$\left.\begin{aligned} 4\pi\mu\frac{\partial\beta_x}{\partial t} &= \rho\left(\frac{\partial^2\beta_x}{\partial x^2} + \frac{\partial^2\beta_x}{\partial y^2}\right) \\ 4\pi\mu\frac{\partial\beta_y}{\partial t} &= \rho\left(\frac{\partial^2\beta_y}{\partial x^2} + \frac{\partial^2\beta_y}{\partial y^2}\right) \end{aligned}\right\} \tag{405}$$

At the surface of the rotor, where $y = 0$, $\beta_y = B\cos\left(\frac{\pi x}{P} - \omega t\right)$. The form of this expression for the normal flux density, $\beta_y$, at the surface of the rotor assumes that a sinusoidal field of maximum density $B$ is moving in the positive direction of $x$ at an electrical angular velocity $\omega$ with respect to the rotor. For simplicity, assume that the mass of rotor iron extends to infinity, and therefore at $y = \infty$, $\beta_y = 0$, $\beta_x = 0$. The solution of the differential equations which satisfies the boundary conditions is

$$\beta_x = \frac{BP}{\pi}\epsilon^{-my}\sqrt{m^2 + n^2}\sin\left(\frac{\pi x}{P} + ny - \omega t - \theta\right) \tag{406}$$

$$\beta_y = B\epsilon^{-my}\cos\left(\frac{\pi x}{P} + ny - \omega t\right) \tag{407}$$

where

$$m = \frac{1}{\sqrt{2}}\frac{\pi}{P}\sqrt{1 + \sqrt{1 + \left(\frac{8\mu f P^2}{\rho}\right)^2}}$$

$$n = \frac{1}{\sqrt{2}}\frac{\pi}{P}\sqrt{-1 + \sqrt{1 + \left(\frac{8\mu f P^2}{\rho}\right)^2}}$$

$$\theta = \tan^{-1}\frac{n}{m}$$

$$2\pi f = \omega$$

$P$ = pole pitch

$\mu$ = permeability

$\rho$ = resistivity (abohms per centimeter cube)

Since $(8\mu fP^2/\rho)^2$ is ordinarily very great compared with unity,

$$m = n = 2\pi\sqrt{\frac{\mu f}{\rho}} \qquad \text{and} \qquad \theta = \frac{\pi}{4} \text{ approximately}$$

Also

$$\beta_x = B\frac{\sqrt{2}\,Pm}{\pi}\epsilon^{-my}\sin\left(\frac{\pi x}{P} + my - \omega t - \frac{\pi}{4}\right) \tag{408}$$

$$\beta_y = B\epsilon^{-my}\cos\left(\frac{\pi x}{P} + my - \omega t\right) \tag{409}$$

The normal flux density, $\beta_y$, and the tangential flux density $\beta_x$, at the surface of the iron are indicated in Fig. 149. The scale for the tangential flux density is one five-hundredth of the scale for the normal flux density. It is interesting to compare Fig. 149 with Fig. 98, page 280, and note the effect produced by laminating the rotor structure. The maximum flux density, $B\sqrt{2}Pm/\pi$, in the $x$ direction occurs at the surface of the rotor at $x = -P/4$ when $\omega t = 0$. If $P = 100$ cm., $f = 120$, $\mu = 500$, $\rho = 15{,}000$, this maximum flux density is 566 times the maximum air-gap density $B$.

If the rotor iron is finely laminated, the maximum density in the $x$ direction is about 1.10 times the maximum air-gap density, $B$ [equation (322), page 281, assuming $P = 2D$].

The difference in magnetic potential, $M$, between the centers of adjacent poles is

$$M = \int_0^P H\,dx = \int_0^P \frac{\beta_x}{\mu}\,dx$$

where $\beta_x$ is the flux density parallel to and at the surface of the rotor at the time $t = 0$.

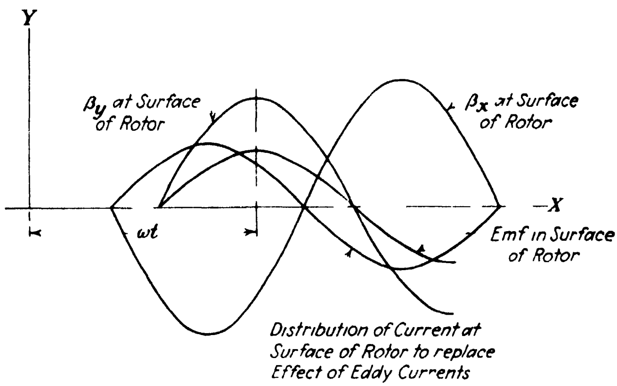



FIG. 149

$$M = \int_0^P \frac{B\sqrt{2Pm}}{\pi\mu}\sin\left(\frac{\pi x}{P} - \frac{\pi}{4}\right)dx$$

$$= \frac{2P^2mB}{\pi^2\mu}$$

If $L$ is the length of the air gap, the magnetomotive force required for the air gap at the centers of the poles is $2BL$.

Hence, the ratio of the magnetomotive force required for the rotor to the magnetomotive force required for the air gap at the centers of the poles is

$$\frac{P^2m}{\pi^2\mu L}$$

If $P = 100$ cm., $f = 120$, $\mu = 500$, $\rho = 15{,}000$, $L = 2.54$ cm.,

$$\frac{P^2m}{\pi^2\mu L} = 10.0$$

The magnetomotive force required for the rotor is 10.0 times the magnetomotive force required for the air gap.

Consider the distribution of the magnetomotive force required for the rotor at time $t = 0$. With respect to the potential at the point $x = -P/4$, the magnetic potential at a point $x$ on the surface of the rotor is

$$\int_{-\frac{P}{4}}^{x} \frac{B_x}{\mu}\, dx = -\frac{\sqrt{2}P^2 mB}{\pi^2 \mu} \cos\left(\frac{\pi x}{P} - \frac{\pi}{4}\right) \tag{410}$$

Thus the distribution of magnetic potential along the surface of the rotor is sinusoidal. Insofar as the effect on the air gap is concerned, the actual rotor can be replaced by a rotor having both infinite permeability and infinite resistivity if it is assumed that there is a distribution of current at the surface of the rotor having the following specifications: The current flows parallel to the shaft, is confined to a thin "skin" at the surface of the rotor, and is distributed so that the current density, $C$, per centimeter at a point $x$ is

$$C = \frac{\sqrt{2}PmB}{4\pi^2 \mu} \sin\left(\frac{\pi x}{P} - \frac{\pi}{4}\right) \tag{411}$$

Since this current distribution is backed by a region of infinite permeability, the magnetic force, $H$, due to the current at a point $x$ on the surface of the rotor, is

$$H = 4\pi C \qquad \text{or} \qquad \frac{\sqrt{2}PmB}{\pi \mu} \sin\left(\frac{\pi x}{P} - \frac{\pi}{4}\right)$$

Thus, the magnetic force due to the current distribution equals in magnitude the magnetic force parallel to and at the surface of the actual rotor. The direction of the current should be chosen so that the magnetic force due to the current is in the same direction as the magnetic force at the surface of the actual rotor. Hence, the distribution of magnetic potential due to the current is equal in magnitude to and has the same direction as the distribution of magnetic potential along the surface of the actual rotor.

For the usual relative values of length of air gap and pole pitch, it can be assumed that the magnetomotive force of the stator

winding plus the magnetomotive force of the current distribution on the surface of the rotor equals the resultant magnetomotive force acting on the gap. The distribution of resultant magnetomotive force per pole is

$$L\beta_y = LB \cos \frac{\pi x}{P} \text{ at } t = 0 \text{ [equation (409), } y = 0]$$

Thus, the stator magnetomotive force, $S$ equals the resultant magnetomotive force minus the magnetomotive force of the current distribution on the surface of the rotor [equation (410)].

$$S = LB \cos \frac{\pi x}{P} + \frac{\sqrt{2}P^2 mB}{\pi^2 \mu} \cos\left(\frac{\pi x}{P} - \frac{\pi}{4}\right)$$

Note that, since the distributions of magnetomotive rotate in the positive direction, the component distribution due to the eddy currents *leads* the resultant magnetomotive force by 45 degrees. For the given numerical values of the coefficients, the ratio of the two components of $S$ is approximately 14. Thus,

$$S = LB(1 + 14/45°).$$

If the magnetizing reactance is $x_\phi$ on the assumption that there are no eddy currents in the rotor and that the permeability is infinite, the combination of the magnetizing reactance with the effects of the eddy currents in the solid rotor is

$$\frac{jx_\phi}{1 + 14/\underline{45°}} = (0.045 + j0.05)x_\phi$$

The resulting impedance to negative sequence equals the sum of this net magnetizing impedance and the leakage impedance of the stator and is $(0.045 + j0.05)x_\phi + Z_a$. If $x_\phi = 0.85$ per unit and $Z_a = 0.02 + j0.15$ per unit, the impedance to negative-sequence for the given numerical values is $0.057 + j0.19$ per unit.

The loss in the solid rotor due to eddy currents can be determined in the following manner. It has been shown that, insofar as the flux distribution in the air gap is concerned, the effect of the eddy currents can be replaced by a distribution of current at the surface of the rotor. Refer to equations (409) and (411) and Fig. 149. In Fig. 149 the positive direction of $B_x$ is toward the right, the positive direction of $B_y$ is toward the rotor, the

positive directions of both current and emf are away from an observer. Current flowing away from an observer produces a magnetic force in the negative direction (toward the left) on the air-gap side of the current. Hence, the current is positive where $B_x$ is negative. The normal flux distribution, $B_y$, at the surface of the rotor moves toward the right and thus generates in the surface of the rotor emfs whose magnitudes are proportional to $B_y$. These emfs are positive where $B_y$ is positive. It will be noted on referring to Fig. 149 that there is a time-phase displacement of $\pi/4$ between the distribution of current and the distribution of emfs. The maximum value of the current per centimeter is $\sqrt{2}PmB/4\pi^2\mu$. The maximum value of the emf is $\omega\phi/2$, where $\phi$, the flux per pole, is $Pl\frac{2}{\pi}B$, and $l$ is the length of the rotor parallel to the shaft. Hence, the power per centimeter due to the distribution of current and the distribution of emf is

$$\frac{1}{2}\left(\frac{\sqrt{2}PmB}{4\pi^2\mu}\right)\left(\frac{2\pi f}{2}Pl\frac{2}{\pi}B\right)\cos\frac{\pi}{4}$$

or

$$\frac{P^2B^2f^{\frac{3}{2}}l}{2\pi\sqrt{\mu\rho}}$$

Therefore, the power loss *per pole* due to eddy currents in the solid rotor is

$$\frac{P^3B^2f^{\frac{3}{2}}l}{2\pi\sqrt{\mu\rho}}10^{-7}\text{ watts}$$

If $P = 100$ cm., $B = 0.05$ of the maximum positive-sequence air-gap-flux density of 7000 gausses, $f = 120$ cycles per second, $l = 350$ cm., $\mu = 500$, $\rho = 15{,}000$, and the eddy current loss per pole due to the negative-sequence field is

$$\frac{100^3 350^2 120^{\frac{3}{2}} 350}{2\pi\sqrt{500 \times 15{,}000}}10^{-7}\text{ or } 327{,}000 \text{ watts per pole}$$

For a 4-pole alternator, the eddy current loss would be about 1300 kilowatts.

The eddy-current loss in the solid rotor can also be determined directly from the expression for the eddy currents. If the

instantaneous eddy-current density (abamperes per square centimeter) at a point $(x,y)$ in the solid rotor is $c$, the differential relation that $c$ must satisfy is

$$4\pi\mu c = \frac{\partial\beta_x}{\partial y} - \frac{\partial\beta_y}{\partial x} \tag{412}$$

On substituting the values of $\beta_x$ and $\beta_y$, equations (408) and (409), the value of $c$ is

$$c = \frac{BP\epsilon^{-my}}{4\pi^2\mu}\left[(m^2 + n^2)\cos\left(\frac{\pi x}{P} + ny - \omega t\right) + \sin\left(\frac{\pi x}{P} + ny - \omega t\right)\right]$$

For the usual values of the coefficients, $m = n$ approximately and $(m^2 + n^2)$, or $8\pi^2\mu f/\rho$, is more than 100. Hence, the instantaneous current density is approximately

$$c = \frac{2BPf}{\rho}\epsilon^{-my}\cos\left(\frac{\pi x}{P} + ny - \omega t\right)$$

The effective or root-mean-square value of the eddy-current density is

$$C_{\text{rms}} = \sqrt{2}\frac{BPf}{\rho}\epsilon^{-my}$$

If $f = 120$, $\mu = 500$, and $\rho = 15{,}000$, the factor $\epsilon^{-my}$ equals 0.1 for $y = \frac{2.3}{4\pi}0.183$ cm. That is, for these values of the constants, both the eddy-current density and the flux density become one-tenth of their values at the surface of the rotor at a point 1.83 mm. below the surface. Therefore, for these values of the constants, nearly all the flux and nearly all the eddy currents are confined to a layer less than 2 mm. thick at the surface of the rotor. This condition entirely justifies the use of rectangular coordinates rather than cylindrical coordinates.

The average heating loss per centimeter cube due to the eddy currents is $\rho C_{\text{rms}}^2$ or

$$\frac{2B^2P^2f^2\epsilon^{-2my}}{\rho}$$

The average heating loss per pole is

$$lP\int^{\infty}\frac{2B^2P^2f^2\epsilon^{-2my}}{\rho}dy$$

or

$$\frac{B^2P^3f^{\frac{3}{2}}l}{2\pi\sqrt{\mu\rho}}10^{-7} \text{ watts}$$

It must be understood that the value of the foregoing analysis depends upon how nearly the assumptions agree with the actual conditions, particularly the assumption in regard to the constancy of the permeability. Actually the permeability varies over a considerable range throughout the region to which the analysis is applied. The justification for the assumption of a constant permeability is that with any other assumption a solution is practically impossible. Therefore, the numerical results that have been computed should be considered in a qualitative sense rather than as being an exact indication of the existing conditions.

**132. Salient Poles.**—When the air gap is approximately uniform, as it is in a well designed 3-phase induction motor, positive- and negative-sequence fundamental currents in the stator windings produce a space distribution of the air-gap flux that contains no third-harmonic component and is often assumed to be substantially free from all other harmonic components. On the other hand, zero-sequence fundamental currents produce an air-gap flux space distribution entirely lacking in fundamental component and in all harmonic components except the third and its multiples. That is, the zero-sequence currents produce a flux distribution which has three times as many poles as that produced by positive- or negative-sequence currents. Furthermore, while the fundamental flux distributions produced by the positive- or negative-sequence currents rotate at synchronous speed, either positively or negatively, the third-harmonic field produced by the zero-sequence currents does not rotate at all but pulsates sinusoidally at fundamental frequency about a fixed position in the air gap. Usually the cylindrical rotor of a synchronous generator is not uniformly slotted and consequently the magnetic circuit is somewhat better along the direct axis of the field poles than it is along the quadrature axis. The difference in the reluctances of the magnetic circuits in these two axes is usually not more than 3 per cent.[1] Thus, the flux distributions produced in the air gap of such a synchronous

[1] WRIGHT, S. H., Determination of Synchronous Machine Constants by Test, *Am. Inst. Elec. Eng.*, **50**, 1331, 1931.

generator by currents in the stator windings are substantially similar to the air-gap flux distributions produced by corresponding currents in the stator windings of an induction motor for which it is assumed that the air gap is uniform. As we have seen, the theory of operation under unsymmetrical conditions of a synchronous generator of this type follows closely the theory which applies to an induction motor.

When the rotor of the synchronous machine is built with salient poles, the reluctance of the magnetic circuit upon which each phase of the stator acts is no longer constant but varies periodically at double synchronous frequency. The flux distribution produced by the current in any set of stator coils will then depend upon the position of the coils with respect to the field poles. Consider a single set of coils arranged as described in Art. 94, page 269, and assume there is flowing in them a direct current of 1 ampere. The air-gap flux distribution produced by this current is shown for three positions of the field poles in Fig. 150, *a*, *b*, and *c*. These distributions consist principally of a fundamental and a third-harmonic component which is relatively much greater when the stator coils are opposite the interpolar space than when the coils are opposite the field poles themselves. The fundamental component of the flux distribution is approximately twice as great when the coils are opposite the poles as when they are opposite the interpolar space. Furthermore, it is important to observe that, when the poles are in the intermediate position (Fig. 150*b*), the fundamental component of the flux distribution is shifted toward the axis of the leading pole so that the distribution is not directly opposite the stator coil as in positions *a* and *c*. Owing to this shift, the fundamental component itself consists of two components one of which, having a maximum value $K_g a_0$, is directly opposite the stator coil and the other, having a maximum value $K_g c_1$, is displaced 90 degrees from the stator coil in the positive direction, as in Fig. 150*b*. If in Fig. 150 the point for which $x = 0$ is chosen at one of the coil sides in phase *a*, the fundamental flux density $\beta$, at a point $x$ radians from this origin, due to a current of 1 ampere in this set of coils, is

$$\beta = K_g[(a_0 + a_1 \cos 2nt) \sin x - c_1 \sin 2nt \cos x] \quad (413)$$

where $K_g$ is a constant of proportionality between the stator

current and the maximum value of the flux density and $a_0$, $a_1$, $c_1$ are constants which are defined in Fig. 150. The angular position of the field poles with respect to the origin is $nt$. Equation (413) is the simplest expression that can be written for the

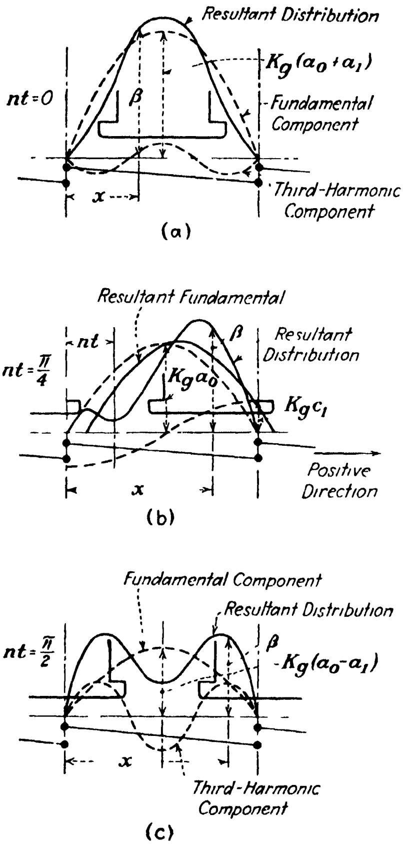



FIG. 150.

flux density, taking into consideration both the variations in magnitude and the shift in space position occuring as the field poles move by the coils. It ignores the space harmonics in the distribution, even though the magnitude of the third-harmonic component may be considerable.[1] Approximate values of the

[1] A more complete discussion of this problem will be found in W. V. Lyon, Application of Harmonic Analysis to the Theory of Synchronous Machines,

coefficients $a_0$, $a_1$, $c_1$ can be computed by assuming the form of the flux distribution for each of several relative positions of the field poles with respect to the armature coils, as was done in the first of the references. More exact values can be obtained from flux plots. The values of these coefficients depend upon ratio of pole arc to pole pitch, the dimensions of the air gap, and the saturation. Reasonable approximate values of the coefficients at no saturation are $a_0 = 0.80$, $a_1 = 0.18$, $c_1 = 0.36$.

The foregoing expression for the flux density $\beta$ is readily transformed into the more useful form

$$\beta = K_g\left[a_0 \sin x + \frac{a_1 + c_1}{2}\sin(x - 2nt) + \frac{a_1 - c_1}{2}\sin(x + 2nt)\right] \quad (414)$$

or the flux distribution consists of three sinusoidal components; the first and largest is stationary; the second, about one-third as great as the first, is rotating at *twice* the speed of the field poles in the *positive* direction; the third, about one-ninth as great as the first, is rotating at *twice* the speed of the field poles but in the *negative* direction.

On referring to Fig. 150*b* it will be noted that the flux density at the same point $x$ due to the current in the next coil on the right is

$$\beta' = K_g\{[a_0 + a_1 \cos 2(nt - \alpha)] \sin(x - \alpha) - c_1 \sin 2(nt - \alpha)\cos(x - \alpha)\}$$

$$= K_g\left[a_0 \sin(x - \alpha) + \frac{a_1 + c_1}{2}\sin(x - 2nt + \alpha) + \frac{a_1 - c_1}{2}\sin(x + 2nt - 3\alpha)\right]$$

Thus, the stationary components of the flux density are displaced from one another by the angle $\alpha$, or the slot pitch. The components which rotate in the positive direction are also displaced from *each other* by the angle $\alpha$. The components which rotate in the negative direction, however, are displaced from *each*

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*Trans. Am. Inst. Elec. Eng.*, **37**, 1477, 1918; R. E. Doherty and C. A. Nickle, Synchronous Machines I, *Trans. Am. Inst. Elec. Eng.*, **45**, 912, 1926.

*other* by an angle $3\alpha$. Therefore, when combining the component flux distributions due to the several coils in one phase, the *fundamental* reduction factor is used for the stationary and positively rotating components while the *third-harmonic* reduction factor is used for the negatively rotating components.

If there are $N_a/3p$ turns per pole per phase, if $K_1$ and $K_3$ are the first- and third-harmonic total reduction factors for breadth and pitch of the armature winding, if the field poles are turning at synchronous angular velocity $\omega$, the fundamental flux density $\beta_a'$ produced by the current $i_a$ in phase $a$ is

$$\beta_a' = K_g \frac{N_a}{3p} i_a \left[ K_1 a_0 \sin x + K_1 \frac{a_1 + c_1}{2} \sin(x - 2\omega t) + K_3 \frac{a_1 - c_1}{2} \sin(x + 2\omega t) \right] \quad (414a)$$

where both $x$ and $\omega t$ are now measured from the center of the $a$ phase belt of conductors. Convenient expressions for the total reduction factors $K_1$, $K_3$, etc. for a 3-phase winding are

$$K_1 = \frac{1.5}{S \sin \frac{90^\circ}{S}} \sin \frac{P}{S} 90^\circ$$

$$K_3 = -\frac{3}{S \sin \frac{270^\circ}{S}} \sin \frac{P}{S} 270^\circ$$

$$K_5 = -\frac{1.5}{S \sin \frac{450^\circ}{S}} \sin \frac{P}{S} 450^\circ$$

where $S$ is the number of slots per pole and $P$ is the coil pitch in slots. These expressions apply strictly only in case there is an integral number of slots per pole per phase. The expression for the flux density produced by the current $i_b$ in the next phase to the right is similar to (414$a$) except that $(x - 120^\circ)$ replaces $x$, and $(\omega t - 120^\circ)$ replaces $\omega t$ for positive-sequence currents, and $(\omega t + 120^\circ)$ replaces $\omega t$ for negative-sequence currents. The time angle $\omega t$ is unchanged for zero-sequence currents. The flux density produced by the current $i_c$ in the third phase can be determined in a similar manner. The resulting flux density in the air gap is the sum of these three component distributions.

**133. Positive Sequence.**—The current in the second phase ($b$) lags the current in the first phase ($a$) by 120 degrees. For positive-sequence fundamental currents, when

$$i_{a1} = \sqrt{2}I_{a1}\sin(\omega t + \theta_1),$$

the resulting fundamental flux density $\beta_1'$ is

$$\beta_1' = \frac{1}{\sqrt{2}}K_gK_1\frac{N_a}{p}I_{a1}\left[a_0\cos(x - \omega t - \theta_1) - \frac{a_1 + c_1}{2}\cos(x - \omega t + \theta_1)\right] \quad (415)$$

Note that $\theta_1$ is the phase angle of the current with respect to the emf that would be generated on open circuit. Both the component fields (415) travel at synchronous speed in the positive direction.

Since the fall in emf generated in phase $a$ of the armature winding by the rotating field is

$$\frac{K_1N_a}{3}\frac{d}{dt}\int_0^{\pi}\beta_1'\,dx$$

$$\text{emf} = \sqrt{2}\omega K_gK_1^2\frac{N_a^2}{3p}I_{a1}\left[a_0\cos(\omega t + \theta_1) - \frac{a_1 + c_1}{2}\cos(\omega t - \theta_1)\right]$$

When $\theta_1$ is zero, this emf is

$$\sqrt{2}\omega K_gK_1^2\frac{N_a^2}{3p}\left(a_0 - \frac{a_1 + c_1}{2}\right)I_{a1}\cos\omega t$$

When $\theta_1$ is $\pi/2$, the emf is

$$\sqrt{2}\omega K_gK_1^2\frac{N_a^2}{3p}\left(a_0 + \frac{a_1 + c_1}{2}\right)I_{a1}\cos\left(\omega t + \frac{\pi}{2}\right)$$

As in the induction motor, the ratio of the emf generated by the air-gap flux to the current that produces the flux is the magnetizing reactance. Thus, when $\theta_1$ is zero, the magnetizing reactance is

$$\omega K_gK_1^2\frac{N_a^2}{3p}\left(a_0 - \frac{a_1 + c_1}{2}\right)$$

This is the magnetizing reactance, $x_{\phi q}$, in the quadrature axis. The synchronous reactance in the quadrature axis is

$$x_q = x_{\phi q} + x_a,$$

where $x_a$ is the leakage reactance of the stator. When $\theta_1$ is $\pm\frac{\pi}{2}$, the magnetizing reactance is

$$\omega K_g K_1^2 \frac{N_a^2}{3p}\left(a_0 + \frac{a_1 + c_1}{2}\right)$$

This is the magnetizing reactance, $x_{\phi d}$, in the direct axis and the synchronous reactance in the direct axis is $x_d = x_{\phi d} + x_a$. While both the magnetizing reactances $x_{\phi q}$ and $x_{\phi d}$ are affected by saturation, the effect on the direct-axis reactance may be greater than the effect on the quadrature-axis reactance (see Art. 150, page 463).

**134. Negative Sequence.**—The current in the second phase ($b$) leads the current in the first phase ($a$) by 120 degrees. For negative-sequence fundamental currents, when

$$i_{a2} = \sqrt{2} I_{a2} \sin(\omega t + \theta_2),$$

the resulting fundamental flux density $\beta_2'$ is

$$\beta_2' = \frac{1}{\sqrt{2}} K_g K_1 \frac{N_a}{p} I_{a2}\left[-a_0 \cos(x + \omega t + \theta_2) + \frac{a_1 + c_1}{2} \cos(x - 3\omega t - \theta_2)\right] \quad (416)$$

The larger of the component fields travels at synchronous speed in the negative direction, while the smaller, which is about one-third as great, travels at three times synchronous speed in the positive direction. Thus, the fields generate emfs in the stator winding of approximately the *same magnitude*, but the *frequency* of the *second component* is three times fundamental frequency. This multiplication of frequency is similar to that which occurs when the rotor winding of an induction motor is unsymmetrical or more specifically when a synchronous machine has a cylindrical rotor with the usual type of single field winding (Art. 130), page 406). That is, unsymmetry in the magnetic circuit produces the same effect on the air-gap flux as unsymmetry in the rotor winding.

The fall in emf generated in phase *a* by the rotating fields (416) produced by the fundamental negative-sequence currents is

$$\text{emf} = \sqrt{2}\frac{x_{\phi d} + x_{\phi q}}{2}I_{a2}\cos(\omega t + \theta_2)$$
$$+ \sqrt{2}\,3\frac{x_{\phi d} - x_{\phi q}}{2}I_{a2}\cos(3\omega t + \theta_2) \quad (417)$$

In the third-harmonic component of this emf the factor 3 is introduced so that the values of the magnetizing reactances are still for fundamental frequency. It should be noted that, since both the component flux distributions (416) rotate at twice synchronous speed with respect to the field poles, the emfs they generate in the field or damper windings are of double frequency.

**135. Zero Sequence.**—For zero-sequence fundamental currents, when $i_{a0} = \sqrt{2}I_{a0}\sin(\omega t + \theta_0)$, the resulting fundamental flux density, $\beta_0'$, in the air gap is

$$\beta_0' = \frac{1}{\sqrt{2}}K_gK_3\frac{N_a}{p}\frac{a_1 - c_1}{2}I_{a0}\Big[\cos(x + \omega t - \theta_0)$$
$$- \cos(x + 3\omega t + \theta_0)\Big] \quad (418)$$

Thus with salient poles, the zero-sequence currents produce two fundamental fields of the same magnitude, both rotating negatively, the one at synchronous speed and the other at three times synchronous speed. *With a uniform air gap,* zero-sequence currents produce no fundamental flux. The fall in emf generated in phase $a$ by these rotating fields is

$$\text{emf} = \sqrt{2}\omega K_gK_3K_1\frac{N_a^2}{3p}\frac{a_1 - c_1}{2}I_{a0}\Big[-\cos(\omega t - \theta_0)$$
$$+ 3\cos(3\omega t + \theta_0)\Big] \quad (419)$$

As yet no name has been given to the magnetizing reactance $\omega K_gK_3K_1\frac{N_a^2}{3p}\frac{a_1 - c_1}{2}$. For a full-pitch winding, this reactance may be 5 or 6 per cent of the direct axis magnetizing reactance $x_{\phi d}$. When the winding pitch is reduced to two-thirds, the reduction factor $K_3$ becomes zero and this reactance also becomes zero, that is, no fundamental air-gap flux is produced.

**136. Third-harmonic Flux Distribution.**—Next to the fundamental component of the flux distribution in the air gap, the third-

harmonic component is by far the most important. Although we shall not consider in detail the effect that it produces, it is well to note the relative magnitude of the third-harmonic flux produced by each of the symmetrical components of the fundamental current in the stator windings. If the same assumption is made in regard to the way in which the magnitude of the third-harmonic flux distribution varies as was made in the case of the fundamental flux distribution, the third-harmonic flux density $\beta'''$ in the air gap due to 1 ampere in one phase of the stator winding is

$$\beta''' = K_g \frac{N_a}{3p}\left[ K_3 a_0''' \sin 3x + K_1 \frac{a_1''' + c_1'''}{2} \sin(3x - 2\omega t) + K_5 \frac{a_1''' - c_1'''}{2} \sin(3x + 2\omega t)\right]$$

where the angular velocity of the field poles is $\omega$ and $x$ is measured from the center of the phase belt to the point at which the flux density is given. The first of these third-harmonic components of flux density is stationary; the second, which is usually the largest of the three, rotates at two-thirds the speed of the field poles and in the *same* direction; while the third, which is usually the smallest, rotates at two-thirds the speed of the field poles but in the *opposite* direction. Reasonable values of the coefficients when the machine is operating in an unsaturated condition are $a_0''' = 0.40$, $a_1''' = -0.32$, $c_1''' = -0.26$. For these values of the coefficients, it is evident that the third component of the third-harmonic flux density is so small that it may be ignored. The resulting flux distribution due to the currents in all three phases of the stator will depend upon the sequence order of the currents just as with the fundamental flux distribution.

**137. Positive Sequence.**—For positive-sequence fundamental currents, when $i_{a1} = \sqrt{2}I_{a1} \sin(\omega t + \theta_1)$, the resulting third-harmonic flux density $\beta_1'''$ is

$$\beta_1''' = \frac{1}{\sqrt{2}} K_g K_1 \frac{N_a}{p} \frac{a_1''' + c_1'''}{2} I_{a1} \cos(3x - 3\omega t - \theta_1)$$

This third-harmonic field rotates positively at synchronous speed. The fall in emf generated in phase $a$ of the stator winding by this flux distribution is

$$\text{emf} = \sqrt{2}\omega K_g K_1 K_3 \frac{N_a^2}{3p} \frac{a_1''' + c_1'''}{2} I_{a1} \cos(3\omega t + \theta_1)$$

Since this emf is due to a third-harmonic in the flux distribution, it does not appear across the terminals of the machine. The reactance $\omega K_g K_1 K_3 \frac{N_a^2}{3p} \frac{a_1''' + c_1'''}{2}$ may be about 15 per cent of the direct-axis magnetizing reactance $x_{\phi d}$.

**138. Negative Sequence.**—For negative-sequence fundamental current, when $i_{a2} = \sqrt{2}I_{a2} \sin(\omega t + \theta_2)$, the resulting third-harmonic flux density $\beta_2'''$ is

$$\beta_2''' = -\frac{1}{\sqrt{2}} K_g K_1 \frac{N_a}{P} \frac{a_1''' + c_1'''}{2} I_{a2} \cos(3x - \omega t + \theta_2)$$

This third-harmonic field rotates positively at one-third synchronous speed. The fall in emf generated in the $a$ phase of the stator winding by this flux distribution is

$$\text{emf} = -\sqrt{2}\omega K_g K_1 K_3 \frac{N_a^2}{3p} \frac{a_1''' + c_1'''}{6} I_{a2} \cos(\omega t - \theta_2)$$

Although this emf is of fundamental frequency it is due to a third-harmonic in the flux distribution and hence does not appear across the line terminals of the machine. The reactance

$$\omega K_g K_1 K_3 \frac{N_a^2}{3p} \frac{a_1''' + c_1'''}{6}$$

is but one-third as great as in the case of positive sequence and is probably not more than approximately 5 per cent of the direct-axis magnetizing reactance $x_{\phi d}$.

**139. Zero Sequence.**—For zero-sequence fundamental currents, when $i_{a0} = \sqrt{2}I_{a0} \sin(\omega t + \theta_0)$, the resulting third-harmonic flux density is

$$\beta_0''' = \frac{1}{\sqrt{2}} K_g K_3 \frac{N_a}{p} a_0''' I_{a0} \Big[ \cos(3x - \omega t - \theta_0) - \cos(3x + \omega t + \theta_0) \Big]$$

There are two component third-harmonic fields, both of which rotate at one-third synchronous speed, the first positively and

the second negatively. The fall in emf generated in the $a$ phase of the stator winding by the flux distribution $\beta_0'''$ is

$$\text{emf} = \sqrt{2}\omega K_g K_3^2 \frac{N_a^2}{3p} \frac{a_0'''}{3} I_{a0} 2 \cos(\omega t + \theta_0)$$

For a full-pitch winding, the reactance $\omega K_g K_3^2 \frac{N_a^2}{3p} \frac{a_0'''}{3}$ may be approximately 5 per cent of the direct-axis magnetizing reactance $x_{\phi d}$.

**140. Interrelation of the Different Sequences.**—It should be noted that, in general, whether the air gap is uniform or nonuniform, a fundamental component of the air-gap flux density generates in a symmetrical 3-phase armature winding emfs of either positive- or negative-sequence phase order but never of zero-sequence phase order; that is, the emfs cannot be instantaneously equal in each of the three phases. Furthermore, a third-harmonic component in the air-gap flux always generates in the armature winding emfs of zero-sequence phase order. When the air-gap is uniform, positive- and negative-sequence currents produce no third-harmonic flux in the air gap, while zero-sequence currents produce no fundamental flux. Moreover, positive-sequence currents produce a positively rotating field and negative-sequence currents produce a negatively rotating field. Thus, in the "ideal" synchronous machine, there is no relation between the current of one sequence and the emf of another sequence generated by the air-gap flux. This, however, is not in general true of the salient-pole machine. For example, zero-sequence fundamental current produces a negatively rotating field which generates a negative-sequence fundamental emf in the armature. Again, negative-sequence fundamental current produces a third-harmonic field which rotates so as to generate a zero-sequence fundamental emf in the armature. It is interesting to note that, when the winding pitch is two-thirds, there is no interrelation between the currents and emfs of different sequences, since for this pitch the reduction factor $K_3$ is zero. Fortunately, the mutual reactances which relate the currents and emfs of different sequences are never more than approximately 5 per cent of the direct-axis magnetizing reactance $x_{\phi d}$. In the following discussion we shall neglect the effects of these third-harmonic fields.

**141. Reaction of Rotor.**—Negative-sequence currents in the armature produce two fundamental fields, one of which rotates at synchronous speed in the negative direction and the other, at three times synchronous speed in the positive direction. The frequency of the currents induced in the rotor circuits by each of these fields is twice fundamental frequency. In the armature winding, the fields generate negative-sequence emfs of fundamental frequency and positive-sequence emfs of triple frequency. Were the rotor circuits of the symmetrical polyphase type, the currents induced in them by the negatively rotating field would themselves produce a field in the air-gap which would rotate synchronously with the inducing field. This would also be true of the field that rotates in the positive direction at three times synchronous speed. Since the rotor circuits are not alike in the two quadrature axes, although they are symmetrical about these axes, the rotor currents induced by the negatively rotating field will themselves produce *two* fields, one rotating synchronously with this negatively rotating inducing field and the other rotating synchronously with the positively rotating inducing field. Likewise, the rotor currents induced by the field which rotates in the positive direction at three times synchronous speed will themselves produce *two* fields, one rotating synchronously with this positively rotating inducing field and the other rotating synchronously with the negatively rotating inducing field. This action of the currents induced in the rotor circuit produces an interrelation between the fundamental and triple frequency emfs generated in the armature windings by the two oppositely rotating fields in the air gap. The general method of analysis we shall adopt is to replace each of the rotating fields produced by the stator currents by its sinusoidally pulsating components in the direct and quadrature rotor axes. Having determined the rotor currents and the fields produced by them in these two axes, we shall then be able to write the expressions for the resulting emfs generated in the armature winding and to equate these emfs to the applied potentials. These equations determine the armature currents and the negative-sequence equivalent circuit of the machine.

Since the positively rotating field generates triple-frequency emfs in the armature windings, there will also be positive-sequence currents of this frequency in the armature. If

$$i_a = \sqrt{2}I_{a1}''' \sin(3\omega t + \theta_1'''),$$

the resulting flux density in the air-gap due to the positive-sequence triple-frequency currents is

$$\beta = \frac{1}{\sqrt{2}}K_g K_1 \frac{N_a}{p} I_{a1}''' \left[ a_0 \cos(x - 3\omega t - \theta_1''') - \frac{a_1 + c_1}{2} \cos(x + \omega t + \theta_1''') \right] \quad (420)$$

The emf generated in phase $a$ of the armature by the rotation of the fields (420) is

$$\text{emf} = \sqrt{2}\frac{3(x_{\phi d} + x_{\phi q})}{2} I_{a1}''' \cos(3\omega t - \theta_1''') + \sqrt{2}\frac{(x_{\phi d} - x_{\phi q})}{2} I_{a1}''' \cos(\omega t + \theta_1''') \quad (421)$$

Thus, the resultant fundamental flux density in the air gap produced by the negative-sequence fundamental current and the attendant positive-sequence triple-frequency current is

$$\beta_a = \frac{1}{\sqrt{2}}K_g K_1 \frac{N_a}{p} \Big\{ -a_0 I_{a2}' \cos(x + \omega t + \theta_2') - \frac{a_1 + c_1}{2} I_{a1}''' \cos(x + \omega t + \theta_1''') + \frac{a_1 + c_1}{2} I_{a2}' \cos(x - 3\omega t - \theta_2') + a_0 I_{a1}''' \cos(x - 3\omega t - \theta_1''') \Big\} \quad (422)$$

Each of these rotating fields is equivalent to two sinusoidally pulsating fields whose axes coincide with the direct and quadrature axes of the rotor. On referring to Fig. 151 it will be observed that the total pulsating component $\phi_d$ of the flux in the direct axis is

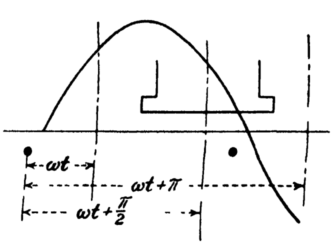



FIG. 151.

$$\phi_d = \int_{\omega t}^{\omega t + \pi} \beta_a \, dx$$

and that the total component $\phi_q$ of the flux in the quadrature axis is

$$\phi_q = \int_{\omega t + \frac{\pi}{2}}^{\omega t + \frac{3\pi}{2}} \beta_a \, dx$$

It follows that

$$\phi_d = \sqrt{2}K_gK_1\frac{N_a}{p}\left(a_0 + \frac{a_1 + c_1}{2}\right)\left[I_{a2}' \sin(2\omega t + \theta_2') + I_{a1}''' \sin(2\omega t + \theta_1''')\right] \quad (423)$$

$$\phi_q = \sqrt{2}K_gK_1\frac{N_a}{p}\left(a_0 - \frac{a_1 + c_1}{2}\right)\left[I_{a2}' \sin\left(2\omega t + \theta_2' + \frac{\pi}{2}\right) - I_{a1}''' \sin\left(2\omega t + \theta_1''' + \frac{\pi}{2}\right)\right] \quad (424)$$

**142. Equivalent Rotor Winding.**—The next step in the analysis is to replace the actual rotor circuits by an equivalent symmetrical 3-phase winding which has the same effective turns as the actual armature winding. This 3-phase winding is made equivalent to the actual rotor circuits by adjusting the leakage impedances of each phase with respect to the armature winding, so that the currents induced in the equivalent 3-phase winding by the air-gap flux will themselves produce the same fundamental component field in the air gap as is produced by the induced currents in the actual rotor circuits. This is the same procedure followed in Art. 120, page 351, when analyzing the behavior of an unsymmetrical cage winding. The analysis in the present case, however, is much more difficult as the air gap is no longer uniform, the damper bars, if there are any, are frequently of different shapes and of different materials, and in the direct axis there are other circuits to consider, such as the field winding and the conducting rings which support the field winding. Probably the most accurate method for determining the necessary constants of the component parts of the actual rotor circuits is by means of flux plots. Lacking these, reasonable assumptions in regard to the flux distributions must be made so that the constants can be calculated. In each axis the problem is essentially that of a transformer having a primary winding (the armature winding) and several independent short-circuited secondaries (the rotor circuits). It is then necessary to determine the leakage impedance of a single equivalent short-circuited secondary so that the reaction produced on the primary by the current induced in the *equivalent* secondary is the same as the combined reactions of the currents induced in the *actual* second-

aries. This must be done for the rotor circuits in both direct and quadrature axes.[1] By following the method used in Art. 121, page 358, these two equivalent secondaries in the direct and quadrature rotor axes can then be replaced by a symmetrical 3-phase winding arranged exactly like the armature winding but having different leakage impedances in its phases. Thus, in Fig. 152, the symmetrical windings *bn*, *fn*, *gn* have leakage impedances $(\frac{3}{2}Z_{bd}'' - \frac{1}{2}Z_{bq}'')$, $Z_{bq}''$ and $Z_{bq}''$, as indicated. Phase *b* is placed directly opposite the pole in the direct axis.

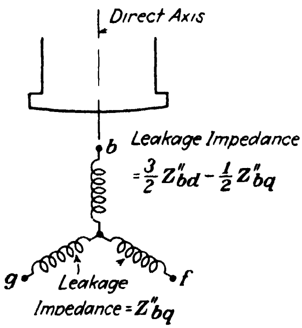



FIG. 152

For a pulsating field in the direct axis *d*, each of the emfs induced in phases *f* and *g* is then one-half of the emf induced in phase *b*, and the terminals *f* and *q* are at the same potential. Since the impedances in phases *f* and *g* are equal, the currents in them are also equal. For the given values of the phase impedances, it follows that the current in each phase is the same as if the leakage impedance in *each* phase were $Z_{bd}''$. Thus, in the direct axis, the reaction of the rotor currents is the same as if the rotor circuits were replaced by a symmetrical 3-phase winding having a leakage impedance $Z_{bd}''$ per phase. Since phase *b* is directly opposite the pole in the direct axis, neither emf nor current will be induced in it by a pulsating field in the quadrature axis *q*. Thus, the currents induced in phases *f* and *g* by the field in the quadrature axis are the same as if the impedances in *each* phase were $Z_{bq}''$. That is, the reaction of the rotor currents in the quadrature axis is the same as if the rotor circuits were replaced by a symmetrical 3-phase winding having a leakage impedance $Z_{bq}''$ per phase. The resulting magnetomotive force developed by the currents induced in a symmetrical 3-phase winding by a pulsating field is independent of the position of the field with respect to the winding. Thus, the effect of the equivalent rotor winding in the quadrature axis can be computed as if one of the three phases were located *in* the quadrature

[1] A detailed procedure is described in a paper by T. M. Linville, Starting Performance of Salient-pole Synchronous Motors, *Am. Inst. Elec. Eng.*, **49**, 531, 1930.

axis. That is, exactly the same procedure should be followed when analyzing the reaction of the rotor in the quadrature axis as when analyzing the reaction in the direct axis.

**143. Equivalent Circuit.**—The *rise* in generated emf per phase in the equivalent 3-phase rotor winding in the direct axis is

$$e_d = -K_1\frac{N_a}{3}\frac{d\phi_d}{dt}$$

and in the quadrature axis is

$$e_q = -K_1\frac{N_a}{3}\frac{d\phi_q}{dt}$$

The expressions for these emfs reduce to

$$e_d = -\sqrt{2}\,2x_{\phi d}\Big[I_{a2}'\cos(2\omega t + \theta_2') + I_{a1}'''\cos(2\omega t + \theta_1''')\Big] \quad (425)$$

$$e_q = -\sqrt{2}\,2x_{\phi q}\Big[I_{a2}'\cos\left(2\omega t + \theta_2' + \frac{\pi}{2}\right) - I_{a1}'''\cos\left(2\omega t + \theta_1''' + \frac{\pi}{2}\right)\Big] \quad (426)$$

Note that $x_{\phi d}$ and $x_{\phi q}$ are the magnetizing reactances in the two axes at *fundamental* frequency. The current in each of these equivalent rotor circuits is determined by the rise in generated emf and the *total self-impedance* of the circuit. If the leakage impedance of the circuit in the direct axis is $Z_{bd}''$ at the actual rotor frequency, the total self-impedance of the rotor circuit in the direct axis is $j2x_{\phi d} + Z_{bd}''$. The currents in the rotor circuits in the two axes are

$$i_{bd} = -\sqrt{2}\frac{2x_{\phi d}}{|j2x_{\phi d} + Z_{bd}''|}\Big[I_{a2}'\cos(2\omega t + \theta_2' - \alpha_d) + I_{a1}'''\cos(2\omega t + \theta_1''' - \alpha_d)\Big] \quad (427)$$

$$i_{bq} = -\sqrt{2}\frac{2x_{\phi q}}{|j2x_{\phi q} + Z_{bq}''|}\Big[I_{a2}'\cos\left(2\omega t + \theta_2' + \frac{\pi}{2} - \alpha_q\right) - I_{a1}'''\cos\left(2\omega t + \theta_1''' + \frac{\pi}{2} - \alpha_q\right)\Big] \quad (428)$$

where $\alpha_d$ is the angle of the impedance $j2x_{\phi d} + Z_{bd}''$ and $\alpha_q$ is the angle of the impedance $j2x_{\phi q} + Z_{bq}''$. The component

flux density $\beta_d$ in the air gap produced by the current $i_{bd}$ in the equivalent rotor winding is

$$\beta_d = \frac{1}{2}K_g K_1 \frac{N_a}{p}\left(a_0 + \frac{a_1 + c_1}{2}\right) i_{bd} \sin(x - \omega t) \tag{429}$$

The fall in generated emf in phase $a$ of the armature winding due to this flux distribution is

$$e_{ad} = K_1 \frac{N_a}{3} \frac{d}{dt} \int_0^{\pi} \beta_d \, dx$$

The expression for this emf becomes [see equation (427)]

$$e_{ad} = \sqrt{2} \frac{2x_{\phi d}^2}{|j2x_{\phi d} + Z_{bd}''|} \left\{ \frac{I_{a2}'}{2}\left[\sin(\omega t + \theta_2' - \alpha_d) + 3\sin(3\omega t + \theta_2' - \alpha_d)\right] + \frac{I_{a1}'''}{2}\left[\sin(\omega t + \theta_1''' - \alpha_d) + 3\sin(3\omega t + \theta_1''' - \alpha_d)\right] \right\} \tag{430}$$

Similarly, the fall in generated emf, $e_{aq}$, in phase $a$ of the armature winding, due to the quadrature field $\beta_q$, is

$$e_{aq} = K_1 \frac{N_a}{3} \frac{d}{dt} \int_0^{\pi} \beta_q \, dx$$

where

$$\beta_q = \frac{1}{2}K_g K_1 \frac{N_a}{p}\left(a_0 - \frac{a_1 + c_1}{2}\right) i_{bq} \sin\left(x - \omega t - \frac{\pi}{2}\right)$$

The expression for this emf becomes [see equation (428)]

$$e_{aq} = \sqrt{2} \frac{2x_{\phi q}^2}{|j2x_{\phi q} + Z_{bq}''|} \left\{ \frac{I_{a2}'}{2}\left[\sin(\omega t + \theta_2' - \alpha_q) - 3\sin(3\omega t + \theta_2' - \alpha_q)\right] - \frac{I_{a1}'''}{2}\left[\sin(\omega t + \theta_1''' - \alpha_q) - 3\sin(3\omega t + \theta_1''' - \alpha_q)\right] \right\} \tag{431}$$

The resulting fall in potential across phase $a$ of the stator winding consists of fundamental and of triple-frequency components due to the combined action of the armature and rotor currents.

The fundamental components of potential are:

1. The armature leakage impedance ($Z_a$) drop due to the negative-sequence armature current $\sqrt{2}I_{a2}' \sin(\omega t + \theta_2)$.

2. The fall in potential due to the component rotating field produced by the negative-sequence current [first term of equation (417), page 427].

3. The fall in potential due to the component rotating field produced by the positive-sequence triple-frequency current $\sqrt{2}I_{a1}''' \sin(3\omega t + \theta_1''')$ [second term of equation (421), page 432].

4. The falls in potential due to the component fields produced by the rotor currents [first and third terms of equation (430) and equation (431), page 436].

The triple-frequency components of potential are:

1. The armature leakage impedance $Z'''$ drop due to the positive-sequence triple-frequency armature current,

$$\sqrt{2}I_{a1}''' \sin(3\omega t + \theta_1''').$$

2. The fall in potential due to the component rotating field produced by the triple-frequency current [first term of equation (421), page 432]

3. The fall in potential due to the component rotating field produced by the negative-sequence fundamental current [second term of equation (417), page 427].

4. The falls in potential due to the component fields produced by the rotor currents [second and fourth terms of equation (430) and equation (431), page 436].

Consider the expressions for the negative-sequence fundamental current, $\sqrt{2}I_{a2}' \sin(\omega t + \theta_2')$, and the positive-sequence triple-frequency current, $\sqrt{2}I_{a1}''' \sin(3\omega t + \theta_1''')$. In each of these expressions the time angle $\omega t$ is zero at the moment the phase of the armature winding in which the given currents flow is opposite a field pole (see Fig. 150, page 422). When these currents are represented by vectors, the vector representing the triple-frequency current rotates at three times the speed of the vector representing the fundamental current. It will be observed, however, that *at the time* $\omega t = 0$ all the component emfs in equation (417), page 427, equation (421), page 432, equation (430), page 436, and equation (431), page 436, can be expressed in terms of the *vectors* which represent the negative-

sequence fundamental current and the positive-sequence triple-frequency current. At *other* times in the cycle, the vector substitution is *not valid* in as much as the triple-frequency vectors do *not* maintain a fixed phase relation with the fundamental frequency vectors. When these components are collected and written in vector form for the time $t = 0$,

$$V_{a2}' = I_{a2}'\left(Z_a + j\frac{x_{\phi d} + x_{\phi q}}{2} + \frac{1}{2}\frac{2x_{\phi d}^2}{j2x_{\phi d} + Z_{bd}''} + \frac{1}{2}\frac{2x_{\phi q}^2}{j2x_{\phi q} + Z_{bq}''}\right) + I_{a1}'''\left(j\frac{(x_{\phi d} - x_{\phi q})}{2} + \frac{1}{2}\frac{2x_{\phi d}^2}{j2x_{\phi d} + Z_{bd}''} - \frac{1}{2}\frac{2x_{\phi q}^2}{j2x_{\phi q} + Z_{bq}''}\right) \quad (432)$$

$$V_{a1}''' = I_{a1}'''\left(Z_a''' + j\frac{3(x_{\phi d} + x_{\phi q})}{2} + \frac{3}{2}\frac{2x_{\phi d}^2}{j2x_{\phi d} + Z_{bd}''} + \frac{3}{2}\frac{2x_{\phi q}^2}{j2x_{\phi q} + Z_{bq}''}\right) + I_{a2}'\left(j\frac{3(x_{\phi d} - x_{\phi q})}{2} + \frac{3}{2}\frac{2x_{\phi d}^2}{j2x_{\phi d} + Z_{bd}''} - \frac{3}{2}\frac{2x_{\phi q}^2}{j2x_{\phi q} + Z_{bq}''}\right) \quad (433)$$

In these equations $Z_a'$ is the armature leakage impedance at fundamental frequency, $Z_{bd}''$ and $Z_{bq}''$ are the leakage impedances of the equivalent rotor circuit in the two axes at double fundamental frequency, $Z_a'''$ is the armature leakage impedance at three times fundamental frequency. The triple-frequency terminal potential $V_{a1}'''$ of the armature equals the product of the triple-frequency armature current $I_{a1}'''$ and the triple-frequency impedance of the supply network as viewed from the terminals of the armature. It must be assumed that this network is symmetrical, for otherwise continued reflections will take place between the armature and field circuits resulting in a series of harmonics. If the reactances are assumed to be proportional to the frequency, the impedance $Z_{bd}''$, for example, is of the form $r_{bd}'' + j2x_{bd}$, where $x_{bd}$ is the leakage reactance of the direct-axis rotor circuit at fundamental frequency. Similarly, $Z_a'''$ is of the form $r_a''' + j3x_a$, where $r_a'''$ is the resistance of the armature at *triple* frequency and $x_a$ is the leakage reactance at *fundamental* frequency. Divide the equation (433) by 3. Let $Z_{bd}''/2 = Z_{bd}$ and $Z_{bq}''/2 = Z_{bq}$ and note, for example, that

$$jx_{\phi d} + \frac{x_{\phi d}^2}{jx_{\phi d} + Z_{bd}} = \frac{jx_{\phi d}Z_{bd}}{jx_{\phi d} + Z_{bd}}$$

Let

$$Z_a + \frac{jx_{\phi d}Z_{bd}}{jx_{\phi d} + Z_{bd}} = Z_d$$

Note that $Z_d$ is the impedance of a short-circuited transformer having primary and secondary leakage impedances of $Z_a$ and $Z_{bd}$ and a magnetizing reactance of $x_{\phi d}$. The impedance $Z_d$

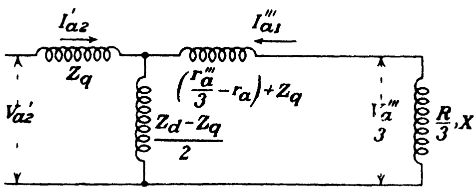



FIG. 153

is the so-called "subtransient impedance" in the direct axis. Also, let

$$Z_a + \frac{jx_{\phi q}Z_{bq}}{jx_{\phi q} + Z_{bq}} = Z_q$$

$$\frac{Z_a'''}{3} + \frac{jx_{\phi q}Z_{bq}}{jx_{\phi q} + Z_{bq}} = Z_q'$$

Note that $Z_q'$ is equal to $Z_q$ except for the difference between $r_a$ and $r_a'''/3$. This is also true of $Z_d'$ and $Z_d$. This impedance $Z_q$ is the so-called "subtransient impedance" in the quadrature axis.

If these changes in notation are made, the two equations (432) and (433) become

$$\left.\begin{aligned} V_{a2}' &= I_{a2}'\frac{Z_d + Z_q}{2} + I_{a1}'''\frac{Z_d - Z_q}{2} \\ \frac{1}{3}V_{a1}''' &= I_{a1}'''\frac{Z_d' + Z_q'}{2} + I_{a2}'\frac{Z_d - Z_q}{2} \end{aligned}\right\} \qquad (434)$$

Equations (434) show that the impedance of the salient-pole synchronous machine to negative sequence can be represented by the equivalent circuit of Fig. 153. It is important to observe that the equations and the equivalent circuit strictly apply to a

phase of the armature only at the moment the phase is directly opposite a field pole. That is, it is only at the moment $\omega t = 0$ that the equations correctly indicate the proper relation between the vectors $I_{a2}'$ and $I_{a1}'''$ which represent the negative-sequence fundamental current and the positive-sequence triple-frequency current. The triple-frequency component of the armature terminal potential is $V_{a1}'''$ and $R + j3X$ is the impedance of the supply network at triple frequency as viewed from the terminals of the armature. Observe that if $Z_d$ equals $Z_q$, impedance of the pillar of the $T$ becomes zero and the circuit reduces to that of the *ideal* synchronous machine shown in Fig. 141. If the resistances of the stator and rotor circuits are neglected, the equivalent circuit of Fig. 153 becomes that of Fig. 154, where $X_d$ and $X_q$ are the reactances of the impedances $Z_d$ and $Z_q$. The equivalent impedance of the power network to which the machine is connected is $(R/3) + jX$. Consider two limiting cases, the first in which this impedance is zero and the second in which it is infinite. In the first case there will be a triple-frequency current in the stator but no triple-frequency potential at the stator terminals. In the second case there will be no triple-frequency current in the stator but a triple-frequency potential will appear at the terminals of the stator. The ratio of the component currents in the first case is

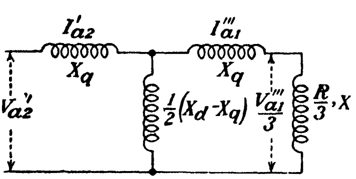



FIG. 154

$$\frac{I_{a1}'''}{I_{a2}'} = \frac{\frac{1}{2}(X_d - X_q)}{X_q + \frac{1}{2}(X_d - X_q)} = \frac{X_d - X_q}{X_d + X_q} \tag{435}$$

If the ratio $X_q/X_d$ is 2, the triple-frequency component of the current is one-third the fundamental current. If the ratio is 3, the triple frequency current is one-half the fundamental current. Similarly, the ratio of the component potentials in the second case is

$$\frac{V_{a1}'''}{3V_{a2}'} = \frac{X_d - X_q}{X_d + X_q} \tag{436}$$

If the ratio $X_q/X_d$ is 2, the triple-frequency component $V_{a1}'''$ of the potential at the stator terminals is equal to the fundamental component. If the ratio is 3, the triple-frequency

component $V_{a1}'''$ is 50 per cent greater than the fundamental component.

The limiting value of the reactance of the machine to negative sequence in the first case is

$$X_q + \frac{\frac{1}{2}(X_d - X_q)X_q}{\frac{1}{2}(X_d - X_q) + X_q} = \frac{2X_dX_q}{X_d + X_q} \tag{437}$$

and in the second case is

$$X_q + \tfrac{1}{2}(X_d - X_q) = \tfrac{1}{2}(X_d + X_q) \tag{438}$$

That is, in the first case the reactance to negative sequence is twice the parallel reactance of $X_d$ and $X_q$, while in the second case it is the arithmetic mean of $X_d$ and $X_q$. If the ratio $X_q/X_d$ is 2, the ratio of the two limiting values of the reactance to negative sequence is 8/9; if $X_q/X_d$ is 3, the ratio of the limiting values is 3/4. If the resistance of the power network is neglected and its reactance equals the geometric mean of $X_d$ and $X_q$, or $X = \sqrt{X_dX_q}$, the resulting reactance to negative sequence is also $\sqrt{X_dX_q}$. Thus, if the reactance of the power network as viewed from the terminals of the armature is less than $\sqrt{X_dX_q}$, the resulting reactance of the machine to negative sequence lies between $2X_dX_q/(X_d + X_q)$ and $\sqrt{X_dX_q}$. It is never as great as $\frac{1}{2}(X_d + X_q)$. It should be noted, however, that unless $X_d$ differs from $X_q$ by more than 25 per cent, the difference between the greatest value and least value of the reactance to negative sequence is less than 1.25 per cent. These estimates are based on the assumption that resistances in the circuit are negligible in comparison with the corresponding reactances.

The *subtransient*[1] reactances, $X_d$ and $X_q$, in the direct and quadrature axes are important constants of a synchronous machine. When the machine has a laminated-field structure without damper winding or other conducting paths except the field winding, in which currents can be induced by the negative-sequence field, these impedances equal the so-called "transient"

[1] PARK, R. H., and B. L. ROBERTSON, Reactance of Synchronous Machines, *Am. Inst. Elec. Eng.*, **47**, 514, 1928.

WRIGHT, S. H., Determination of Synchronous Machine Constants by Test, *Am. Inst. Elec. Eng.*, **50**, 1331, 1931.

reactances in the direct and quadrature axes. When there are no conducting paths on the field structure in the quadrature axis, the transient reactance in the quadrature axis equals the synchronous reactance in this axis. Owing to the currents that are induced in the closed field winding, the transient reactance in the direct axis is less than the synchronous reactance in this axis. It should be noted that the damper winding is much more effective in the direct axis than in the quadrature axis, particularly when the end rings are not continuous from pole to pole. This is discussed in Art. 120, page 351, in connection with the cage winding of an induction motor. It has been suggested that subtransient reactances be indicated by double primes (″) and that transient reactances be indicated by single primes (′).

If the resistances of the armature and rotor circuits are neglected, an important extension of the Method of Symmetrical Components to the theory of single-phase operation can be made.

**144. Line-to-line Short Circuit.**—When two of the line terminals of a salient-pole generator are short-circuited, either directly or through an impedance, both the terminal potential of the armature and the armature current consist of a fundamental and a series of odd-harmonic components. With a line-to-line short circuit on the terminals *b-c*, the following relations between the positive- and negative-sequence components of terminal potential and armature current are true

$$\left.\begin{aligned} V_{a1} &= V_{a2} \\ I_{a1} + I_{a2} &= 0 \end{aligned}\right\} \qquad (439)$$

These relations are true *individually* for the *fundamental* and for *each harmonic component* of potential and current. If *all resistances* are negligible, the relations between the negative-sequence fundamental terminal potential and armature current and the positive-sequence triple-frequency terminal potential and armature current are correctly indicated in the equivalent circuit of Fig. 154, page 440.

If *negative-sequence triple-frequency* potentials are impressed on the armature terminals, fourth-harmonic currents will be induced in the field circuits. These currents will react and produce fifth-harmonic potentials and currents in the armature.

The relations between the negative-sequence triple-frequency potentials and currents and the reflected positive-sequence fifth-harmonic potentials and currents are indicated in the equivalent circuit of Fig. 155. This equivalent circuit is exactly like the equivalent circuit which relates the negative-sequence fundamental potential and current and the positive-sequence triple-frequency potential and current. Note that, in Fig. 155, the negative-sequence triple-frequency potential is divided by 3 and that the positive-sequence fifth-harmonic potential is divided by 5. The same equivalent circuit also relates the negative-sequence fifth-harmonic potential and current and the positive-sequence seventh-harmonic potential and current, and so on ad infinitum. These equivalent circuits indicate the correct relations between the potentials and currents in *any one* of the armature phases at the moment this phase is directly opposite a field pole, page 437. If phase $a$ is chosen, the positive- and negative-sequence components of potential and current of *each frequency* are related as indicated in equations (434). If the equivalent circuits, Fig. 154 and 155 are connected as in Fig. 156 the conditions in equations (439) are satisfied. Thus

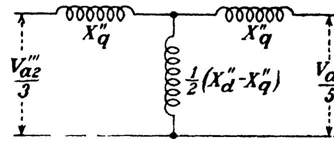



FIG. 155

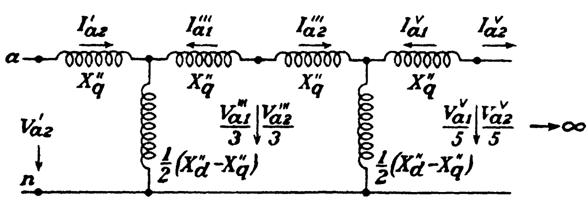



FIG. 156.

the reactance to negative-sequence, $V_{a2}'/I_{a2}'$, is the reactance measured at the terminals $a$-$n$ of this network. This reactance is readily determined by the following procedure. Since there is an infinite number of $T$-component elements in the network, the removal of the first element will not alter the reactance of the remaining elements. Thus in Fig. 156, the reactance between the terminals $a$-$n$ equals the reactance of the remaining elements, $X_2$. That is,

$$X_2 = X_q'' + \frac{(X_q'' + X_2)\dfrac{(X_d'' - X_q'')}{2}}{X_q'' + X_2 + \dfrac{X_d'' - X_q''}{2}}$$

From which,

$$X_2 = \sqrt{X_d'' X_q''} \tag{440}$$

Therefore, the reactance of the salient-pole generator to negative sequence during a line-to-line short circuit is the geometric mean of the subtransient reactances in the direct and quadrature axes.

The positive-sequence component, $I_{a1}'$, of the fundamental current in phase $a$ is

$$I_{a1}' = \frac{E_a}{j(X_d + \sqrt{X_d'' X_q''})} \tag{441}$$

where $E_a$ is the open-circuit generated emf in phase $a$, and $X_d$ is the synchronous reactance in the direct axis. The fundamental component, $I_b'$, of the short-circuit current in phase $b$ is

$$I_b' = \frac{E_{bc}}{j(X_d + \sqrt{X_d'' X_q''})} \tag{442}$$

where $E_{bc}$ is the open-circuit generated emf between the terminals $b$-$c$.

If the generator is short-circuited through a reactance $X$ as indicated in Fig. 157, it can be assumed that one-half this reactance is added to the *leakage* reactance of phase $b$, that the other half is added to the *leakage* reactance of phase $c$, and that the armature terminals are then directly short-circuited. The new values of the synchronous reactance in the direct axis and the subtransient reactances in the direct and quadrature axes are

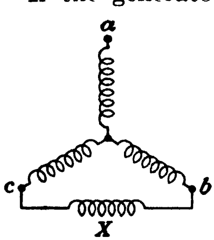


FIG. 157.

$$X_d + \frac{X}{2}, \qquad X_d'' + \frac{X}{2}, \qquad \text{and} \qquad X_q'' + \frac{X}{2}$$

The fundamental component of the short-circuit current in phase $b$ is

$$I_b' = \frac{E_{bc}}{j\left[X_d + \frac{X}{2} + \sqrt{\left(X_d'' + \frac{X}{2}\right)\left(X_q'' + \frac{X}{2}\right)}\right]} \tag{443}$$

We shall now determine the magnitudes and phase relations of the harmonic components of the potential and current. Refer to Fig. 156. Since the currents in parallel branches divide inversely as the impedances of these branches,

$$\frac{I_{a1}'''}{I_{a2}'} = -\frac{\frac{X_d'' - X_q''}{2}}{\frac{X_d'' - X_q''}{2} + X_q'' + X_2}$$

The minus sign appears on the right of the equality sign owing to the assumed directions of $I_{a1}'''$ and $I_{a2}'$ in Fig. 156. Since $X_2 = \sqrt{X_d''X_q''}$, the ratio of the currents readily reduces to

$$\frac{I_{a1}'''}{I_{a2}'} = -\frac{\sqrt{X_d''} - \sqrt{X_q''}}{\sqrt{X_d''} + \sqrt{X_q''}} \tag{444}$$

It also follows that

$$\frac{I_{a2}'''}{I_{a1}'} = -\frac{\sqrt{X_d''} - \sqrt{X_q''}}{\sqrt{X_d''} + \sqrt{X_q''}}$$

Let

$$-\frac{\sqrt{X_d''} - \sqrt{X_q''}}{\sqrt{X_d''} + \sqrt{X_q''}} = a \tag{445}$$ [1]

Since for *each* frequency $k$ the current $I_b^k$ is

$$I_b^k = -j\sqrt{3}I_{a1}^k = +j\sqrt{3}I_{a2}^k$$

it follows that

$$I_b''' = aI_b'$$
$$I_b^{V} = a^2I_b'$$
$$I_b^{VII} = a^3I_b'$$

Therefore, the root-mean-square value, $I_b$, of the short-circuit current is

$$\begin{aligned} I_b &= \sqrt{1 + a^2 + a^4 + a^6 \ldots}\, I_b' \\ &= \sqrt{\frac{1}{1 - a^2}}\, I_b' \\ &= \frac{\sqrt{X_d''} + \sqrt{X_q''}}{2\sqrt[4]{X_d''X_q''}}\, I_b' \end{aligned} \tag{446}$$

[1] See footnotes on pp. 458 and 460.

The phase relations of the harmonic components are determined as follows. Refer to Fig. 156. It will be observed that, owing to the conventions adopted for the currents, the vectors $I_{a2}'$ and $I_{a1}'''$ are in time-phase opposition if the reactance in the pillar of the $T$ is positive. Hence, if $X_d''$ is greater than $X_q''$, the vector $I_{a1}'''$, representing the positive-sequence triple-frequency current, is in time-phase opposition with the vector $I_{a2}'$, representing the negative-sequence fundamental current in phase $a$, *at the moment phase a is directly opposite a field pole.* Since it is generally true that the positive- and negative-sequence components of *each* harmonic frequency of current in phase $a$ are in time opposition, the vectors representing the *positive-sequence* components of *all* frequencies in phase $a$ are in *time-phase at the moment phase a is opposite a field pole.* Hence, since it is generally true that $I_b = -j\sqrt{3}I_{a1}$ for each frequency, the vectors representing the harmonic components of the short-circuit current in $b$-$c$ are in *time-phase at the moment phase a* is opposite a field pole. At this moment, the winding $b$-$c$ is opposite the quadrature axis and the fundamental generated emf, $e_{bc}$, is passing through its maximum value and, since there is no resistance in the circuit, the fundamental current, $i_{bc}$, is passing through its *zero* value. If the *vectors* representing the fundamental and harmonic components are in time-phase at the moment the fundamental is passing through its *zero* value, the fundamental and harmonic components themselves are in time-phase. Therefore, when $X_d''$ is greater than $X_q''$, the harmonic components of the short-circuit current in phase $b$ are in time-phase as indicated in Fig. 158.

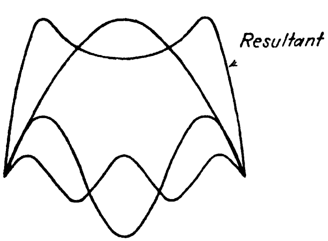



FIG. 158.

Owing to the effect of the conducting paths on the field structure in the direct axis, the subtransient reactance $X_d''$ is usually smaller than the subtransient reactance $X_q''$ in the quadrature axis. If $X_q''$ is greater than $X_d''$, the vector $I_{a1}'''$, representing the positive-sequence triple-frequency current, is in time-phase with the vector $I_{a2}'$, representing the fundamental negative-sequence current in phase $a$, at the moment phase $a$ is directly opposite a field pole. Therefore when $X_q''$ is greater than $X_d''$,

the triple-frequency component of the short-circuit current is in time opposition with the fundamental component. The fifth-harmonic component of the short-circuit current is in time opposition with the triple-frequency component. These phase relations are indicated in Fig. 159. The resulting wave form of the short-circuit current is sharply peaked.[1]

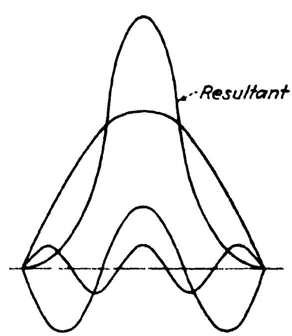



FIG. 159.

On referring to the equivalent network (Fig. 156, page 443) it will be observed that

$$V_{a2}' = jI_{a2}'\sqrt{X_d''X_q''}$$

$$\frac{V_{a2}'''}{3} = jI_{a2}'''\sqrt{X_d''X_q''} = jaI_{a2}'\sqrt{X_d''X_q''}$$

$$\frac{V_{a2}^{V}}{5} = jI_{a2}^{V}\sqrt{X_d''X_q''} = ja^2I_{a2}'\sqrt{X_d''X_q''}$$

The root-mean-square value $V_{a2}$ of the negative-sequence terminal potential of phase $a$ is

$$V_{a2} = jI_{a2}'\sqrt{X_d'' - X_q''}\sqrt{1 + (3a)^2 + (5a^2)^2 + (7a^3)^2 + \cdots} \quad (447)$$

It should be noted that the harmonics are relatively greater in the potential than in the current. Since the terminal potential $V_{ab}$ is three times the negative-sequence component of the terminal potential of phase $a$,

$$V_{ab} = j3I_{a2}'\sqrt{X_d''X_q''}\sqrt{1 + 9a^2 + 25a^4 + 49a^2 + \cdots}$$

Since

$$I_{a2}' = -j\frac{I_b'}{\sqrt{3}}$$

$$= -j\frac{1}{\sqrt{3}}I_b\frac{2\sqrt[4]{X_d''X_q''}}{\sqrt{X_d''} + \sqrt{X_q''}}$$

$$V_{ab} = 2\sqrt{3}I_b\frac{(X_d''X_q'')^{\frac{3}{4}}}{\sqrt{X_d''} + \sqrt{X_q''}}\sqrt{1 + 9a^2 + 25a^4 + 49a^6 + \cdots} \quad (448)$$

[1] DOHERTY, R. E., and C. A. NICKLE, Synchronous Machines IV, *Am. Inst. Elec. Eng.*, **47**, 470, 1928.

where $V_{ab}$ is the root-mean-square potential between the terminals $a$-$b$, and $I_b$ is the root-mean-square current in the short-circuited phase $b$. This relation between the terminal potentials and the short-circuit current can also be written

$$\frac{V_{ab}}{\sqrt{3}I_b} = \sqrt{X_d''X_q''}\frac{\sqrt{1 + 9a^2 + 25a^4 + 49a^6 + \cdots}}{\sqrt{1 + a^2 + a^4 + a^6 + \cdots}} \qquad (449)$$

$$= \sqrt{X_d''X_q''}\sqrt{1 + 8_a^2 + 16_a^4 + 24_a^6 + \cdots}$$

If $X_q''$ is less than $3X_d''$, $a$ is less than 0.27 and

$$V_{ab}/\sqrt{3}I_b = \sqrt{X_d''X_q''}(1 + 4a^2) \text{ approximately.}$$

If $X_q''$ is approximately twice as great as $X_d''$ the ratio $a$ is approximately 0.17 and $V_{ab}/\sqrt{3}I_b$ is approximately 11 per cent greater than the reactance, $\sqrt{X_d''X_q''}$, to negative sequence.

$$\text{If } X_q'' = 2.5X_d'', \qquad \frac{V_{ab}}{\sqrt{3}I_b} = 1.2\sqrt{X_d''X_q''}$$

$$\text{If } X_q'' = 1.5X_d'', \qquad \frac{V_{ab}}{\sqrt{3}I_b} = 1.04\sqrt{X_d''X_q''}$$

With an "ideal" synchronous machine the reactance to negative sequence (neglecting the resistance) can be measured by the single-phase short-circuit method and is $V_{ab}/\sqrt{3}I_b$. (Art. 126, page 382.) However, with a salient-pole machine this method of measuring the reactance to negative sequence gives a value which is 20 per cent too great if $X_q'' = 2.5X_d''$.

**Illustration.**—The following per unit data are for a 30,000 kva salient-pole generator without dampers. Since this generator is without dampers, $X_d'' = X_d'$ and $X_q'' = X_q$.

$$X_d = 1.15 \qquad X_d' = 0.33 \qquad X_q = 0.83$$

Hence by equation (445) $a = 0.226$ and $\sqrt{1 + a^2 + a^4 + \cdots} = 1.026$.

By equations (442) and (446)

$$I_b = \frac{1.026E_{bc}}{j[1.15 + \sqrt{(0.33)(0.83)}]} \text{ rms}$$

If this generator is assumed to have "ideal" characteristics and a reactance to negative-sequence, $X_2 = \sqrt{X_d'X_q}$, the short-circuit current is

$$I_b = \frac{E_{bc}}{j[1.15 + \sqrt{(0.33)(0.83)}]}$$

The assumption of ideal characteristics and a reactance to negative sequence of $\sqrt{X_d'X_q''}$ is justified when computing the line-to-line short-circuit current in this manner.

**145. Line-to-neutral Short Circuit.**—If *all resistances* are negligible, the procedure used in Art. 144 will also determine relations between the potentials and currents in the case of a line-to-neutral short circuit, provided it is assumed that zero-sequence currents produce neither positive- nor negative-sequence potentials (see Art. 140, page 430). If phase $a$ is short-circuited, the following relations are true for each harmonic component of potential and current

$$\left.\begin{aligned} V_{a0} + V_{a1} + V_{a2} &= 0 \\ I_{a0} = I_{a1} &= I_{a2} \end{aligned}\right\} \qquad (450)$$

Before the $T$-component elements of the network can be connected so as to satisfy these equations (450), it is necessary to adopt the opposite convention to that in Fig. 153, for positive values of the positive-sequence potentials and currents. Thus the equations (434), page 439, and the equivalent circuit of Fig. 153, become

$$\left.\begin{aligned} V_{a2}' &= j\left(\frac{X_d'' + X_q''}{2}\right)I_{a2}' + j\left(\frac{X_q'' - X_d''}{2}\right)I_{a1}''' \\ \frac{1}{3}V_{a1}''' &= j\left(\frac{X_d'' + X_q''}{2}\right)I_{a1}''' + j\left(\frac{X_q'' - X_d''}{2}\right)I_{a2}' \end{aligned}\right\} \qquad (451)$$

and the equivalent circuit of Fig. 160. Note that the positive directions of $V_{a1}'''$ and $I_{a1}'''$ have been reversed and that $X_d''$ and $X_q''$ have been interchanged. If the $T$-component elements are connected through the reactance, $X_0$, to zero sequence as indicated in Fig. 161, the conditions required by equations (450) are satisfied. Note that $X_0$ is the reactance to zero sequence at fundamental frequency and therefore that $V_{a0}''' = j3X_0I_{a0}'''$. Thus

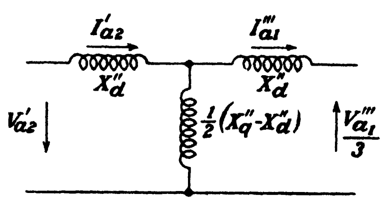



FIG. 160.

$$\frac{V_{a0}'''}{3} = jX_0 I_{a0}'''$$

At fundamental frequency,

$$V_{a0}' + V_{a1}' + V_{a2}' = 0$$

Also,

$$V_{a1}' = -E_a + jX_d I_{a1}'$$

where $E_a$ is the rise in generated emf and $X_d$ is the synchronous

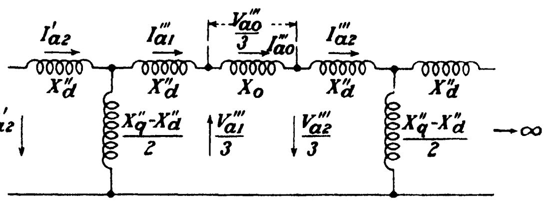



FIG. 161.

reactance in the direct axis. It is assumed that this emf, $E_a$, is sinusoidal. Thus,

$$-E_a + jX_d I_{a1}' + jX_0 I_{a0}' + V_{a2}' = 0$$

This condition is satisfied if the network of Fig. 161 is modified as indicated in Fig. 162.

On referring to Fig. 161 and Fig. 162 it will be observed that

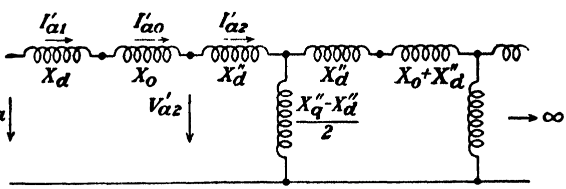



FIG. 162.

each T-component element consists of the three reactances, $\left(\frac{X_0}{2} + X_d''\right)$, $\left(\frac{X_0}{2} + X_d''\right)$ and $\frac{1}{2}(X_q'' - X_d'')$. By following the procedure used in Art. 144, the resulting reactance of the network consisting of an infinite number of these T-component elements can be shown to be

$$\sqrt{\left(\frac{X_0}{2} + X_d''\right)\left(\frac{X_0}{2} + X_q''\right)} \qquad (452)$$

Therefore,

$$I_{a1}' = \frac{E_a}{j\left[X_d + \frac{X_0}{2} + \sqrt{\left(\frac{X_0}{2} + X_d''\right)\left(\frac{X_0}{2} + X_q''\right)}\right]} \tag{453}$$

Since in general,

$$I_{a1}' = \frac{E_a}{j(X_1 + X_0 + X_2)},$$

the reactance to negative-sequence $X_2$ during a line-to-neutral short circuit is (see Fig. 161)

$$X_2 = \sqrt{\left(\frac{X_0}{2} + X_d''\right)\left(\frac{X_0}{2} + X_q''\right)} - \frac{X_0}{2} \tag{454}$$

It is readily shown that [see equation (444)]

$$\frac{I_{a1}'''}{I_{a2}'} = \frac{\sqrt{X_q'' + \frac{X_0}{2}} - \sqrt{X_d'' + \frac{X_0}{2}}}{\sqrt{X_q'' + \frac{X_0}{2}} + \sqrt{X_d'' + \frac{X_0}{2}}} \tag{455}$$

It will be observed that the effect of the reactance to zero sequence is to reduce the relative magnitudes of the harmonics.

**Illustration.**—The following data are for a 30,000-kva salient-pole generator without dampers. Since this generator is without dampers, $X_q'' = X_q$, $X_d = 1.15$, $X_0 = 0.093$, $X_d' = 0.33$, and $X_q = 0.83$.

$$I_{a1}' = \frac{E_a}{j\left[1.15 + \frac{0.093}{2} + \sqrt{\left(\frac{0.093}{2} + 0.33\right)\left(\frac{0.093}{2} + 0.83\right)}\right]}$$
$$= \frac{E_a}{j1.77}$$

The ampere value, $I_{a1}$, of the positive-sequence current is $\frac{1}{\sqrt{1 - a^2}} I_{a1}'$, where $a = \frac{I_{a1}'''}{I_{a2}'}$ [see equation (455)]. Consequently $I_{a1} = \frac{1.02E_a}{j1.77}$. If it is assumed that this generator has "ideal" characteristics and a reactance to negative sequence

$X_2 = \sqrt{X_d'X_q}$, the positive-sequence component of the short-circuit current is

$$I_{a1} = \frac{E_a}{j[1.15 + 0.093 + \sqrt{(0.33)(0.83)}]} = \frac{E_a}{j1.77}$$

In this case the assumption of "ideal" characteristics is justified when computing the line-to-neutral short-circuit current in this manner.

If the armature is short-circuited through a reactance $X$, it can be assumed that this reactance is added to the leakage reactance of the armature and that the armature is then directly short-circuited. The values of the synchronous reactance in the direct axis, the reactance to zero sequence and the subtransient reactances in the direct and quadrature axes become

$$X_d + X \qquad X_0 + X \qquad X_d'' + X \qquad X_q'' + X$$

The positive-sequence component of the fundamental current is

$$I_{a1}' = \frac{E_a}{j\left[X_d + X + \dfrac{X_0 + X}{2} + \sqrt{\left(\dfrac{X_0 + X}{2} + X_d'' + X\right)\left(\dfrac{X_0 + X}{2} + X_q'' + X\right)}\right]} \qquad (456)$$

**146. Measurement of Positive-sequence Constants.**—The resistance of the armature to alternating current is commonly determined from measurements made while the armature is short-circuited. The necessary data can be obtained by measuring the power input to the shaft of the machine or by the retardation method. In either case, the proper allowance is made for the friction, windage, and core losses due to the rotating magnetic field in the air gap. This rotating field is so small at short circuit that the loss due to it is frequently neglected.

The armature leakage reactance[1] $x_a$ and the synchronous reactance $X_d$ in the direct axis are determined by means of the well-known open-circuit and zero power-factor characteristics.

[1] ROGERS, T. A., Test Values of Armature Leakage Reactance, *Am. Inst. Elec. Eng.*, 700, 1935.

MARCH, L. A., and S. B. CRARY, Armature Leakage Reactance of Synchronous Machines, *Am. Inst. Elec. Eng.*, 378, 1935.

Also see footnote on p 334.

The armature leakage reactance has been commonly assumed to equal the Potier reactance, which is that portion of the synchronous reactance in the direct axis that is theoretically independent of the saturation. The armature leakage reactance consists of three parts: the slot leakage, the coil-end leakage, and a part due to the harmonic fluxes in the air gap which are produced by the fundamental armature current and which rotate so as to generate in the armature winding an emf of fundamental frequency. If these harmonic fluxes induce currents in the amortisseur bars on the field poles which react and alter the magnitude of the fluxes, this portion of the armature leakage reactance will depend to some extent upon the amortisseur bars and their arrangement. Furthermore, the magnitude of these harmonic fluxes depends somewhat upon the power factor and the saturation. The slot leakage is reduced if the flux density in the armature teeth becomes abnormally high. Of the three component parts, the coil-end leakage is probably the most nearly constant. The Potier reactance is determined by constructing the well-known Potier triangle. In this construction it is assumed that the triangle is the same for short circuit as it is for normal terminal potential. However, if the Potier triangle is constructed for different terminal potentials, it will be found that both the reactance and the reaction are subject frequently to some variation. Two possible causes of these variations are the effect of the field leakage and the effect produced on the saturation of the main magnetic circuit by the third-harmonic component of the air-gap flux produced by the armature current in a salient pole machine. Although the third-harmonic flux generates a third-harmonic emf in the armature windings this emf does not appear in the *terminal* potential, but, this flux, nevertheless, does increase the air-gap flux and hence increases the saturation of the main magnetic circuit. Thus, a given resultant magnetomotive force produces less sinusoidal air-gap flux under load conditions than it would produce on open circuit, or under load conditions the open-circuit characteristic does not represent the true relation between the emf generated by the air-gap flux and the net magnetomotive force necessary to produce this flux.[1] Since the field leakage depends directly upon the value of the field current, this leakage is greater when the

[1] LYON, W. V., Application of Harmonic Analysis to the Theory of Synchronous Machines, *Am. Inst. Elec. Eng.*, **37**, 1477, 1918.

zero-power-factor characteristic is taken than it is at points on the open-circuit characteristic for the same terminal potential. Thus, the increase in field current above its open-circuit value is not due wholly to the zero-power-factor current in the armature but should be attributed in part to the effect of the increased field leakage. That is to say, the Potier triangle as commonly constructed determines an armature reactance and an armature reaction both of which include effects that do not depend directly upon the value of the armature current. For this reason, it has been suggested that it is best to construct the triangle for as great a saturation as is possible since it is claimed that the Potier reactance is then more nearly equal to the leakage reactance computed from design data On the other hand, it may be said that the Potier triangle as usually determined at normal saturation offers an empirical means for including the effects of increased saturation and increased field leakage. When the Potier reactance has been determined, the unsaturated values of the synchronous reactance $X_d$ and of the magnetizing reactance $X_{\phi d}$ in the direct axis are obtained as in Fig. 139. It is commonly assumed that the magnetizing reactance is due to the *fundamental component* of the air-gap flux distribution produced by the armature current. The unsaturated value of the synchronous reactance $X_q$ in the quadrature axis can be determined by noting when the machine pulls out of step while running as a synchronous motor without field excitation.[1] When a synchronous machine operates without field excitation, the power it develops is $I_d I_q(X_d - X_q)$, where $I_d$ and $I_q$ are the direct and quadrature components of the armature current. For the usual values of armature resistance, the maximum power occurs at a power angle of approximately 45 degrees. The power angle, $\theta$, is the time-phase angle between the terminal potential and the emf that would be generated in the armature on open circuit by the field flux. If the armature resistance $r$ is so small that $r^2$ is negligible in comparison with $X_d X_q$, the expression for the maximum power is

$$\frac{3}{2}V^2\left(\frac{X_d - X_q}{X_d X_q}\right)\left[1 - \frac{r(X_d - X_q)}{X_d X_q}\right] \qquad (457)$$

where $V$ is the terminal potential per phase.

[1] KINGSLEY, C., JR., *Am. Inst. Elec. Eng.*, p. 572, 1935.

At pull-out the armature copper loss is approximately

$$\frac{3V^2}{2}\left(\frac{X_d^2 + X_q^2}{X_d^2X_q^2}\right)r$$

Since the power input $P$ equals the sum of power developed, armature copper loss, and core loss $P_c$,

$$P - P_c = \frac{3V^2}{2}\left(\frac{X_d - X_q + 2r}{X_dX_q}\right)$$

Hence,

$$X_q = \frac{X_d + 2r}{1 + \frac{2X_d}{3V^2}(P - P_c)} \tag{458}$$

As the terminal potential is reduced, it is necessary merely to note the terminal potential $V$ per phase and the power input $P$ at the moment the motor pulls out of step. The motor can be operated either without shaft load or when delivering a few per cent of its rated load. The value of the core loss can be estimated if desired, though it is often so small that it can be neglected.

Both the direct-axis and quadrature-axis synchronous reactances can be measured by the so-called "slip test" provided satisfactory data can be obtained. This is not always possible, owing to the difficulty in operating the machine at a *small* and *constant* slip. If the field poles are well laminated and there are no conducting circuits on them in the quadrature axis, the quadrature-axis synchronous reactance equals the quadrature-axis reactance measured by applying a single-phase potential to the armature winding while the rotor is blocked, as in Art. 147, III.

**147. Measurement of Impedance to Negative Sequence.**[1]—In Arts. 125 and 126, page 380, two methods were suggested for measuring the impedance to negative sequence of an ideal synchronous machine. We have seen that the actual machine differs usually from the ideal in two important respects. Either the field circuits are not of the symmetrical polyphase type or the air gap is not uniform due to the salient-pole construction. Both these conditions produce the same effect on the rotating magnetic fields in the air gap and both modify in the same way the equivalent circuit of the machine to negative sequence. The

[1] THOMAS, W. A., Negative-Sequence Reactance of Synchronous Machines, *Am. Inst. Elec. Eng.*, 1378, 1936.

unsymmetry in the rotor produces a triple-frequency current in the armature winding and in the power network to which the machine is connected. Thus the impedance of the machine to negative sequence must depend to some extent upon the constants of the power network to which the machine is connected. Under actual operating conditions, negative-sequence current flows only when the power network is made unsymmetrical, as by some fault or connected load. The circuits on both sides of the air gap are then unsymmetrical and the method of symmetrical components does not strictly apply because of the continued reflections between the stator and rotor circuits. Therefore, it is necessary to assume either that the rotor circuits are symmetrical or that the power network to which the stator is connected is symmetrical. The first of these assumptions is the simpler as it is equivalent to assuming that the machine is of the "ideal" type, in which case its equivalent circuit depends only upon the constants of the machine and is entirely independent of the power network to which the machine is connected. If the unsymmetry on the rotor side is great enough, it may be desirable to choose the second and probably more accurate assumption. One should be guided in his choice of assumptions by either the values of the design constants or the data obtained from test. Of the several tests suggested for determining the required machine constants, we will consider four, two of which have been discussed already in connection with the ideal synchronous machine.

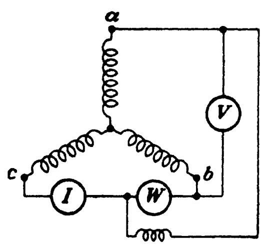


FIG. 163

I. *Synchronous machine operating as generator with two of the line terminals short-circuited through an ammeter and wattmeter as indicated in Fig.* 163. If the rotor circuits are unsymmetrical, both potential and current will consist of a fundamental and a series of odd harmonics of diminishing amplitudes. Thus, it is important in this test to record the wave forms of both potential and current. If these wave forms depart substantially from sinusoidal, the constants computed on the assumption that the potential and current are sinusoidal may be of doubtful value for accurately predicting the performance of the machine when it is connected to a power network.

II. *Negative-sequence potentials applied to the machine while it is being driven at synchronous speed in the positive direction.* In this case, triple-frequency currents will flow in the armature, due either to the effect of the salient poles or to the effect of a single field winding if the air gap is uniform. If the generator supplying the negative-sequence potential does not have symmetrical field circuits, there will be further reflections, resulting in a series of harmonic currents, as in the first test. Owing, however, to the added reactance of this generator, the amplitudes of these harmonics will probably be insignificant. The measurement is made in the customary manner with voltmeter, ammeter, and wattmeters. An oscillograph record of the current supplied to the generator under test should also be obtained in order to determine the relative magnitude of the third-harmonic component. The impedance of the machine to negative sequence is the ratio of fundamental applied potential to fundamental current. Ignoring a 30 per cent third harmonic introduces an error of only 4.5 per cent in the impedance to negative sequence. It is only when the unsymmetry in the rotor circuits is very pronounced that the third-harmonic current needs be considered. In this method, the impedance is measured directly rather than indirectly and, since the currents in the armature and field circuits have their proper frequencies, both the resistances and reactances of these circuits have their normal negative-sequence values. In this test the magnetic circuit is unsaturated and the core loss is relatively small.

III. *Balanced potentials applied to the stator while the rotor is locked.* The rotor is locked in different positions and the maximum and minimum values of the current in one phase of the armature are recorded. The equivalent circuit of Fig. 153 applies in this case also, except that, since the frequencies of the stator and rotor currents are now the same, the resistance of the rotor at fundamental frequency replaces one-half its resistance at twice fundamental frequency. Similarly the resistance of the armature at fundamental frequency replaces one-third its resistance at triple frequency. In test II, the armature current consists of two components. The first component is a negative-sequence current of fundamental frequency and the second is a positive-sequence current of triple frequency. When the rotor is locked, the positive-sequence component becomes

one of fundamental frequency and therefore must be combined vectorially with the negative-sequence component in order to determine the actual current. The phase relation between these two components in any phase is fixed by the position of the field poles with respect to this phase. Thus, by altering the position of the field poles, the current in this phase will assume values between the numerical sum and the numerical difference of the two components. When the resistances in the equivalent circuit are neglected, we have seen that the first or negative-sequence component $I_2$ of the stator current is

$$\frac{V_2(X_d + X_q)}{2X_dX_q}$$

and that the ratio of the two components is

$$\frac{(X_d - X_q)}{(X_d + X_q)}.$$

From this it follows that the greatest and least values of the armature current are $V/X_q$ and $V/X_d$, where $V$ is the applied potential per phase.[1] Thus, in this test we have a convenient method for measuring these important reactance constants. The ratio of the greatest to the least current gives a measure of the unsymmetry of the rotor circuits. The resistance determined by power measurement is probably not the correct value of the resistance to negative sequence, inasmuch as the frequency of the rotor currents and of the rotor flux is fundamental frequency instead of being double fundamental frequency, as is the frequency of the negative-sequence rotor currents and rotor flux variation.

IV. *Single-phase potential applied to the stator while the rotor is locked.* The single-phase potential can be applied as shown in Fig. 164, *a* or *b*. In the following discussion we shall assume that the connection shown in Fig. 164*a* is used. Since the rotor is symmetrical about the direct and quadrature axes, the currents as indicated by the ammeters in phases *b* and *c* are equal when phase *a* is either opposite the poles or opposite the interpolar space.

With a uniform air gap, the circular field produced by symmetrical currents in a symmetrical 3-phase winding is opposite any phase of the winding when the current in that phase is

[1] Note that $a$ [Eq. (445), p. 445] equals $(\sqrt{b} - 1)/(\sqrt{b} + 1)$, where $b$ equals $X_d/X_q$ and $X_d$ and $X_q$ are defined in Fig. 154, p. 440.

passing through its maximum value. At this time, the current in each of the other two phases is one-half its maximum value. Since this momentary distribution of the currents is the same as the distribution of the currents in the synchronous machine being tested and since in this machine the air gap is symmetrical about both direct and quadrature axes, even though it is not uniform, the air-gap flux produced by the armature currents in the synchronous machine being tested is centered on either the direct or quadrature axis, depending upon the position of the field poles with respect to phase *a*. The magnetizing reactances associated with the air-gap flux for the two positions of the field poles with respect to phase *a* are the so-called "direct- and

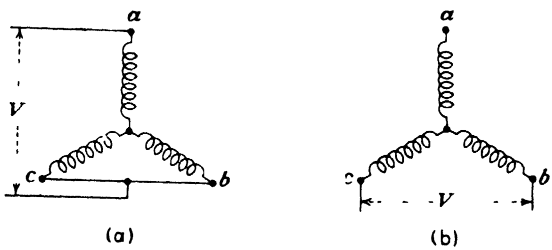



FIG. 164.

quadrature-magnetizing reactances." Their relation to the design constants of a salient pole machine is developed in Arts. 132, 133. When the field poles are opposite phase *a*, currents are induced in the direct-axis rotor circuits but not in the circuits centered on the quadrature axis. The reaction of the rotor currents is then determined solely by the impedance of the rotor circuits in the direct axis, and the potential and current relations are the same as if the air gap were uniform and the rotor circuits were symmetrical. Thus, in this single-phase test, the synchronous machine behaves exactly like an induction motor with its rotor blocked and a single-phase potential impressed on the stator terminals, as in Fig. 164*a*. The positive- and negative-sequence components of this applied single-phase potential $V$ are for phase *a*, $V_1 = V_2 = V/3$ the positive- and negative-sequence components of the current in this phase are $I_1 = I_2 = I/2$. Note that these values of the symmetrical components of potential and current depend upon the symmetry obtained when phase *a* lies either in the direct or the

quadrature axis. The impedance of the machine to negative-sequence is the vector ratio $\frac{V_2}{I_2} = \frac{2}{3}\frac{V}{I}$.

If the connection of Fig. 164*b* is used, the impedance to negative sequence is $\frac{V_2}{I_2} = \frac{1}{2}\frac{V}{I}$. The impedance to negative sequence usually has different values in the direct and quadrature axes.[1] The equivalent circuits which represent the conditions in the two axes are shown in Fig. 165, *a* and *b*. Note that in these equivalent circuits the resistance $r_{bd}$ of the rotor branch is at fundamental frequency. In the equivalent circuit which represents the impedance to negative sequence during the actual operation of the machine in the network, the resistance of the

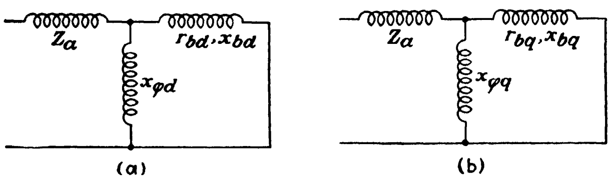



FIG. 165

rotor branch is $r_{bd}''/2$, where $r_{bd}''$ is the resistance of the rotor at twice fundamental frequency. Thus, it is apparent that the single-phase blocked test does not give an accurate method for measuring the resistance of the machine to negative sequence. Since the resistances are usually small in comparison with the corresponding reactances, the reactances to negative sequence, as determined by the single-phase blocked test, closely approximate their true values. When there is little difference between the measured values of the impedances in the direct and quadrature axes, the operating characteristics of the machine under unbalanced conditions approach those of the "ideal" machine.

**148. Zero-sequence Impedance.**—It has been noted that the slot leakage reactance is the same for the three sequences if the winding pitch is unity but that the effect of a fractional pitch is much more pronounced in reducing the reactance to zero sequence than in reducing the reactance to positive or negative sequence. The other parts of the leakage reactance are also reduced by employing a fractional pitch, but for these the effect upon the

[1] Note that *a* [Eq. (445), p. 445] equals $(\sqrt{b} - 1)/(\sqrt{b} + 1)$, where *b* equals the ratio of the currents for the direct- and quadrature-axis positions.

reactance to zero sequence is not always greater than it is upon the reactance to positive or negative sequence. Since fractional-pitch windings are used almost without exception, the leakage reactance to zero sequence is frequently less than the leakage reactance to positive or negative sequence.

The resulting reactance to zero sequence also depends upon the rotating fields in the air gap and on the extent to which these fields are modified by the currents they induce in the rotor circuits. With a cylindrical rotor, zero-sequence current produces a third-harmonic pulsating field in the air gap. If the field winding is uniformly distributed and occupies two-thirds of the rotor surface, no emf will be generated in it by this third-harmonic field since the third-harmonic reduction factor for such a winding is zero. In this case, the resulting reactance to zero sequence equals the sum of the leakage reactance to zero sequence and the magnetizing reactance due to this third-harmonic field. This magnetizing reactance is zero if the armature winding pitch is two-thirds. With salient-pole construction, zero sequence emfs are generated in the armature by rotating fields produced by currents of all three sequences. Of these emfs the largest is the third-harmonic emf due to the action of the positive-sequence current. The next in importance is the emf of fundamental frequency due to the action of the zero-sequence current itself. The least important is the emf of fundamental frequency due to the action of the negative-sequence current. Since all these emfs are directly proportional to the third-harmonic reduction factor, they will vanish when the winding pitch is two-thirds. Only a portion, and in many cases only a small portion, of the third-harmonic fields which generate these zero-sequence emfs link with the field winding and thus little current will be induced in the field winding. Furthermore, the flux produced by this induced current is distributed sinusoidally and therefore does not generate a zero-sequence emf in the armature winding. That is, when the field poles are not equipped with dampers, the resulting reactance to zero sequence very closely equals the sum of the leakage reactance to zero sequence and the magnetizing reactance to zero sequence. When the field poles are equipped with dampers, the currents induced in the dampers produce a third-harmonic field in the air gap which modifies the corresponding field produced by the zero-sequence armature currents. Thus, in the

zero-sequence equivalent circuit, the magnetizing reactance is shunted by the leakage impedance of the damper to zero sequence. That is, the addition of a damper may produce a considerable reduction in the impedance of the machine to zero sequence.

**149. Measurement of Impedance to Zero Sequence.**—Three methods have been suggested for measuring the unsaturated value of this impedance to zero sequence. In the first and second methods the armature windings are connected either in series or parallel (see Art. 127, page 386). The machine should be run at synchronous speed, particularly for the parallel connection. In each case, the impedance per phase to zero sequence is $V_0/I_0$. A wattmeter indicates the total zero-sequence power. The impedance can be separated into its resistance and reactance components in the usual manner. When the rotor is not turning, the current will be sinusoidal if the applied potential is sinusoidal, but neither the resistance nor the reactance will have its correct value, though the error will be small if there is little reaction of the rotor circuits. When the rotor is turning at synchronous speed, the stator current will contain harmonic components due to the unsymmetry in the rotor circuits, and it is desirable to make oscillographic measurements in order to determine the stator-current wave form. If the source of potential has a relatively high reactance, the magnitudes of the harmonic currents will be minimized.

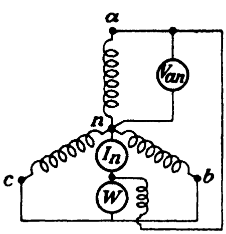


FIG. 166

In the third test, the machine is operated as a synchronous generator with a double line-to-neutral short circuit, as in Fig. 166. The assumed positive directions for fall in potential and current flow are from $a$, $b$, $c$ toward $n$. It is important to note that the following relations are true only when the rotor circuits are symmetrical. Since phases $b$ and $c$ are short-circuited, it is evident that if the impedance of the wattmeter and ammeter are neglected,

$$V_{a1} = V_{a2} = V_{a0} = \frac{V_{an}}{3}$$

where $V_{an}$ is the potential across the open phase. The current

flowing from the neutral is $3I_0$. Thus, the impedance $Z_0$ to zero sequence, or $V_0/I_0$, equals $V_{an}/I_n$. Note that the wattmeter indicates nine times the zero-sequence power per phase. The correction for the effect of the *combined impedance* $Z$ of the wattmeter and ammeter can be made if desired. Since $V_a = V_{an}$, $V_b = -3I_0Z$, and $V_c = -3I_0Z$, it follows that

$$V_0 = \frac{V_{an}}{3} - 2I_0Z$$

whence

$$Z_0 + 2Z = \frac{V_{an}}{I_n}$$

That is, the correction consists in subtracting twice the combined impedance of the two instruments from the uncorrected impedance.

From the foregoing discussion, it is evident that oscillographic measurements form an essential part of any thorough investigation of the impedances of a synchronous machine to zero-sequence and to negative sequence.

**150. Effect of Saturation on Reactance to Positive Sequence.**— The effects of saturation in a synchronous machine, even when the machine is operating under balanced load conditions, are difficult to determine from characteristic data taken at no load. Saturation may occur in the armature core and the armature teeth, in the field poles and yoke of a salient pole machine, and in different parts of the cylindrical rotor of a steam-turbine-driven alternator. Whether all parts or only some of the parts of the magnetic circuit are saturated is determined by the particular load condition under which the alternator is operating.

Consider a machine having a cylindrical rotor. The magnetomotive force due to the armature current is then independent of the power factor at which the machine operates. The simplest assumption that can be made in regard to the effects of saturation is that saturation occurs only in the main magnetic circuit which is mutual to both field and armature magnetomotive forces. There is no saturation in those parts of the magnetic circuit which are acted upon by either the field current or the armature current independently. Thus, for example, it is assumed that the field leakage flux has no effect upon the saturation of the machine. Since the air-gap flux is the only flux that is mutual

to both the field and armature circuits, it is assumed that the saturation of the machine is determined solely by the air-gap flux, which in turn is determined by the combined magnetomotive forces of the field and armature currents. It is further assumed that the distribution of the flux in the air gap is sinusoidal, except for such harmonics due to armature current as generate emfs of fundamental frequency in the armature winding. A typical open-circuit characteristic (occ), together with a zero-power factor characteristic (zpc) for an armature current $I_a$ are shown in Fig. 167. Consider a point on the zero-power-factor characteristic at which the armature terminal potential (per phase) is $V$ and the field current is $I_F$. Assuming that the leakage reactance of the armature is $x_a$, the emf $E_{ag}$ generated in the armature by the resultant air-gap flux is $E_{ag} = V + I_a x_a$. This emf, $E_{ag}$, determines the saturation at which the machine is operating. If this saturate condition of the mutual magnetic circuit were maintained *constant*, the armature emf would rise to the value $E_F$, as indicated in Fig. 167, when the armature circuit is opened and the field current is maintained constant. The rise in armature emf divided by the armature current, $I_a$, is the saturated value of the synchronous reactance $x_s$ for the saturation determined by $E_{ag}$. That is,

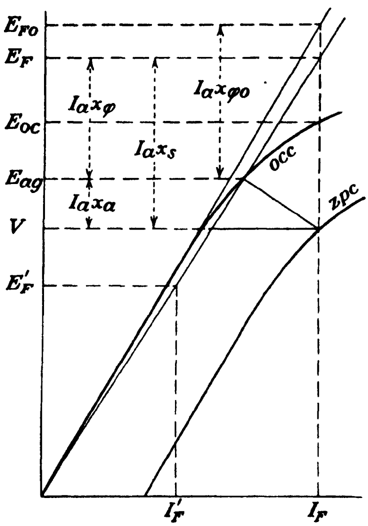


FIG. 167.

$$x_s = \frac{E_F - V}{I_a}. \tag{459}$$

Furthermore, for the value of the saturation determined by the air-gap emf $E_{ag}$ (Fig. 167), the excitation component emf due to the field current is *proportional* to the field current. That is, a field current $I_F'$ will produce a component emf $E_F'$. Therefore, for a *given* value of saturation determined by a *given* air-gap emf, $E_{ag}$, the machine may be treated as if it were oper-

ating in the unsaturated state, provided the synchronous reactance is determined by equation (459) and provided the component emf due to the field current is assumed to be proportional to the field current, the constant of proportionality being equal to the ratio of the given air-gap emf to the field current producing it on open circuit. The foregoing analysis is the same as the "general method" of analyzing an alternator having a cylindrical rotor, except that a magnetizing reactance $x_\phi$ replaces the armature magnetomotive force $A$.[1] The armature magnetomotive force

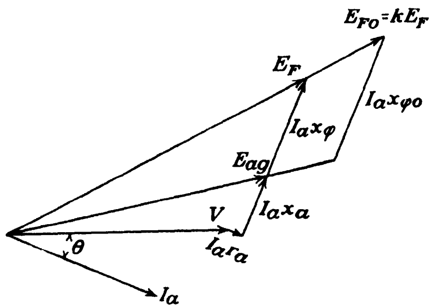



FIG. 168.

produces a fundamental component of the air-gap flux which generates an emf in the armature winding. The magnetizing reactance equals the ratio of this emf to the armature current. Refer to Fig. 167. The saturated value of the magnetizing reactance $x_\phi$ is $(E_F - E_{ag})/I_a$.

The synchronous reactance has been illogically defined as the rise in potential $(E_{0c} - V)$ to the point on the open-circuit characteristic divided by the armature current, or

$$x_s = \frac{(E_{0c} - V)}{I_a}$$

(see Fig. 167). The use of this value of the synchronous reactance is properly considered an empiricism, since this conception of synchronous reactance is not founded on sound theory.

The vector diagram applying to a synchronous generator when operating under balanced conditions and delivering a current $I_a$ per phase at a terminal potential $V$ per phase and a power factor $\cos\theta$ is in Fig. 168. The emf generated by the air-gap flux is

[1] See LAWRENCE, R. R., "Principles of Alternating Current Machinery," p. 84, 2d ed., 1920.

$E_{ag}$, the saturated value of the magnetizing reactance corresponding to a saturation determined by $E_{ag}$ is $x_\phi$, and the component emf due to the field current is $E_F$. The relation between the emf $E_F$ and the field current $I_F$ is determined by the straight line drawn through the origin and the point on the open-circuit characteristic for which the emf is $E_{ag}$. If the machine were without saturation, the emf $E_{F0}$, due to the field current $I_F$, would be determined by the air-gap line. Accordingly $E_{F0} = kE_F$, where $k$ is the saturation factor. The value of the saturation factor, $k$, is defined as the ratio of the armature emf on the air-gap line to the armature emf on the open-circuit characteristic, both for the same value of the field current. If the machine were without saturation the value of the magnetizing reactance, $x_{\phi 0}$, would be determined from short-circuit data. Thus, if the unsaturated value of the synchronous reactance, as determined at short circuit, is $x_{s0}$, the unsaturated value of the magnetizing reactance is $(x_{s0} - x_a)$. Just as $E_{F0} = kE_F$, so $x_{\phi 0} = kx_\phi$. The following vector relation can be written

$$V + I_a\left(z + j\frac{x_{\phi 0}}{k}\right) = \frac{E_{F0}}{k} \qquad (460)$$

Of this equation it should be noted that the leakage impedance, $z_a$, is (assumed) constant, the magnetizing reactance, $x_{\phi 0}$, is constant, and the emf, $E_{F0}$, is directly proportional to the field current, the constant of proportionality being fixed by the air-gap line. The saturation factor, $k$, alone is a variable parameter of the machine. The technique of using equation (460) is described by C. Kingsley, Jr., for the condition of *balanced* operation.[1] Under *unbalanced* operation occasioned by short circuit, the air-gap flux is often less than the critical value determined by the point $e$, Fig. 139, page 377, and the machine operates in an unsaturated state. If, under any unbalanced condition of operation, a solution using the unsaturated values of the machine constants shows that the air-gap emf is greater than the critical value, the saturation factor, $k$, can be determined and a corrected solution can be obtained by applying the principles indicated in equation (460).

[1] KINGSLEY, C., JR., Saturated Synchronous Reactance, *Am. Inst. Elec. Eng.*, **54**, 300, 1935.

Owing to field leakage and to harmonic components in the air-gap flux, the relation between the air-gap flux on open circuit and the field current[1] is not the same as under load conditions. If deemed necessary, the effects of field leakage and of harmonics can be estimated and a corrected curve can be drawn representing the relation between the air-gap flux under load conditions and the field current.[1] It is doubtful whether this refinement would ever be considered necessary in a problem involving unsymmetrical short circuits or unsymmetrical load conditions.

In salient-pole machines, saturation affects differently the magnetizing reactances in the direct and quadrature axes. As an approximation it may be assumed that saturation affects *only* the direct-axis magnetizing reactance, or it may be assumed that saturation affects both the direct- and quadrature-axis magnetizing reactances in the same proportion. There is no generally accepted method of determining the effect of saturation on these two magnetizing reactances. In problems involving unbalanced short circuits, the machines usually operate in an unsaturated condition and the unsaturated values of the machine constants should be used.

**151. Effect of Saturation on Impedance to Negative Sequence.** An approximate method for estimating the effect of saturation upon the magnetizing reactance was discussed in Art. 115, page 335. If the magnetizing reactance in the negative-sequence equivalent circuit is reduced by saturation, the resulting impedance to negative sequence will also be reduced, though not in the same proportion. For example, if the armature leakage reactance is 0.10, the unsaturated magnetizing reactance 0.80, and the field leakage reactance 0.20, the unsaturated value of the reactance to negative sequence is $0.10 + \frac{(0.8)(0.2)}{0.8 + 0.2}$, or 0.26, provided all the losses are ignored. If, owing to saturation, the magnetizing reactance is reduced to one-fourth its unsaturated value, the saturated value of the reactance to negative sequence becomes 0.20, or, in this case, a reduction of 75 per cent in the magnetizing reactance reduces the reactance to negative sequence by only 23 per cent.

[1] Under load conditions the *field current* is equivalent of the resultant mmf due to the actual field and armature currents.

The effect of saturation upon the impedance to negative sequence can be studied if two preferably identical machines $A$ and $B$, electrically connected in series as in Fig. 169, are mechanically coupled to the same source of power so their generated

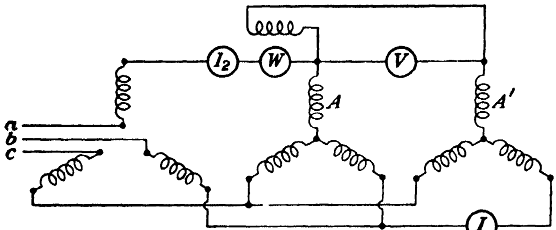



FIG. 169.

emfs are in time-phase. Their excitations can then be adjusted so there is no potential at the terminals $a$, $b$, $c$. If negative-sequence emfs are applied at these terminals, negative-sequence currents will flow in both machines. It is thus possible to adjust

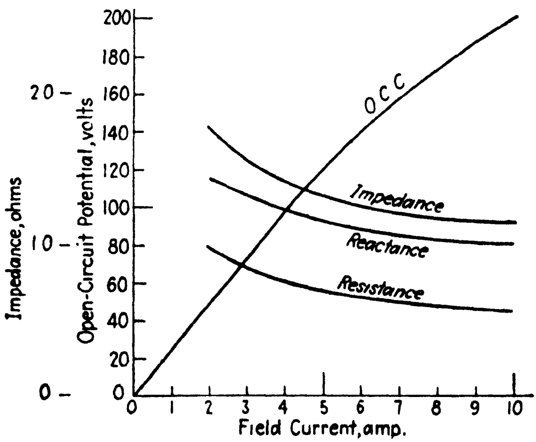



FIG. 170

independently both the saturation and the magnitude of the negative-sequence current The saturation is determined by the excitations of the generators $A$ and $B$ and the value of negative-sequence current is controlled by adjusting the balanced potentials of the source of power. If the machines are identical, it can reasonably be assumed that their impedances to negative sequence are each equal to one-half the impedance measured at

the terminals *a*, *b*, *c*. The negative-sequence potential at the terminals of the generator *A* can be measured directly by the voltmeter *V* if the excitation of the third generator *A′* is adjusted so that no current flows in the ammeter *I* (see Art. 85, page 235). Since the potential indicated by the voltmeter *V* is three times the negative-sequence potential of phase *a*, it is suitable for use in the measurement of the negative-sequence power. Thus, if the potential coil of the wattmeter *W* is connected in parallel with the voltmeter *V*, the wattmeter indicates the total negative-sequence power supplied electrically to generator *A*. The variation in the impedance to negative sequence for different values of the negative-sequence current and for different field currents is indicated in Fig. 170. Both the resistance and reactance to negative sequence show similar characteristics. Since these data were taken on a 5-kw., 230-volt, 60-cycle laboratory generator, it would be unwise to draw any general conclusions in regard to the variation of the negative-sequence constants of large power generators, other than to note the possibility of such variations and the need for investigating their magnitude.

## CHAPTER X

# POWER NETWORKS

Fortescue was led to his development of the method of symmetrical components by studying the behavior of an induction machine when operating under unbalanced conditions. The application of the method, however, has been largely directed toward the solution of the conditions that arise when unsymmetrical faults occur on otherwise symmetrical power networks. The component parts of a power network are the transmission lines, the transformers, the connected synchronous and induction machines, and whatever combination of single-phase loads may exist. In the preceding chapters the object has been to analyze the behavior of these component parts and to consider how we can measure or calculate the necessary constants which relate the sequence components of potential and current when the component parts of the network are subjected to unsymmetrical operation. Once the characteristic constants of the component parts of a power network are known, it should be possible to devise a method by which the behavior of the network as a whole can be determined. The particular problem in which we shall interest ourselves is the determination of the steady conditions that result from the application of one or more unsymmetrical faults to a symmetrical 3-phase network. We shall consider also the more general problem in which unsymmetrical groupings of single-phase loads are added to the network Although the method of symmetrical components has been extended to become a powerful tool in the solution of transient problems, such as arise in stability studies and in the determination of the transient currents occasioned by symmetrical and unsymmetrical faults, we shall limit our discussion to a consideration of steady state conditions only. Thus, it will be assumed the speed of each rotating machine is maintained constant. That is, we shall assume there is no variation in either the phase or magnitude of the excitation emf of any synchronous

machine and also that the impedances of the induction machines both to positive- and negative-sequence current are constant.

In every case it will be assumed that the synchronous and induction machines are "ideal" and that the transformer groupings which form a part of the network are symmetrical. Also, it will be assumed that the transmission lines are symmetrically transposed so that they too form symmetrical connecting links in the network. When combinations of single-phase loads are connected to and considered a part of the network, it must be assumed that they are symmetrically arranged and have fixed impedances. The full significance of these assumptions is discussed in Art. 157, page 482.

**152. Effect of Unsymmetrical Fault.**—We shall now consider the effect of an unsymmetrical fault on a network consisting of

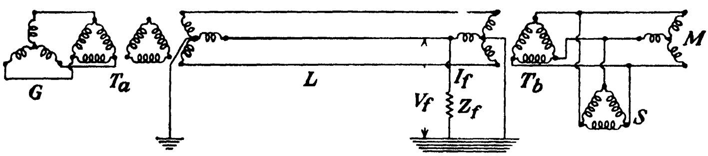



FIG. 171.

the simple combination of component parts in Fig. 171. A synchronous generator $G$ supplies power through a bank of 3-winding transformers $T_a$ to a 3-phase transmission line $L$. At the other end of the line is a bank of two-winding transformers $T_b$, to whose secondaries are connected a 3-phase induction motor $M$ and a balanced arrangement of single-phase loads $S$. A line-to-ground fault, having an impedance $Z_f$, occurs on the high-tension side of the transformers $T_b$ as indicated in Fig. 171.

*Relations between the Line Potentials at the Fault and the Fault Current.*—Since current flows into the fault from *one* phase only, the symmetrical components of the fault current, $I_f$, are each $I_f/3$. That is

$$I_{f0} = I_{f1} = I_{f2} = \tfrac{1}{3}I_f \tag{461}$$

The line-to-ground fall in potential $V_f$ across the fault impedance is $I_fZ_f$. If the symmetrical components of this potential are $V_{f0}$, $V_{f1}$, and $V_{f2}$,

$$V_{f0} + V_{f1} + V_{f2} = I_fZ_f \tag{462}$$

or

$$V_{f0} + V_{f1} + V_{f2} = 3I_{f1}Z_f \qquad (463)$$

*Relation between the Symmetrical Components of Potential and Current in the Generator, Transformer, Line, and Load.*—The distribution of potential and current in this network is determined by two general principles:

First: The two Kirchhoff laws governing the distribution of potential and current can be applied to the network for the zero, positive, and negative sequences separately (Art. 3, page 35).

Second: Since the network is symmetrical, current of one sequence causes a potential drop for that sequence alone. In all the following equations it is assumed that all the quantities are referred to the high-tension line-to-neutral potential. It is best to write the equations for the phase on which the fault occurs.

*Positive-sequence Equations.*—The potential equation which applies to the circuit at the left of the fault is

$$E_G - I_{a1}(Z_{G1} + Z_{Ta} + Z_L) = V_{f1} \qquad (464)$$

where $E_G$ = rise in excitation emf from neutral to line
$I_{a1}$ = positive-sequence current in the generator and line in the same direction (toward the fault)
$Z_{G1}$ = synchronous impedance of generator
$Z_{Ta}$ = impedance of transformer bank $T_a$ to positive sequence (see Art. 80, page 208)
$Z_L$ = impedance of transmission line to positive sequence (See Art. 45, page 115)
$V_{f1}$ = positive-sequence rise in potential from neutral to line at the fault

Since the rise in potential from neutral to line is the same as the *fall* in potential from *line to neutral* at the fault, $V_{f1}$ is the same potential that is represented by $V_{f1}$ in equation (462).

Equation (464) can be abbreviated to

$$E_G - I_{a1}Z_{a1} = V_{f1} \qquad (465)$$

The potential equation which applies to the circuit at the right of the fault is

$$0 - I_{b1}\left(Z_{Tb} + \frac{Z_{M1}Z_S}{Z_{M1} + Z_S}\right) = V_{f1} \qquad (466)$$

where $I_{b1}$ = positive-sequence current toward the fault
$Z_{Tb}$ = impedance of transformer bank $T_b$ to positive sequence (see Art. 75, page 198)
$Z_{M1}$ = impedance of induction motor to positive sequence (page 229)
$Z_S$ = impedance of combination of single-phase loads. It is the same to positive as to negative sequence (Art. 24, page 89)

Since the two loads are in parallel their combined impedance is $Z_{M1}Z_S/(Z_{M1} + Z_S)$. There is no excitation emf acting in the circuit at the right of the fault. Equation (466) can be abbreviated to

$$0 - I_{b1}Z_{b1} = V_{f1} \tag{467}$$

*Negative-sequence Equations.*—The potential equation which applies to the circuit at the left of the fault is

$$0 - I_{a2}(Z_{G2} + Z_{Ta} + Z_L) = V_{f2} \tag{468}$$

where $I_{a2}$ = negative-sequence current in generator and line toward the fault
$Z_{G2}$ = impedance of generator to negative sequence (Art. 125, page 380)
$Z_{Ta}$ = impedance of transformer bank $T_a$ to negative sequence. This equals the impedance to positive sequence (Art. 80, page 208)
$Z_L$ = impedance of transmission line to negative sequence. This equals the impedance to positive sequence (Art. 45, page 115)
$V_{f2}$ = negative-sequence fall in potential from line to neutral at the fault.

Equation (468) is similar to equation (464) for positive sequence except that there is no negative-sequence excitation emf in the generator. The equation which applies to the circuit at the right of the fault is

$$0 - I_{b2}\left(Z_{Tb} + \frac{Z_{M2}Z_S}{Z_{M2} + Z_S}\right) = V_{f2} \tag{469}$$

where $I_{b2}$ = negative-sequence current toward the fault
$Z_{Tb}$ = impedance of transformer bank $T_b$ to negative sequence. This equals the impedance to positive sequence

$Z_{M2}$ = impedance of induction motor to negative sequence (page 229)

$Z_S$ = impedance of combination of single-phase loads (Art. 24, page 89)

*Zero-sequence Equations.*—The equation which applies to the circuit at the left of the fault is

$$0 - I_{a0}(Z_{Ta0} + Z_{L0}) = V_{f0} \tag{470}$$

where $I_{a0}$ = zero-sequence current toward the fault

$Z_{Ta0}$ = impedance of transformer bank $T_a$ to zero sequence (Art. 80, page 208)

$Z_{L0}$ = impedance of transmission line to zero sequence (Art. 52, page 130)

$V_{f0}$ = zero-sequence fall in potential from line to neutral at the fault

There is no zero-sequence excitation emf acting in the generator. Furthermore, there is no zero-sequence current in the generator, since not only is the neutral of the generator not grounded but its terminals are connected to a bank of Δ-connected transformers. The equation which applies to the circuit at the right of the fault is

$$0 - I_{b0}Z_{Tb} = V_{f0} \tag{471}$$

where $I_{b0}$ = zero-sequence current toward the fault

$Z_{Tb}$ = impedance of transformer bank $T_b$ to zero sequence. This equals the impedance to either positive or negative sequence (Art. 75, page 198)

$V_{f0}$ = zero-sequence fall in potential from line to neutral at the fault

If equations (468) and (469) are combined,

$$I_{a2} + I_{b2} = -V_{f2}\left(\frac{1}{Z_{G2} + Z_{Ta} + Z_L} + \frac{1}{Z_{Tb} + \dfrac{Z_{M2}Z_S}{Z_{M2} + Z_S}}\right)$$

Since $(I_{a2} + I_{b2})$ is the negative-sequence component $I_{f2}$ of the fault current, this equation can be written

$$I_{f2} = -\frac{V_{f2}}{Z_{N2}} \tag{472}$$

where $Z_{N2}$ is the parallel combination of $(Z_{G2} + Z_{Ta} + Z_L)$

and $\left(Z_{Tb} + \frac{Z_{M2}Z_S}{Z_{M2} + Z_S}\right)$, or $Z_{N2}$ is the impedance of the entire network to negative sequence looking into the network from the point at which the fault occurs. Similarly, if equations (470) and (471) are combined,

$$I_{a0} + I_{b0} = -V_{f0}\left(\frac{1}{Z_{Ta0} + Z_{l0}} + \frac{1}{Z_{Tb}}\right)$$

or

$$I_{f0} = -\frac{V_{f0}}{Z_{N0}} \tag{473}$$

where $Z_{N0}$ is the impedance of the entire network to zero sequence looking into the network from the point at which the fault occurs.

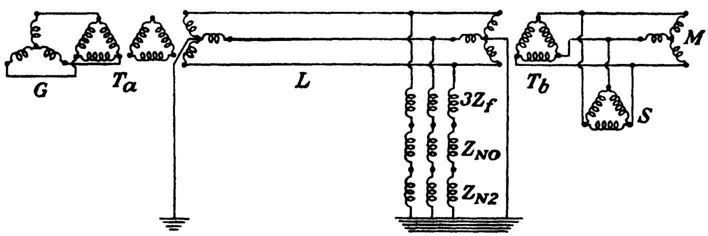



FIG. 172

Equations (461) and (463) determine the relations between the symmetrical components of the fault current and the symmetrical components of the line-to-neutral potential at the fault. Equations (472) and (473) determine the ratio of the negative-sequence components of this potential and current and also the ratio of their zero-sequence components in terms of the impedances of the network to negative sequence and to zero sequence. From these five equations it is possible to determine the ratio of the positive-sequence component of the line-to-neutral potential at the fault to the positive-sequence component of the fault current. For example, if $I_{f2}$, $I_{f0}$, $V_{f2}$, $V_{f0}$ are eliminated from these five equations, we have, for the type of fault considered

$$I_{f1}(Z_{N0} + Z_{N2} + 3Z_f) = V_{f1} \tag{474}$$

Equation (474) together with the positive-sequence equations (465) and (467), which apply to the network, show that the distribution of the positive-sequence currents is the same as the

distribution of the symmetrical currents which would flow if the fault were replaced by a *balanced* load having an impedance $(Z_{N0} + Z_{N2} + 3Z_f)$ per phase. The equivalent circuit is shown in Fig. 172. When the positive-sequence currents have been determined, the negative- and zero-sequence components can be obtained by means of the equations that have been developed. We shall illustrate this procedure by a numerical example, for which the following data are given:

Assume the per unit values of the impedances of the different branches to positive, negative, and zero sequence are

| | Positive sequence | Negative sequence | Zero sequence |
|---|---|---|---|
| $G$ | $j0.70$ | $j0.25$ | |
| $T_a$ | $j0.07$ | $j0.07$ | $j0.06$ |
| $L$ | $j0.15$ | $j0.15$ | $j0.35$ |
| $T_b$ | $j0.08$ | $j0.08$ | $j0.08$ |
| $M$ | $1.8 + j0.8$ | $j0.4$ | |
| $S$ | $1.8 + j0$ | $1.8 + j0$ | |

Assume the excitation emf, $E_G$, of the generator adjusted so that the potential at the load is 1.0 per unit before the fault occurs. Assume the impedances of the load are the same before and after the fault.

The impedance of the load to positive sequence is $0.94 + j0.19$. The impedance of the load to negative sequence is $0.085 + j0.38$. The excitation emf, $E_G$, of the generator is

$$E_G = \frac{j0.70 + j0.07 + j0.15 + j0.08 + 0.94 + j0.19}{0.94 + j0.19} 1.0$$

$$= \frac{0.94 + j1.19}{0.94 + j0.19} = 1.11 + j0.94$$

$$\frac{1}{Z_{N2}} = \frac{1}{j0.25 + j0.07 + j0.15} + \frac{1}{j0.08 + 0.085 + j0.38}$$

$$Z_{N2} = 0.022 + j0.23$$

$$\frac{1}{Z_{N0}} = \frac{1}{j0.06 + j0.35} + \frac{1}{j0.08}$$

$$Z_{N0} = j0.067$$

Assume

$$Z_f = 0.01 + j0$$

$$Z_{N0} + Z_{N2} + 3Z_f = 0.052 + j0.30$$

This impedance should be combined in parallel with the impedance $0.94 + j0.19 + j0.08$, giving an impedance $0.11 + j0.24$. Hence the positive-sequence current, $I_{a1}$, delivered by the generator during the time of the fault is

$$
\begin{aligned}
I_{a1} &= \frac{1.11 + j0.94}{j0.70 + j0.07 + j0\ 15 + 0.11 + j0.24} \\
&= 0\ 89 - j0\ 88
\end{aligned}
$$

By equation (465)

$$
\begin{aligned}
V_{f1} &= (1.11 + j0.94) - (0\ 89 - j0.88)(j0.92) \\
&= 0.30 + j0.12
\end{aligned}
$$

By equation (466)

$$
\begin{aligned}
I_{b1} &= \frac{-(0.30 + j0.12)}{0.94 + j0.27} \\
&= -0.32 - j0\ 033
\end{aligned}
$$

Hence

$$
\begin{aligned}
I_{f1} &= 0.92 - j0.77 - 0.32 - j0.033 \\
&= 0.60 - j0.80
\end{aligned}
$$

Therefore the fault current, $I_f$, is

$$
I_f = 1.80 - j2.40
$$

By equations (461) and (472)

$$
\begin{aligned}
V_{f2} &= -(0.60 - j0.80)(0.022 + j0.23) \\
&= -0.20 - j0.12
\end{aligned}
$$

By equation (468)

$$
\begin{aligned}
I_{a2} &= \frac{0.20 + j0.12}{j0.25 + j0.07 + j0.15} \\
&= 0.26 - j0.43
\end{aligned}
$$

The positive- and negative-sequence currents in the corresponding phase of the generator are (see Art. 75, page 198)

$$
\begin{aligned}
I_{d1} &= -0.88 - j0.89 \\
I_{d2} &= 0.43 + j0.26 \\
I_d &= 0.45 - j0.43
\end{aligned}
$$

In the next phase

$$\begin{aligned} I_e &= \alpha^2 I_{d1} + \alpha I_{d2} \\ &= -0.5(I_{d1} + I_{d2}) - j0.866(I_{d1} - I_{d2}) \\ &= -1.22 + j1.35 \end{aligned}$$

**153. General Method of Analysis.**—In the general analysis of the effect produced by the addition of one or more unsymmetrical faults or circuits to a symmetrical power network, there are two distinct groups of relations that must be determined. The first group consists of the relations that exist among the symmetrical components of the potentials at the fault and of the currents flowing into the fault. These relations depend entirely upon the character of the fault or upon the constants of the unsymmetrical circuit connected to the network. The second group consists of the relations that exist among the symmetrical components of the potentials and currents throughout the symmetrical network. These relations depend entirely upon the character and arrangement of the component parts of the symmetrical network. The combination of these two groups of relations is sufficient to determine the magnitudes of all the symmetrical components of the potentials and currents throughout the network.

**154. Types of Fault and Their Effect upon the Symmetrical Components of Potential and Current.**—There are two general types of unsymmetrical transmission-line faults: the shunt, in which the fault occurs between the line conductors or between a line conductor and earth; the series, in which the fault produces an unsymmetry in the series impedance of the transmission circuit.

**155. Shunt Faults.**—An unsymmetrical shunt fault may occur

(I) between two lines,

(II) between one line and ground,

(III) between two lines and ground,

(IV) between one line and ground, and simultaneously between the other two lines,

When the impedance of the fault is zero, the general relations among the symmetrical components of the line-to-ground potentials at the fault and of the currents supplied to the fault are the same as those developed in Chap. IX applying to a synchronous generator that is unsymmetrically short-circuited.

It will be observed that each type of fault imposes as many conditions upon the fault-currents and the potentials at the fault as there are symmetrical components of the fault current. In all the following equations, the fault currents are $I_a$, $I_b$, $I_c$ and the line-to-neutral potentials at the fault are $V_a$, $V_b$, $V_c$ as in Fig. 173. The positive directions of these currents and potentials are indicated by arrows.

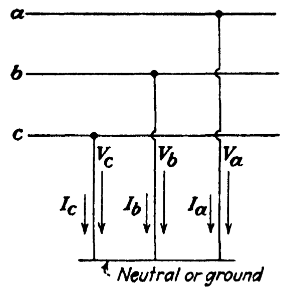


FIG. 173

I. *Line-to-line Fault between Conductors b and c.*—There is no zero-sequence current.

Since $I_a = 0$,

$$I_{a1} + I_{a2} = 0 \tag{1}$$

If the impedance of the fault is zero, $V_{bc} = 0$. That is,

$$-j\sqrt{3}V_{a1} + j\sqrt{3}V_{a2} = 0$$

or

$$V_{a1} = V_{a2} \tag{2}$$

If the impedance of the fault is $Z$, the terminal potential

$$V_{bc} = I_b Z$$

and the relation (2) becomes

$$-j\sqrt{3}V_{a1} + j\sqrt{3}V_{a2} = -j\sqrt{3}I_{a1}Z$$

or

$$V_{a1} - V_{a2} = I_{a1}Z \tag{2}$$

II. *Line-to-ground Fault on Conductor a.*—Since the fault currents $I_b$ and $I_c$ are both zero,

$$I_{a0} = I_{a1} = I_{a2} = I_a/3 \qquad (1), (2)$$

If the impedance of the fault is zero, $V_a = 0$ and

$$V_{a0} + V_{a1} + V_{a2} = 0 \tag{3}$$

If the impedance of the fault is $Z$, the terminal potential $V_a = I_a Z$ and

$$V_{a0} + V_{a1} + V_{a2} = 3I_{a0}Z \tag{3}$$

III. *Line-to-ground Fault on Conductors b and c Simultaneously.* Since $I_a = 0$,

$$I_{a0} + I_{a1} + I_{a2} = 0 \tag{1}$$

If the impedance of each fault is zero, both $V_b$ and $V_c$ are zero and

$$V_{a0} = V_{a1} = V_{a2} = \frac{V_a}{3} \qquad (2),\ (3)$$

If the impedance of each fault is $Z$, the terminal potentials are

$$V_b = I_b Z \qquad V_c = I_c Z$$

If these potentials and currents are expressed in terms of the symmetrical components of the potential $V_a$ and of the current $I_a$,

$$\begin{aligned} V_{a0} + \alpha^{-1}V_{a1} + \alpha^{-2}V_{a2} &= Z(I_{a0} + \alpha^{-1}I_{a1} + \alpha^{-2}I_{a2}) \\ V_{a0} + \alpha V_{a1} + \alpha^2 V_{a2} &= Z(I_{a0} + \alpha I_{a1} + \alpha^2 I_{a2}) \end{aligned}$$

The sum and difference of these two equations give

$$2V_{a0} - V_{a1} - V_{a2} = Z(2I_0 - I_{a1} - I_{a2}) \tag{2}$$

$$V_{a1} - V_{a2} = Z(I_{a1} - I_{a2}) \tag{3}$$

Or,

$$V_{a0} - I_{a0}Z = V_{a1} - I_{a1}Z = V_{a2} - I_{a2}Z$$

IV. *Line-to-ground Fault on Conductor a and Simultaneously a Line-to-line Fault on Conductors b and c.*—If the impedances of the faults are zero, $V_a = 0$ and

$$V_{a0} + V_{a1} + V_{a2} = 0 \tag{1}$$

Since $V_{bc}$ is also zero, we have, as in the first case,

$$V_{a1} = V_{a2} \tag{2}$$

Since no current flows to ground from either conductor $b$ or conductor $c$, the sum of the line currents $I_b$ and $I_c$ is zero. If these currents are expressed in terms of the symmetrical components of $I_a$,

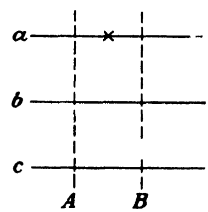
$$(I_{a0} + \alpha^{-1}I_{a1} + \alpha^{-2}I_{a2}) + (I_{a0} + \alpha I_{a1} + \alpha^2 I_{a2}) = 0$$

or

$$2I_{a0} - I_{a1} - I_{a2} = 0 \tag{3}$$

**156. Series Faults.**—In general, the currents flowing into the series fault on one side equal the currents flowing out of the fault

on the other side. For example, in Fig. 174, the current in the $a$ phase is the same at the left of $A$ as at the right of $B$. This is also true of the other two phases. Consequently the symmetrical components of the currents at the left of $A$ equal the corresponding symmetrical components of the currents at the right of $B$. Should a shunt fault occur simultaneously at $A$, the differences (if the positive direction of current flow is toward the right on both sides of the fault) between the symmetrical components at the left of $A$ and the corresponding components at the right of $B$ are the symmetrical components of the currents flowing into the shunt fault.



Series fault between A and B
FIG 174

An unsymmetrical series fault may affect either (I) one transmission conductor or (II) two transmission conductors.

I. *Fault in One Conductor.*—If the fault adds impedance in series with the $a$ conductor alone, the potential drop across the fault in the $b$ and in the $c$ conductors is zero. Thus, if the potential drop across the fault in the $a$ conductor is $V_{AB}$

$$V_{a0} = V_{a1} = V_{a2} = \frac{V_{AB}}{3} \qquad (1), (2)$$

where $V_{a0}$, $V_{a1}$, $V_{a2}$ are the symmetrical components of $V_{AB}$. If the impedance of the fault is $Z$,

$$(I_{a0} + I_{a1} + I_{a2})Z = V_{AB} \qquad (3)$$

If the impedance of the fault is infinite, that is, if the line conductor $a$ is broken, the third relation becomes

$$I_{a0} + I_{a1} + I_{a2} = 0 \qquad (3)$$

Compare these relations with those in cases II and III of the shunt faults.

II. *Fault in Two Conductors.*—If the fault adds impedance in series with the $b$ and $c$ conductors, the potential drop $V_a$ across the fault in the $a$ conductor is zero. Thus

$$V_{a0} + V_{a1} + V_{a2} = 0 \qquad (1)$$

If the fault impedance, $Z$, is the same in conductor $b$ and in conductor $c$,

$$I_bZ = V_b \qquad \text{and} \qquad I_cZ = V_c$$

If the potentials and currents are expressed in terms of their symmetrical components, it follows, as in Case III of the shunt faults, that

$$(2I_{a0} - I_{a1} - I_{a2})\,Z = 2V_{a0} - V_{a1} - V_{a2} \qquad (2)$$

$$(I_{a1} - I_{a2})Z = V_{a1} - V_{a2} \qquad (3)$$

Or,

$$V_{a0} - I_{a0}Z = V_{a1} - I_{a1}Z = V_{a2} - I_{a2}Z$$

If the impedances of both faults are infinite, that is, if both the $b$ and $c$ line conductors are broken

$$2I_{a0} - I_{a1} - I_{a2} = 0 \qquad (2)$$

$$I_{a1} - I_{a2} = 0 \qquad (3)$$

That is

$$I_{a0} = I_{a1} = I_{a2} = \frac{I_a}{3}$$

Compare these relations with those in Case III of the shunt faults. It will be observed that, unless the system neutral is grounded on both sides of the fault, the breaking of two line conductors will completely interrupt the flow of current through the fault.

**157. Significance of Assumptions as Applied to the General Balanced Network.**—The assumptions made in regard to the symmetry of the network are of vital importance in the solution of any problem and their significance should be carefully considered. Thus, the emfs generated by rotation in the "ideal" synchronous machines are symmetrical and have neither zero-sequence nor negative-sequence components. Since all the impedances in the network are balanced in the three phases, positive-sequence currents produce only positive-sequence components of potential, negative-sequence currents produce only negative-sequence components of potential, and zero-sequence currents produce only zero-sequence components of potential. This is true of the "ideal" synchronous machines and of all other parts of the network.

There is no relation whatsoever between the currents of one sequence and the potentials of a different sequence *within the network itself.* That is, the positive-sequence components of potential throughout the network depend only upon the positive-sequence components of the currents and the excitations of the connected synchronous machines, and are in no way

affected by the negative-sequence or zero-sequence components of the current. Since there are neither negative-sequence nor zero-sequence emfs (generated by rotation in the synchronous machines) impressed upon the network, the negative-sequence components of potential throughout the network depend only upon negative-sequence components of current, and zero-sequence components of potential depend only upon zero-sequence components of current.

When an unsymmetrical shunt fault or load is connected to this network at some point, unsymmetrical currents flow from the network into the fault or load. The line-to-ground potentials at the fault and the currents that flow into the fault are characterized, in general, by zero-, positive-, and negative-sequence components. Let these components be represented by $V_0$, $V_1$, $V_2$ and by $I_0$, $I_1$, $I_2$. Of the total vector volt-ampere input ($\underline{V}_0 I_0 + \underline{V}_1 I_1 + \underline{V}_2 I_2$) to the fault, only the positive-sequence component $\underline{V}_1 I_1$ is produced by the synchronous machines in the network. If the impedance of the fault is zero, the total vector volt-ampere input to the fault is also zero and $\underline{V}_1 I_1 = -\underline{V}_0 I_0 - \underline{V}_2 I_2$. Thus, it may be said that the positive-sequence component $\underline{V}_1 I_1$ of the vector volt-amperes is transformed at the fault into zero- and negative-sequence components which are supplied to the network, and are there dissipated.

In general, the currents and potentials in the network will also consist of zero-, positive- and negative-sequence components. If, for example, it were possible to abstract from each and every branch of the network the zero- and negative-sequence components of current without altering the positive-sequence component of current in any branch, the positive-sequence components of line-to-ground potential at every point in the network including the point at which the fault occurs would be unaltered and all the zero and negative-sequence components of potential would vanish. This result can actually be accomplished by replacing the fault with a *symmetrical* load which has an impedance ($V_1/I_1$) per phase equal to the ratio of the positive-sequence component of potential at the fault to the positive-sequence component of the fault current. When the unsymmetrical fault or load is replaced by a symmetrical load having the impedance $V_1/I_1$ per phase, the symmetrical currents that flow in each part of the network are the same as the positive-

sequence components of the current in that part during the time that the unsymmetrical fault or load is connected to the network. Likewise, the potential to neutral at each point in the network is the same as the positive-sequence component of the line-to-ground potential at that point during the time that the unsymmetrical fault or load is connected to the network.

Again, if the excitations of the synchronous machines are withdrawn and the positive- and negative-sequence components of current are abstracted from each and every branch of the network without altering the zero-sequence component of current in any branch, the positive- and negative-sequence components of potential will vanish everywhere in the network without altering the zero-sequence components of potential. This result can also be accomplished by applying to the network, in place of the fault, zero-sequence line-to-ground potentials $V_0$ and at the same time withdrawing the excitations from all the connected synchronous machines in order that there shall be no positive-sequence components of current. The zero-sequence line-to-ground potentials applied to the network in place of the fault will then produce a current in every branch in which it is possible for zero-sequence current to flow, this current being equal to the zero-sequence component of the current in that branch during the time that the unsymmetrical fault is connected to the network. The line-to-ground potentials at every point in the network will be the same as the zero-sequence component of the potential at that point during the time that the unsymmetrical fault is connected to the network.

Similarly, if negative-sequence line-to-ground potentials $V_2$ are applied to the network in place of the unsymmetrical fault and at the same time the excitations of the connected synchronous machines are withdrawn, the current in every branch of the network is the same as the negative-sequence component of the current in that branch during the time that the unsymmetrical fault is connected to the network. Again, the line-to-neutral potential at every point in the network is the same as the negative-sequence component of the line-to-neutral potential at that point during the time that the unsymmetrical fault is connected to the network.

These conceptions form the basis of a powerful method for determining the sequence components of the currents and poten-

tials in a symmetrical network subjected to one or more unsymmetrical faults or loads.

All these deductions depend upon the symmetry of the network itself. Should any of its branches be unsymmetrical, in consequence of which current of one sequence will produce potential of a different sequence (see Art. 24, page 89), the foregoing relations do not hold. Such a case, in which there is an interrelation between the sequence components within the network itself, is illustrated in Art. 87, page 241, in which, the unsymmetrical network consists of an unsymmetrical feeder through which power is supplied to an induction motor. When the unsymmetrical network has many branches, the number of simultaneous equations is so great that the computation becomes extremely laborious.

**158. Positive-, Negative-, and Zero-sequence Networks.**—When a symmetrical network is subjected to an unsymmetrical fault, the relations between the positive-sequence components of potential and current throughout the network are determined by the excitation emfs of the synchronous machines, the impedances of the branches of the network to positive-sequence, and the positive-sequence components of potential and current at the fault. The relations between the negative-sequence components of potential and current are determined solely by the impedances of the branches of the network to negative sequence. Similarly, the relations between the zero-sequence components of potential and current are determined solely by the impedances of the branches of the network to zero sequence. The impedances of the branches of any symmetrical network will assume these different sequence values depending upon the sequence of the currents in them. Thus, when positive-sequence currents flow in the network, all the branches, including those in which the excitation emfs are generated, have impedances to positive sequence and the network is then designated as the "positive-sequence network."

When negative-sequence currents flow in the network, the same branches have impedances to negative sequence and the network is then designated as the "negative-sequence network."

Again, when zero-sequence currents flow in the network, the impedances of those branches in which the currents flow assume

their values to zero sequence and the network is designated as the "zero-sequence network."

Since within the actual network there is no interrelation among the different sequences, the three sequence networks can be set up independently. Furthermore, since each of these networks is symmetrical, one phase only needs be considered.

Thus, each network can be represented by a single-phase diagram similar to that drawn commonly to represent a *sym-*

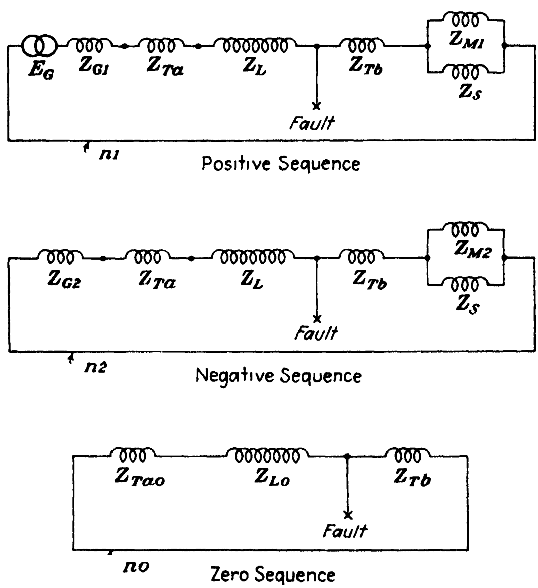
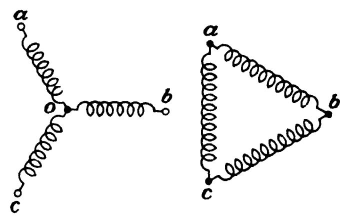



FIG. 175

*metrical* 3-phase power network. One side of this diagram represents a neutral conductor or earth which has zero impedance. The three sequence-component networks of the simple distribution circuit described in Art. 152, page 471, are shown in Fig. 175, in which these neutrals are $n_1$, $n_2$, $n_0$. The positive-sequence diagram is exactly like the one which would be used to represent the circuit before the fault occurs. The arrangement of the branches in the negative-sequence diagram is identical with that in the positive-sequence diagram, but the excitation emfs generated in the synchronous machines are absent, although the circuits in which these emfs are generated are included in the diagram.

The impedances and charging admittances of the transmission lines, the impedances of the transformers, and the equivalent impedances of the balanced single-phase loads are the same in both networks, but the impedances of all the rotating machines are different in the two networks.

Since the power distribution system is usually arranged so that zero-sequence current does not flow in all of its branches, the zero-sequence network will, in general, have fewer branches than either the positive- or the negative-sequence network. Moreover, the impedances of all the system branches, excepting the two-winding transformers, are different from the impedances of the corresponding branches in the positive- and negative-sequence networks. In regard to transmission lines, it should be noted that *all* the impedance of the conductor-earth loop is included in the impedance of the line conductor.




$$y_{ab} = \frac{y_{ao}\, y_{bo}}{y_{ao} + y_{bo} + y_{co}}$$

FIG. 176

**159. Analysis of the Component Networks.**—There are but five fundamental electrical principles that need be used in the analysis of power networks. They are the relations between potential and current for resistance, inductance, and capacitance and the two Kirchhoff Laws in regard to the sum of the currents flowing toward a point and the fall in potential around a closed circuit. From the Kirchhoff Laws have been derived the principles relating to the series and parallel combinations of circuits and the $Y - \Delta$ and more general star-mesh transformations. It is important to note that the impedances of series circuits and the admittances of parallel circuits are always combined *vectorially*. The star-mesh transformation applies only to branches in which no excitation emfs are acting. It is illustrated in Fig 176. In a three-branch star or $Y$ circuit, the mesh admittance $y_{ab}$, for example, is

$$y_{ab} = \frac{y_{a0}y_{b0}}{y_{a0} + y_{b0} + y_{c0}}$$

The general star and mesh circuits are exactly equivalent as viewed from the terminals $a$, $b$, $c$, $d$, etc., if the admittance of each mesh branch equals the product of the admittances of the

two star branches that it spans, divided by the sum of the admittances of all the star branches. Since, in the general mesh circuit, a branch must connect each and every pair of terminals, excluding the neutral of the star, it requires $n(n - 1)/2$ mesh branches to replace $n$ star branches. By this replacement process the neutral of the star is eliminated. It is always possible to determine a mesh equivalent to any given star, but conversely it is, in general, possible to determine a star equivalent to a given mesh only when there are *three* branches. Note that it is better to express the circuit constants as admittances rather

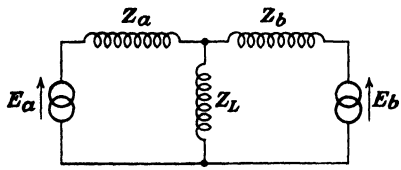



FIG. 177.

than as impedances when making the star-mesh transformation. This is also true when combining circuits in parallel.

By means of the series, parallel, and star-mesh transformations, it is possible to eliminate as many junction points in the network as desired. Thus, both the zero- and negative-sequence networks can be reduced individually to an equivalent mesh-connected circuit whose terminals are the neutral point of the network and the points at which the faults occur. In the case of a single fault, both the negative and zero-sequence networks can be reduced to a single impedance connecting the neutral and the point at which the fault occurs. When faults occur at two points, the equivalent network can assume either the $\Delta$ or the $Y$ form. Should faults occur at three points, the equivalent network has four terminals and will then require six mesh branches to represent it. In this case there is, in general, no equivalent star circuit.

The principle of superposition which applies when there is a linear relation between cause and effect, such as is assumed to exist between the applied emfs and currents in a power network, frequently leads to a simpler solution than would a direct application of the two Kirchhoff Laws alone. It has already been noted that the method of symmetrical components itself depends upon the principle of superposition.

If there are several emfs acting in a network, it can be assumed that the current flowing in any branch is the sum of the currents that would be produced in that branch by the independent action of each of the emfs. For example, the current that is supplied to the load impedance $Z_L$ in Fig. 177 is the sum of the currents that $E_a$ and $E_b$ will produce independently. If the emf $E_b$ is removed, the branches $b$ and $L$ are in parallel and the total impedance of the circuit is $Z_a + \frac{Z_b Z_L}{Z_b + Z_L}$. The total current supplied by the emf $E_a$ is

$$\frac{E_a}{Z_a + \frac{Z_b Z_L}{Z_b + Z_L}}$$

The portion of this current which flows in the load impedance $Z_L$ is

$$\frac{E_a}{Z_a + \frac{Z_b Z_L}{Z_b + Z_L}} \frac{Z_b}{Z_b + Z_L}$$

or

$$\frac{E_a Z_b}{Z_a Z_b + Z_L(Z_a + Z_b)}$$

Similarly, the current which would be supplied to the load impedance by the emf $E_b$ if it alone were acting is

$$\frac{E_b Z_a}{Z_a Z_b + Z_L(Z_a + Z_b)}$$

Thus, the actual current flowing in the load when both emfs are acting is

$$\frac{E_a Z_b + E_b Z_a}{Z_a Z_b + Z_L(Z_a + Z_b)}$$

If the network has many branches, it is usually less laborious to determine the current in any branch by this method of superposition than to apply the two Kirchhoff Laws directly and solve the resulting simultaneous equations.

Thévénin's Theorem, which is derived from the Principle of Superposition, offers an interesting and extremely useful method of analysis. By this principle we can determine the current

which will flow in a branch of impedance $Z$ when it is connected across two points $a$-$b$ of a network in which several emfs are acting. Owing to these emfs, there is a potential $E_{ab}$ across the points $a$-$b$ before the impedance is connected to them. With the switch open as indicated in Fig. 178, it can be assumed that the resulting current in this impedance, which is then zero, is due to the action of the network emfs and the potential $E_{ab}$. If the potential $E_{ab}$ is withdrawn, that is, if the switch is closed, the current in the impedance is due wholly to the action of the network emfs. Designate this current by $I$. Again, if the network emfs are withdrawn, the current in the impedance is

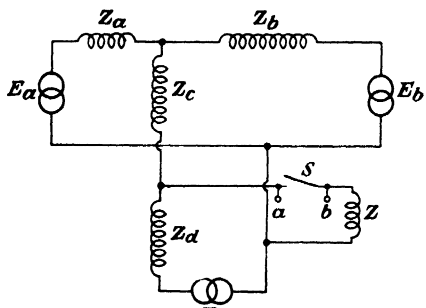


FIG. 178

due wholly to the potential $E_{ab}$, which must be assumed to be maintained by some generating source connected across the open switch. It will be observed that the component of the current due to $E_{ab}$ is $E_{ab}/(Z + Z_{Nab})$, where $Z_{Nab}$ is the impedance of the network between the points $a$-$b$ when all the network emfs are withdrawn. Since the two components of the resulting current in the impedance $Z$ are in opposite directions,

$$0 = I - \frac{E_{ab}}{Z + Z_{Nab}}$$

or

$$I = \frac{E_{ab}}{Z + Z_{Nab}}$$

Since $IZ$ is the resulting potential $V_{ab}$ across $a$-$b$,

$$\dot{E}_{ab} - IZ_{Nab} = V_{ab}$$

For example, consider the Wheatstone-bridge circuit shown in

Fig. 179, whose branches have the impedances $M$, $N$, $P$, $X$, and $G$ as indicated. The current $I_G$ in the galvanometer can be determined as follows: Let $E_{ab}$ represent the potential across the points $a$-$b$ before the galvanometer is connected between the arms of the bridge. Observe that

$$E_{ab} = E\left(\frac{M}{M + N} - \frac{P}{P + X}\right)$$

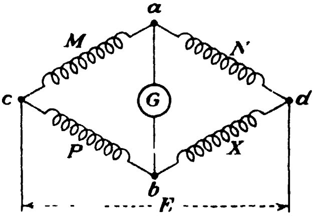


FIG. 179.

When the applied emf $E$ is withdrawn, the points $c$-$d$ become short-circuited if the impedance in the generating source is negligible, and the impedance $Z_{Nab}$ of the bridge arms between the points $a$-$b$ is

$$Z_{Nab} = \frac{MN}{M + N} + \frac{PX}{P + X}$$

Thus, the galvanometer current will be

$$I_G = \frac{E\left(\frac{M}{M + N} - \frac{P}{P + X}\right)}{G + \frac{MN}{M + N} + \frac{PX}{P + X}}$$

The current that the generating source $E$ supplies to the bridge is fixed by the impedance of the bridge between the points $c$-$d$. This impedance is readily determined if the $\Delta$ branches $M$, $P$, $G$ are replaced by their equivalent $Y$ branches. A somewhat less direct solution can be obtained by replacing the $Y$ branches $M$, $G$, $N$ by their equivalent $\Delta$. The impedance of one branch of an equivalent $Y$ circuit equals the product of the impedances of the $\Delta$ branches between which the particular $Y$ branch lies divided by the sum of the impedances of all the $\Delta$ branches (see Chap. II, page 70). Thus the impedances of the equivalent $Y$ branches which connect the points $c$-$a$-$b$ are

$$\frac{MP}{M + P + G}, \qquad \frac{MG}{M + P + G}, \qquad \frac{PG}{M + P + G}$$

It follows that the impedance of the bridge between the points $a$-$d$ is

$$\frac{MP}{M+P+G} + \frac{\left(\dfrac{MG}{M+P+G} + N\right)\left(\dfrac{PG}{M+P+G} + X\right)}{\dfrac{(M+P)G}{M+P+G} + N + X}$$

By applying these principles, it is possible to reduce the positive-, negative-, and zero-sequence components of any power network to their simplest equivalent forms. Experience alone must be the guide in choosing the method or combination of methods whose application will result in the most direct solution of the problem in hand.

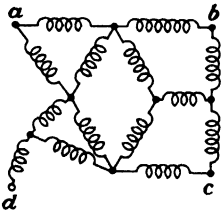



FIG. 180.

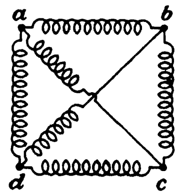



FIG. 181.

**160. Network Analyzer.**—If the network has many branches, this process of reduction becomes of necessity very laborious. For this reason recourse is made, whenever possible, to a reconstruction in miniature of the three sequence networks. This is accomplished by the so-called "network analyzer." The arrangements of the branches in each miniature network is exactly like that in the actual network of the same sequence. Since the impedances and emfs in the miniature network bear a constant ratio to those in the actual network, the currents in the miniature network are proportional to the corresponding currents in the actual network.

For example, consider a network, having four principal terminals, $a$, $b$, $c$, $d$, that has been set up in miniature on the network analyzer (see Fig. 180). This miniature network may represent the actual network for zero, negative, or positive sequence, provided it is assumed that all of the excitation emfs have been withdrawn from the positive-sequence network. Insofar as the terminals $a$, $b$, $c$, $d$ are concerned, the multibranched network of Fig. 180 is equivalent to the six-branch network of Fig. 181, if the impedances between each corresponding pair of terminals in Figs. 180 and 181 are equal.[1] The impedances, or admittances,

[1] STARR, F. M., Equivalent Circuits, *Am. Inst. Elect. Eng.*, p. 287, 1932.

of the branches in the equivalent circuit (Fig 181) can be measured as indicated in Fig. 182, provided the impedances of the ammeters are negligible in comparison with the impedances of the branches of the network. It will be observed, in this case, that the ammeters short-circuit the branch impedances $Z_{bc}$, $Z_{bd}$, $Z_{cd}$, and hence no current flows in each of these impedances. Therefore, it follows that the impedances in the other three branches are

$$Z_{ab} = \frac{E}{I_{ab}} \qquad Z_{ac} = \frac{E}{I_{ac}} \qquad Z_{ad} = \frac{E}{I_{ad}}$$

or

$$Y_{ab} = \frac{I_{ab}}{E} \qquad Y_{ac} = \frac{I_{ac}}{E} \qquad Y_{ad} = \frac{I_{ad}}{E}$$

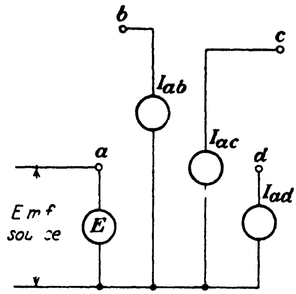


FIG 182

The angles of these impedances or admittances can be determined by means of wattmeters. The values of the other three impedances $Z_{bc}$, $Z_{bd}$, $Z_{cd}$ can be determined in a similar manner.

An interesting illustration is the application of this method to a two-winding transformer, or coupled circuit, when it is essential that there should be no electrical connection between the primary and secondary windings. Refer to Fig. 183. Note that the currents $I_{ac}$ and $I_{ad}$ are equal in magnitude and opposite in time-phase and that they equal the *short-circuit* current in the secondary winding. Also, note that the current $I_{ab}$ is the primary current when the secondary winding is short-circuited. The equations applying to the transformer, or coupled circuit, are

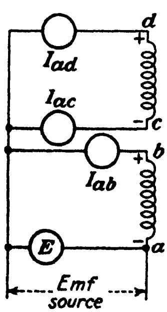


FIG. 183.

$$E = I_{ab}Z_p + I_{cd}Z_{ps}$$
$$0 = I_{cd}Z_s + I_{ab}Z_{ps}$$

where $Z_p$ and $Z_s$ are the self-impedances of the primary and secondary windings and $Z_{ps}$ is the mutual impedance between them. From these equations,

$$\frac{E}{I_{ab}} = Z_p - \frac{Z_{ps}^2}{Z_s}$$
$$\frac{E}{I_{ac}} = Z_{ps} - \frac{Z_pZ_s}{Z_{ps}}$$

If the primary ($p$) is short-circuited and potential is applied to the secondary ($s$),

$$\frac{E}{I_{cd}} = Z_s - \frac{Z_{ps}^2}{Z_p}$$

Therefore, the impedances of the branches of the equivalent mesh circuit for a transformer, or coupled circuit, whose primary and secondary are *not* connected electrically, are

$$\left.\begin{aligned} Z_{ab} &= Z_p - \frac{Z_{ps}^2}{Z_s} \\ Z_{cd} &= Z_s - \frac{Z_{ps}^2}{Z_p} \\ Z_{ac} = Z_{bd} &= -\frac{Z_pZ_s}{Z_{ps}} + Z_{ps} \\ Z_{ad} = Z_{bc} &= \frac{Z_pZ_s}{Z_{ps}} - Z_{ps} \end{aligned}\right\} \tag{475}$$

These values of $Z_{ac}$ or $Z_{bd}$ and $Z_{ad}$ or $Z_{bc}$ assume that the mutual reactance coupling between the primary, *a-b*, and the secondary,

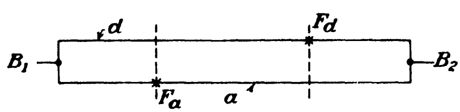



FIG. 184.

*c-d*, is positive. Owing to the fact that two of these six impedances have negative resistances, there may be some difficulty in representing the transformer by this equivalent circuit in the network analyzer (see Art. 174, page 528). There would be no difficulty, however, in applying the star-mesh transformation to a network of which this equivalent circuit formed a part.

When there is mutual impedance between parts of the network as in the case of a double-circuit transmission line, there are two forms of equivalent circuits which can be used if the network is set up on the network analyzer. If the two parts of the circuit are connected, as at a bus point, a $T$-equivalent circuit can be used. This was illustrated in Art. 59, page 152. Referring again to that illustration, assume there are *two* simultaneous faults at $F_a$ and $F_b$ as indicated in Fig. 184, which shows only the $a$ and $d$ conductors. The two circuits are bussed at $B_1$ and $B_2$. The transmission circuits between $B_1$ and $F_a$ can be

represented by an equivalent circuit as in Fig. 185*a*, in which $Z_a'$ is the self-impedance of the *a* circuit, $Z_d'$ is the self-impedance of the *d* circuit, and $Z_{ad}'$ is the mutual impedance between the (*a,b,c*) and (*d,e,f*) circuits Likewise the transmission circuits between $B_2$ and $F_d$ can be represented by a similar *T*-equivalent circuit, Fig. 185*b*, in which the constants are $Z_a''$, $Z_d''$, $Z_{ad}''$. These two equivalent circuits are connected by an equivalent circuit which represents the two transmission circuits between $F_a$ and $F_d$. Since the *a* and *d* transmission circuits are insulated from each other in the part of the circuit between $F_a$ and $F_d$ the *T* form of equivalent circuit cannot be used to represent them.

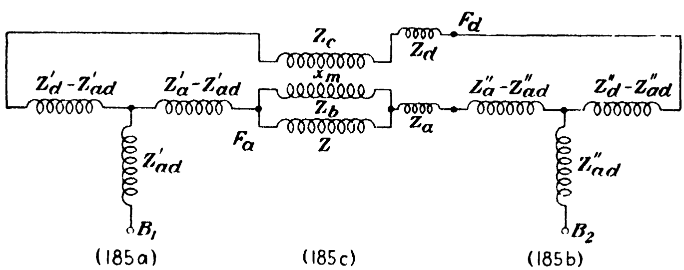



FIG. 185

However, a properly adjusted mutual impedance may be used as the connecting link as indicated in Fig. 185*c*. The purpose of the self-impedance $Z$, which is connected in parallel with $Z_b$, is to provide a means for adjusting the value of the mutual impedance between the *a* and *d* circuits in this section of the transmission circuit. In this manner, it is possible to provide an adjustable *mutual impedance* without the necessity of having a *mutual reactance* which is itself adjustable. If $Z_{ad}$ is the desired value of the mutual impedance between the *a* and *d* circuits for this section of the transmission line, it is readily shown that the necessary value of $Z$ is

$$Z = \frac{Z_b Z_{ad}}{jx_m - Z_{ad}}$$

where $Z_b$ is the self-impedance of one coil and $x_m$ is the value of the mutual reactance in Fig. 185*c*. The proper values of the *self-impedances* of the *a* and *d* circuits in this section can be obtained by adding adjustable impedances $Z_a$ and $Z_d$ as indicated in Fig. 185*c*.

The transmission circuit between $F_a$ and $F_d$ can also be replaced by a six-branch mesh circuit as in Fig. 186. If the star-mesh transformation is applied four times so as to eliminate the points $m$, $n$, $p$ and $q$, the entire transmission circuit will be reduced to a six-branch mesh circuit having terminals at $B_1$, $B_2$, $F_a$, and $F_d$.

Provision is made in the network analyzer for measuring the current, potential, and power wherever their values are desired. Thus, the positive-, negative-, and zero-sequence component networks can be set up independently on the network analyzer and all the required measurements can be made directly on the analyzer, thus determining the impedances and the sequence components of potentials and currents for the actual network.

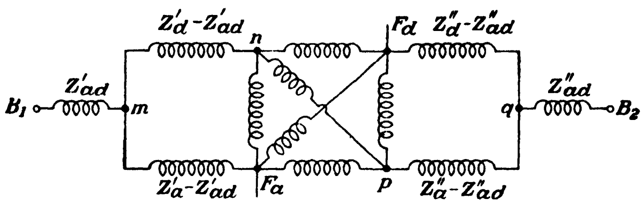



FIG. 186.

We shall now consider the application of these methods of analysis in a number of typical cases to determine the fault current and the distribution of current and potential throughout the network. It is particularly interesting to compare the different procedures that may be used to obtain the required solution.

**161. Single Shunt Fault.**—This is the simplest of the cases we shall discuss. It has been considered already in some detail in connection with the effect produced by an unsymmetrical short circuit on the operation of a synchronous machine. In every case, the first step in the analysis is to set up the zero-, positive-, and negative-sequence networks, either diagrammatically as an aid in making the required computations or on the network analyzer. In the latter case, the circuit constants can be measured as well as the sequence components of the potential and current. We will assume that this has been done and that the rectangles designated $N_0$, $N_1$, $N_2$, in Fig. 187, represent the component networks. The neutral terminals of these networks are $n_0$, $n_1$, $n_2$, and the point at which the fault occurs is $a$. In the case of a single fault, both the zero- and negative-sequence

networks can be reduced to a single impedance connecting the neutral and the point at which the fault occurs, and the value of each of these impedances can be calculated or can be measured. Denoting them by $Z_{N0}$ and $Z_{N2}$,

$$V_0 = -I_0 Z_{N0} \tag{476}$$
$$V_2 = -I_2 Z_{N2} \tag{477}$$

where $V_0$, $V_2$, $I_0$, $I_2$ are the zero- and negative-sequence components of the fall in line-to-neutral potential at the fault and the current in the same direction *through* the fault, *not* through the equivalent network. If these two equations (476) and (477) are combined with those of Art. 155, page 478, which determine the relations between the potentials and currents at the fault, it will always be possible to eliminate the zero and negative-sequence components of potential and current and in this manner determine the relation that must exist between the positive-sequence components of potential and current at the fault. The ratio of the positive-sequence components of potential and current at the fault gives the value of an impedance. If this impedance is connected to the positive-sequence network between the neutral and the point at which the fault occurs, the currents and potentials that are computed or are measured in the miniature network are the positive-sequence components of the currents and potentials at the corresponding points in the actual network. For a line-to-line fault this impedance is $Z_{N2}$; for a single line-to-ground fault the impedance is $Z_{N0} + Z_{N2}$; for a double line-to-ground fault the impedance is $\frac{Z_{N0}Z_{N2}}{Z_{N0} + Z_{N2}}$. If the impedance of the fault is $Z_f$, these three impedances become $(Z_{N2} + Z_f)$, $(Z_{N0} + Z_{N2} + 3Z_f)$, and $Z_f + \frac{(Z_f + Z_{N0})(Z_f + Z_{N2})}{2Z_f + Z_{N0} + Z_{N2}}$. For a simultaneous line-to-ground fault on conductor $a$ and a line-to-line fault on conductors $b$ and $c$, the impedance to connect to the positive-sequence network is $\frac{Z_{N0}Z_{N2}}{Z_{N0} + 4Z_{N2}}$.

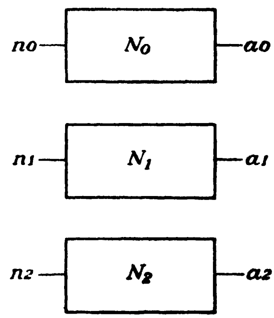


FIG. 187.

The positive-sequence component of the fault current is readily computed by applying Thévénin's Theorem. If the initial, or prefault, line-to-neutral potential at the point where the fault occurs is $E_{a1}$, the positive-sequence component, $I_1$, of the fault current is

$$I_1 = \frac{E_{a1}}{Z + Z_{N1}} \qquad (478)$$

where $Z$ is the impedance of the combination of the zero- and negative-sequence networks which replaces the fault, and $Z_{N1}$ is the impedance of the network measured between the line and neutral at the point where the fault occurs when all of the excitation emfs of the synchronous machines are withdrawn. This relation can be expressed as

$$V_1 = E_{a1} - I_1 Z_{N1} \qquad (479)$$

where $V_1$ is the resulting positive-sequence line-to-neutral potential at the fault. Thus, it may be said that with respect to a single fault, the positive-sequence network is equivalent to a single emf, $E_{a1}$, in series with an impedance, $Z_{N1}$, where $E_{a1}$ and $Z_{N1}$ have the values given in equation (478).

The foregoing analysis is like the analysis applying to the "ideal" synchronous machine operating with unsymmetrical faults or loads (Art. 128, page 387). The prefault potential $E_{a1}$ corresponds to the excitation emf of the generator and the network impedances correspond to the impedances of the generator to zero, positive, and negative sequence.

**Illustration.**—Consider the double-circuit line described in Fig. 46, page 152. Assuming there is a single fault one-third the length of the line from one end, the equivalent circuit for positive, negative, or zero sequence is in Fig. 48. If the line is rated at 50,000 kva., the unit of impedance is

$$\frac{66{,}000}{\sqrt{3}} \div \frac{50{,}000}{\sqrt{3} \times 66} = 87.1 \text{ ohms.}$$

Referred to this base (87.1 ohms) the per unit values of the reactances of the branches in the equivalent circuits are indicated in Figs. $A$, $B$, and $C$. In order to simplify the computation the resistances will be neglected.

At each end of the line there are synchronous machines connected to the line through $Y$-$\Delta$ transformers, the neutrals on the high-tension sides being grounded. The transformers

have reactances of 0.07 per unit. The per unit reactances of the synchronous machines are indicated in Figs. *A* and *B*. The per unit values of reactances of the transformers and the synchronous machines are based on a rating of 50,000 kva. Assume that before the fault occurs the synchronous machines

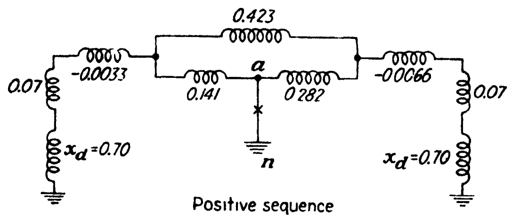



FIG. *A*.

are each operating at rated kilovolt-amperes and rated potential on the high-tension sides of the transformers. The prefault potential $E_a$ at the fault point is determined by the vector diagram of Fig. *D*, from which $E_a = 0.9955$ per unit.

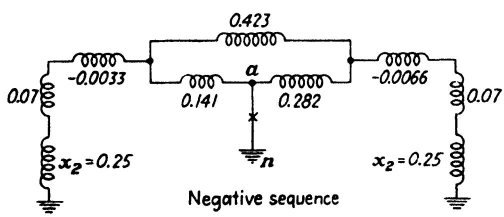



FIG. *B*.

Assume the synchronous machines have "ideal" characteristics and that the magnitudes and relative phase of their excitation emfs are not altered by the fault. It is important to note that under these assumptions the computed currents are the steady

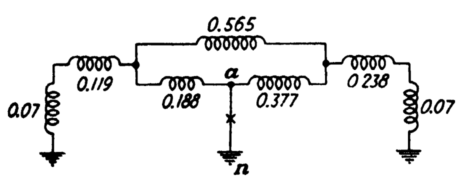



FIG. *C*.

state currents that would flow if the magnitudes and relative phase displacement of the excitation emfs were the same before and after the fault. Actually the change in load on the synchronous machines due to the effect of the fault alters the relative

phase displacement of the excitation emfs even though their magnitudes are maintained constant. In any case the value of $E_a$ used in the computation of the fault currents is the value the line-to-neutral potential at the point of fault would have if the fault were removed without changing either the magnitude or phase of any excitation emf acting in the network. If it is assumed that the *positive-sequence* components of the *terminal* potentials of each synchronous machine are maintained constant by potential regulators, the prefault potential $E_a$ is nearly constant for any load if the load is not greater than 50,000 kva. Under this assumption the reactances $x_d$ should be omitted from the positive-sequence equivalent circuit (Fig. *A*).

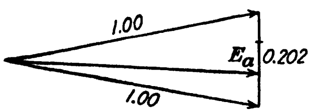


FIG. *D*.

The transient fault currents are sometimes computed on the assumption that the emfs of the synchronous machines "behind" their transient reactances are maintained constant both in magnitude and phase before and after the fault. In this case the synchronous reactances $x_d$ in the positive-sequence network (Fig. *A*) are replaced by the transient reactances $x_d'$. The computed value of the current is further multiplied by a decrement factor to account for the decay of the transient current. If the synchronous machine has ideal characteristics there is also, in general, a damped direct component in the transient

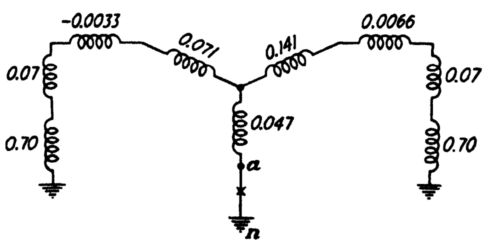


FIG. *E*.

current. Notwithstanding the importance of transient conditions they are not discussed in the present volume.

The sequence networks of Figs. *A*, *B*, and *C* are readily reduced by replacing the central Δ branches in each of these networks by their equivalent *Y* branches. The impedance of any one of the equivalent *Y* branches equals the product of the impedances

of the two Δ branches between which it lies divided by the sum of the impedances of the three Δ branches. Thus Fig. *A* reduces to Fig. *E*. The reactance of the network between the fault point *a* and the neutral *n* is 0.481 per unit. Therefore the positive-sequence network with the two excitation emfs acting is equivalent to an emf $E_a = 0.9955$ in series with a reactance $x = 0.481$. In a similar manner the negative-sequence network (Fig. *B*) reduces to a reactance of 0.256 per unit between the

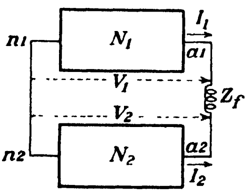



FIG. 188.

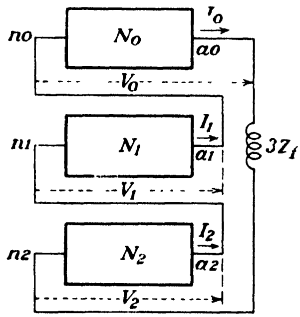



FIG. 189.

fault point *a* and the neutral *n*. The zero-sequence network (Fig. *C*) reduces to a reactance of 0.243 per unit between *a* and *n*.

Therefore the positive-sequence fault current $I_{f1}$ is

$$I_{f1} = \frac{0.9955}{0.481 + 0.256 + 0.243}$$
$$= 1.015 \text{ per unit}$$

Since for a line-to-ground fault the symmetrical components of the fault current are equal, the fault current $I_f$ is

$$I_f = 3.05 \text{ per unit}$$

It is always possible to make either a direct connection, or a connection through the transformers, between the three sequence component networks which will satisfy the required relations between the sequence components of potential and current at the fault. Thus, it will be observed that Figs. 188, 189, and 190 satisfy the conditions in Cases I, II, and III under Art. 155, page 478. If the network analyzer has enough units, all three sequence networks can be set up simultaneously and interconnected as indicated. If the currents and potentials are measured with the aid of current and potential transformers,

the *sum* of the sequence components of the current and potential can be recorded on a single ammeter and voltmeter by following the procedure used in measuring the zero-sequence current and potential (Arts. 11, 14, pages 56, 65). Thus, the actual current and potential at any point in the network can be measured for the phase to which the fault equations of Art. 155 apply. When there are not enough units in the network analyzer to permit setting up the component networks simultaneously, the zero- and negative-sequence networks can be replaced by their equivalent single impedances. It will then be necessary to determine the symmetrical components of the potentials and currents independently. Once the positive-sequence components of the potential and current at the fault are determined, the zero- and negative-sequence components of the potential at the fault can be computed by means of the general relations developed in Art. 155, page 478. If these potentials are applied to the zero- and negative-sequence networks at the point where the fault occurs, the corresponding components of potential and current throughout the network can be computed or measured. We find this method of interconnecting the sequence networks particularly useful when faults occur at more than one point.

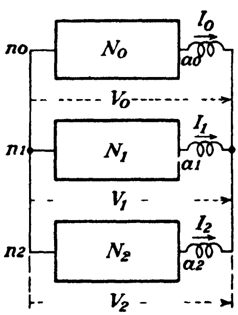



FIG 190

**162. Unbalanced Load.**—We shall now briefly consider the effect of adding to the network an unbalanced arrangement of single-phase loads at a point where the line-to-neutral potential is $E_a$. The loads are connected between the line conductors and have admittances $y_a$, $y_b$, $y_c$, where $y_a$ is the admittance of the load connected between the $a$ and $b$ line conductors. The positive- and negative-sequence components of the line current supplied to this load from the line conductor $a$ are

$$I_1 = 3(V_1 y_{b0} - V_2 y_{b2})$$
$$I_2 = 3(V_2 y_{b0} - V_1 y_{b1}) \quad \text{[see equation (63), page 86]}$$

where $V_1$ and $V_2$ are the positive- and negative-sequence components of the resulting line-to-neutral potential at the load for conductor $a$. There is also the general relation

$$V_2 = -I_2 Z_{N2}$$

If $V_2$ and $I_2$ are eliminated from these three equations,

$$\frac{V_1}{I_1} = \frac{1 + 3y_{b0}Z_{N2}}{3y_{b0} + 9Z_{N2}(y_{b0}^2 - y_{b1}y_{b2})}$$

Noting that $3y_{b0} = y_a + y_b + y_c$, and that

$$3(y_{b0}^2 - y_{b1}y_{b2}) = (y_ay_b + y_by_c + y_cy_a)$$

$$\frac{V_1}{I_1} = \frac{1 + (y_a + y_b + y_c)Z_{N2}}{(y_a + y_b + y_c) + 3Z_{N2}(y_ay_b + y_by_c + y_cy_a)} \qquad (480)$$

Thus, we have determined the impedance per phase of a symmetrical load which if connected to the network will take a current equal to the positive-sequence current supplied to the actual load. This current can be determined either by computation or by measurement on the network analyzer. If desirable, the admittances of the loads in equation (480) can be replaced by their impedances. Whether it is better to use the admittances or the impedances depends upon the character of the loads. If, for example, there is no load connected across one pair of line conductors, the impedance of that phase of the load is infinite and the admittance is zero. The symmetrical components of the load impedances would be all infinite in this case and in order to obtain a solution of the problem, the final result for the symmetrical components of the currents must be written in terms of the *actual* load impedances and the limit then determined as one impedance approaches infinity. This procedure is unnecessary if the load admittances are used, since if one of them is zero there is no difficulty in determining and in using their symmetrical components. The most general type of three-phase load is one that is $Y$ connected, with its neutral grounded through an impedance. The analysis can be carried out by using equations (55), page 82, or equations (56), page 85, in conjunction with the network equations (476), (477), and (478). The grounding impedance can be considered a part of the zero-sequence network if three times its value is added to the impedance of the zero-sequence network.

Thévénin's Theorem is of considerable assistance in formulating the computation of the positive-sequence current. That is

$$E_1 = V_1 + I_1Z_{N1}$$

where $E_1$ is the line-to-neutral potential before the load is added and $Z_{N1}$ is the impedance of the positive-sequence network

measured at this point, with the excitation emfs withdrawn. The combination of this equation with the first three equations gives for the positive- and negative-sequence components of the resulting line-to-neutral potential at the load

$$V_1 = \frac{1 + 3y_{b0}Z_{N2}}{1 + 3y_{b0}(Z_{N1} + Z_{N2}) + 9Z_{N1}Z_{N2}(y_{b0}^2 - y_{b1}y_{b2})}E_1 \quad (481)$$

$$V_2 = \frac{3y_{b1}Z_{N2}}{1 + 3y_{b0}(Z_{N1} + Z_{N2}) + 9Z_{N1}Z_{N2}(y_{b0}^2 - y_{b1}y_{b2})}E_1 \quad (482)$$

This establishes the relation between the line-to-neutral potentials at this point in the network before and after the unbalanced load is added.

**163. Simultaneous Shunt Faults at Two Points.**—Faults sometimes occur simultaneously at two different points which may be widely separated. The general process of analysis is similar to that used in the case of a single fault, but the symmetrical components of the network currents must now adjust themselves to meet the conditions imposed by the faults at two points instead of at one point. Many illustrations have already been given showing that it is best to write the symmetrical component equations for the phase that is symmetrically located with respect to the unbalance. Since the fault equations at *both* faults must be written for the *same* phase, it may or may not be possible to choose a phase which is symmetrically located with respect to the unbalance for both faults. Thus, these double faults may be divided into two classes, one for which the symmetry holds and one for which it does not hold. When both faults are symmetrical with respect to the same phase, they belong to the first class. If the phases at one fault are *a*, *b*, *c*, and the corresponding phases at the other fault are *d*, *e*, *f*, the first class of faults will include any one of the following group:

A line-to-ground fault on *a*
A line-to-line fault on *b-c*
A double line-to-ground fault on *b-c*
A line-to-ground fault on *a* simultaneously with a line-to-line fault on *b-c*

and one of the following group:

A line-to-ground fault on *d*
A line-to-line fault on *e-f*
A double line-to-ground fault on *e-f*

A line-to-ground fault on *d* simultaneously with a line-to-line fault on *e-f*.

All other combinations of unbalanced faults belong to the second class. The reason for making this distinction is that it is somewhat simpler to obtain a solution for faults of the first class, either by computation or by measurement on the network analyzer, than it is to obtain a solution for faults of the second class. It is assumed that the faults are not separated by a group of $Y$-$\Delta$-connected transformers. Faults that occur on opposite sides of such a transformer group are discussed in Art. 175, page 529. The general relations between the symmetrical components of the potential and the fault current at *a-b-c* have been given already in Art. 155, page 478. The corresponding relations between the symmetrical components of the potential and fault current at *d-e-f* are obtained for faults belonging to the first class by substituting *d* for *a* in the equations of Art. 155, page 478. When the fault at *a-b-c* is symmetrical with respect to phase *a* and the fault at *d-e-f* is symmetrical with respect to either phase *e* or phase *f*, it belongs to the second class. If, for example, the fault at *d-e-f* is symmetrical about phase *e*, the relation between the symmetrical components of the potential and the fault current for phase *e* are obtained by substituting *e* for *a* in the equations of Art. 155, page 478. These equations must then be transformed and expressed in terms of the symmetrical components of the potential and the fault current for phase *d*. In the following equations it is assumed that the impedance of the fault is zero.

I. *Line-to-line Fault between Conductors f and d.*—The relations for phase *e*

$$I_{e1} + I_{e2} = 0$$
$$V_{e1} = V_{e2}$$

become the relations for phase *d*

$$\left.\begin{aligned} \alpha^2 I_{d1} + \alpha I_{d2} &= 0 \\ \alpha^2 V_{d1} &= \alpha V_{d2} \end{aligned}\right\} \tag{483}$$

II. *Line-to-ground Fault on Conductor e.*—The relations for phase *e*

$$I_{e0} = I_{e1} = I_{e2}$$
$$V_{e0} + V_{e1} + V_{e2} = 0$$

become the relations for phase $d$

$$\left.\begin{aligned} I_{d0} &= \alpha^2 I_{d1} = \alpha I_{d2} \\ V_{d0} + \alpha^2 V_{d1} + \alpha V_{d2} &= 0 \end{aligned}\right\} \tag{484}$$

III. *Line-to-ground Fault on Conductors f and d Simultaneously.* The relations for phase $e$

$$\begin{aligned} I_{e0} + I_{e1} + I_{e2} &= 0 \\ V_{e0} = V_{e1} &= V_{e2} \end{aligned}$$

become the relations for phase $d$

$$\left.\begin{aligned} I_{d0} + \alpha^2 I_{d1} + \alpha I_{d2} &= 0 \\ V_{d0} = \alpha^2 V_{d1} &= \alpha V_{d2} \end{aligned}\right\} \tag{485}$$

IV. *Line-to-ground Fault on Conductor e and Simultaneously a Line-to-line Fault between Conductors f and d.*—The relations for phase $e$

$$\begin{aligned} V_{e0} + V_{e1} + V_{e2} &= 0 \\ V_{e1} &= V_{e2} \\ 2I_{e0} - I_{e1} - I_{e2} &= 0 \end{aligned}$$

become the relations for phase $d$

$$\left.\begin{aligned} V_{d0} + \alpha^2 V_{d1} + \alpha V_{d2} &= 0 \\ \alpha^2 V_{d1} &= \alpha V_{d2} \\ 2I_{d0} - \alpha^2 I_{d1} - \alpha I_{d2} &= 0 \end{aligned}\right\} \tag{486}$$

If the fault is symmetrical with respect to phase $f$ instead of phase $e$, the preceding equations (483), (484), (485), and (486) should be altered by interchanging the rotating operators $\alpha$ and $\alpha^2$.

**164. Zero-sequence and Negative-sequence Equivalent Circuits.**—It has been shown how both the zero- and negative-sequence networks can be reduced to equivalent $\Delta$ or $Y$ circuits whose terminals are the neutral and the two points at which the faults occur. We shall designate these points as $n$, $a$, and $d$ with the proper subscript to denote the sequence of the network. The $Y$ forms of these networks have been chosen and are shown in Fig. 191.[1] The impedances of the branches are indicated,

[1] The choice of the equivalent $Y$ circuit is in accordance with the method proposed by Edith Clark, Simultaneous Faults, *Am. Inst. Elec. Eng.*, **50**, 919, 1931.

and can be computed from the network constants, as discussed in Art. 159, page 487, or can be measured on the network analyzer, by noting the potential, current and power when potential is applied to each pair of terminals successively. If these measured impedances are $Z_{an}$, $Z_{dn}$, $Z_{ad}$, the branch impedances are

$$C = \tfrac{1}{2}(Z_{an} - Z_{dn} + Z_{ad}),$$

$D = \frac{1}{2}(-Z_{an} + Z_{dn} + Z_{ad})$, and $S = \frac{1}{2}(Z_{an} + Z_{dn} - Z_{ad})$. By this procedure the impedances of the branches of the zero- and negative-sequence networks can be determined. We must

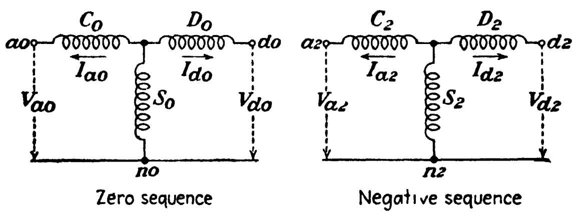



FIG. 191.

now establish the relations between the zero- and negative-sequence components of potential and current demanded by these two networks. This can be done by applying either the Kirchhoff Laws directly or by the method of superposition. The positive direction of the fault current is *toward* the fault as indicated, while the positive direction of the fall in potential is from the line to either neutral $n_0$ or $n_2$. Thus,

$$I_{a0} = \frac{-V_{a0}}{C_0 + \dfrac{D_0 S_0}{D_0 + S_0}} + \frac{V_{d0}}{D_0 + \dfrac{C_0 S_0}{C_0 + S_0}} \frac{S_0}{C_0 + S_0}$$

or

$$I_{a0} = \frac{-V_{a0}(D_0 + S_0) + V_{d0}S_0}{C_0D_0 + D_0S_0 + S_0C_0} \tag{487}$$

Similarly

$$I_{d0} = \frac{-V_{d0}(C_0 + S_0) + V_{a0}S_0}{C_0D_0 + D_0S_0 + S_0C_0} \tag{488}$$

$$I_{a2} = \frac{-V_{a2}(D_2 + S_2) + V_{d2}S_2}{C_2D_2 + D_2S_2 + S_2C_2} \tag{489}$$

$$I_{d2} = \frac{-V_{d2}(C_2 + S_2) + V_{a2}S_2}{C_2D_2 + D_2S_2 + S_2C_2} \tag{490}$$

These equations correspond to equations (476) and (477), page 497, for the case of a single fault.

**165. General Relations between Positive-sequence Components of Potential and Current at the Faults.**—In the case of a single fault, the ratio of the positive-sequence component of the potential at the fault to the positive-sequence component of the fault current is determined by the impedance of a combination of the zero- and negative-sequence networks. When there are two faults, the positive-sequence component of the potential at each of them must depend upon the positive-sequence components of the currents flowing into *both* faults. The method of establishing the relationship between these potentials and currents is similar to that used when there is but a single fault, however. In all there are twelve quantities which must eventually be determined—the three symmetrical components of the potential of the current at each fault. Of these the zero- and negative-sequence components of potential and current can be eliminated from equations (487), (488), (489), (490) and the equations relating the symmetrical components of the potentials at the faults and the fault currents determined by the character of each fault. When this elimination is completed, there will remain two relations between the positive-sequence components of the potentials at the faults and the fault currents, or, in general,

$$V_{a1} = kI_{a1} + mI_{d1} \tag{491}$$
$$V_{d1} = nI_{a1} + lI_{d1} \tag{492}$$

Note that the impedances $k$, $l$, $m$, $n$ depend only upon the impedances in the zero- and negative-sequence networks.

For example, in case there are line-to-line faults of zero impedance between $b$ and $c$ and between $e$ and $f$

$$V_{a1} = V_{a2} \qquad\qquad V_{d1} = V_{d2}$$

and

$$I_{a1} + I_{a2} = 0 \qquad\qquad I_{d1} + I_{d2} = 0$$

If these values of $V_{a2}$, $I_{a2}$, $V_{d2}$, $I_{d2}$ are substituted in equations (489) and (490), there results

$$V_{a1} = (C_2 + S_2)I_{a1} + S_2I_{d1}$$
$$V_{d1} = S_2I_{a1} + (D_2 + S_2)I_{d1}$$

Thus, in this case

$$k = C_2 + S_2 \qquad m = S_2$$
$$n = S_2 \qquad l = D_2 + S_2 \tag{493}$$

Note that $(C_2 + S_2)$ is the impedance of the negative-sequence network measured between $a$ and $n$ and that $(D_2 + S_2)$ is the impedance measured between $d$ and $n$. Again, if there are line-to-ground faults of zero impedance on $a$ and on $d$,

$$V_{a0} + V_{a1} + V_{a2} = 0 \qquad V_{d0} + V_{d1} + V_{d2} = 0$$

and

$$I_{a0} = I_{a1} = I_{a2} \qquad I_{d0} = I_{d1} = I_{d2}$$

If all but the positive-sequence components of potential and current are eliminated from these six equations and the equations (487) to (490) inclusive, there results

$$V_{a1} = (C_0 + S_0 + C_2 + S_2)I_{a1} + (S_0 + S_2)I_{d1}$$
$$V_{d1} = (S_0 + S_2)I_{a1} + (D_0 + S_0 + D_2 + S_2)I_{d1}$$

In this case

$$\left.\begin{aligned} k &= C_0 + S_0 + C_2 + S_2 \qquad m = S_0 + S_2 \\ n &= S_0 + S_2 \qquad l = D_0 + S_0 + D_2 + S_2 \end{aligned}\right\} \tag{494}$$

This elimination is readily accomplished in the following manner. Substitute $I_{a1}$ for $I_{a0}$ and $I_{d1}$ for $I_{d0}$ in equations (487) and (488) and eliminate $V_{d0}$. This gives

$$-V_{a0} = (C_0 + S_0)I_{a1} + S_0 I_{d1}$$

Treat the equations (489) and (490) in a similar manner, giving

$$-V_{a2} = (C_2 + S_2)I_{a1} + S_2 I_{d1}$$

Since

$$V_{a1} = -V_{a0} - V_{a2}$$
$$V_{a1} = (C_0 + C_2 + S_0 + S_2)I_{a1} + (S_0 + S_2)I_{d1}$$

Note that in each of the two preceding cases $m$ and $n$ are equal, which is true of all faults of the first class.

For faults of the second class, in which both the faults are not symmetrical about the same phase, the relations between the symmetrical components of the potentials at the faults and the fault currents given on pages 505 and 506 must be used at one of

the faults. Thus, if there are line-to-line faults of zero impedance between conductors *b-c* and conductors *f-d*

$$I_{a1} + I_{a2} = 0 \qquad \alpha^2 I_{d1} + \alpha I_{d2} = 0$$

and

$$V_{a1} = V_{a2} \qquad \alpha^2 V_{d1} = \alpha V_{d2}$$

If these values of $V_{a2}$, $I_{a2}$, $V_{d2}$, $I_{d2}$ are substituted in equations (487) and (488), page 507, there results

$$\begin{aligned} V_{a1} &= (C_2 + S_2)I_{a1} + \alpha S_2 I_{d1} \\ V_{d1} &= \alpha^2 S_2 I_{a1} + (D_2 + S_2)I_{d1} \\ k &= C_2 + S_2 \qquad m = \alpha S_2 \\ n &= \alpha^2 S_2 \qquad l = D_2 + S_2 \end{aligned} \tag{495}$$

In this case, where one fault is symmetrical about conductor *a* and the other about conductor *e*, the impedances *k* and *l* are the same as if both faults were symmetrical about the same phase but the impedances *m* and *n* are no longer equal. This inequality is due to the effect of the rotating operators $\alpha$ and $\alpha^2$. Note that, if $S_2$ is an inductive impedance, the *resistance* part of $\alpha S_2$ is a negative quantity. If the faults are symmetrical about *a* and *f*, the rotating operators $\alpha$ and $\alpha^2$ should be interchanged. Again, if there are line-to-ground faults of zero impedance on conductors *a* and *e*,

$$V_{a0} + V_{a1} + V_{a2} = 0 \qquad V_{d0} + \alpha^2 V_{d1} + \alpha V_{d2} = 0$$

and

$$I_{a0} = I_{a1} = I_{a2} \qquad I_{d0} = \alpha^2 I_{d1} = \alpha I_{d2}$$

If all but the positive-sequence quantities are eliminated from these six equations and equations (487) and (490) inclusive, there results

$$\begin{aligned} V_{a1} &= (C_0 + S_0 + C_2 + S_2)I_{a1} + (\alpha^2 S_0 + \alpha S_2)I_{d1} \\ V_{d1} &= (\alpha S_0 + \alpha^2 S_2)I_{a1} + (D_0 + S_0 + D_2 + S_2)I_{d1} \end{aligned} \tag{496}$$

If the faults are on conductors *a* and *f*, the rotating operators $\alpha$ and $\alpha^2$ are interchanged but otherwise the relations are unaltered.

**166. Equivalent Fault Circuits.**—When *m* and *n* are equal, equations (491) and (492), page 508, are like those which relate the potentials and currents in a coupled circuit—a transformer, for example. That is, these equations express the relations

existing between the potentials and currents in the circuit of Fig. 192, which is the standard equivalent of a coupled circuit. Thus, if this circuit is connected to the positive-sequence network at the points $a_1$ and $d_1$, where the two unbalanced faults occur, the currents and potentials throughout the network will equal the positive-sequence components of the currents and potentials at the corresponding points in the actual network under the

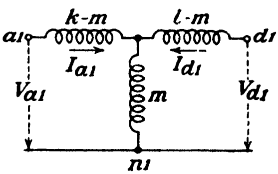


FIG. 192.

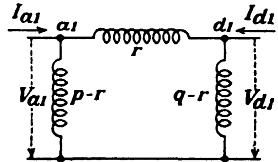


FIG. 193.

unbalanced fault conditions. The general positive-sequence equations can also be written

$$\left.\begin{aligned} I_{a1} &= pV_{a1} - rV_{d1} \\ I_{d1} &= -sV_{a1} + qV_{d1} \end{aligned}\right\} \tag{497}$$

where

$$\left.\begin{aligned} p &= \frac{l}{kl - mn} \qquad & r &= \frac{m}{kl - mn} \\ s &= \frac{n}{kl - mn} \qquad & q &= \frac{k}{kl - mn} \end{aligned}\right\} \tag{498}$$

If $m$ and $n$ are equal, $r$ and $s$ are also equal and these equations lead directly to the equivalent $\pi$ form of the $T$ circuit, Fig. 192. The equivalent $\pi$ circuit is shown in Fig. 193.

For faults belonging to the second class, the impedance constants $m$ and $n$ are not equal and it is not possible to connect the circuit of Fig. 192 to the positive-sequence network as a substitute for the two faults. There are several ways in which this difficulty can be overcome. By adding and subtracting $nI_{a1}$ and $mI_{d1}$ the general positive-sequence equations (491) and (492) become

$$\begin{aligned} V_{a1} &= (k - n)I_{a1} + nI_{a1} + mI_{d1} \\ V_{d1} &= (l - m)I_{d1} + nI_{a1} + mI_{d1} \end{aligned}$$

These equations can be rewritten as

$$V_{a1} = (k - n)I_{a1} + \frac{m+n}{2}(I_{a1} + I_{d1}) + \frac{m-n}{2}(I_{d1} - I_{a1})$$
$$V_{d1} = (l - m)I_{d1} + \frac{m+n}{2}(I_{a1} + I_{d1}) + \frac{m-n}{2}(I_{d1} - I_{a1}) \tag{499}$$

The equivalent circuit corresponding to these equations is shown in Fig. 194. This is similar to the circuit of Fig. 192 except that

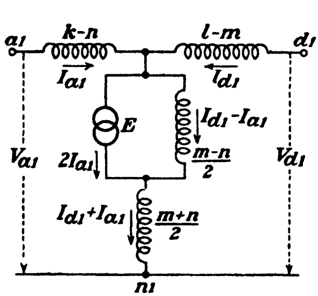


FIG. 194.

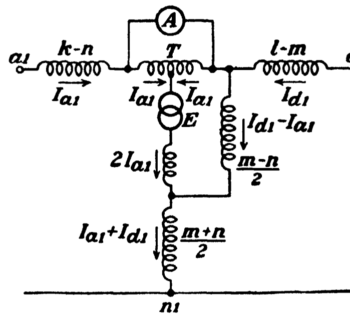


FIG. 195.

the emf $E$ is inserted and adjusted in magnitude and phase until the current through it is $2I_{a1}$. To aid in this adjustment it is convenient to add a stabilizing impedance $Z$ in series with this emf $E$ and a two-to-one autotransformer $T$ and ammeter $A$ as indicated in Fig. 195. When this circuit is properly adjusted, the ammeter $A$ indicates zero. The equivalent $\pi$ circuit of Fig. 193 can also be modified so that it corresponds to equations (497) when the admittances $r$ and $s$ are not equal. The equivalent circuit of Fig. 193 then becomes that of Fig. 196. When the emf $E$ is properly adjusted to equal $2V_{a1}$, both in magnitude and phase, the voltmeter $V$ indicates zero. The modified form of equations (497) which apply to this circuit are

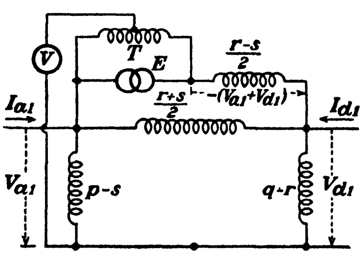


FIG. 196.

$$\left.\begin{aligned} I_{a1} &= (p - s)V_{a1} + \frac{r + s}{2}(V_{a1} - V_{d1}) - \left(\frac{r - s}{2}\right)(V_{a1} + V_{d1}) \\ I_{d1} &= (q - r)V_{d1} - \frac{r + s}{2}(V_{a1} - V_{d1}) + \left(\frac{r - s}{2}\right)(V_{a1} + V_{d1}) \end{aligned}\right\} \tag{500}$$

When either of these circuits (Fig. 195 or Fig. 196) is connected to the positive-sequence network at the fault points, $a_1$ and $d_1$, and neutral $n_1$, the currents and potentials in the network equal the positive-sequence components of the currents and potentials at the corresponding points in the actual network. These equivalent circuits would be used only when the solution of the problem in hand is obtained by means of the network analyzer. A method which may be pursued in computing the symmetrical components of the currents is discussed in Arts. 167 to 170 inclusive.

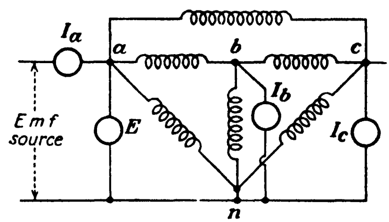


FIG. 197.

**167. Extension of Thévénin's Theorem to Simultaneous Faults.**—The following solution for problems involving simultaneous faults is based upon the direct application of the Superposition Principle after the manner of Thévénin. That is to say, the solution is expressed in terms of the circuit constants and the *prefault line-to-neutral potentials* at the points where the faults occur.

The constants of the equivalent circuits for zero, positive, or negative sequence can be determined either by computation (Art. 159, page 487) or by measurement (Art. 160, page 492). A slightly different method of measuring the branch admittances of the equivalent mesh circuit than the method used in Art. 160, page 492, is to be preferred. The arrangement of the meters is indicated in Fig. 197, in which the neutral of the network is $n$ and the points at which the faults occur are $a$, $b$, $c$. The network may be set up for either zero, negative, or positive sequence, provided all the excitation emfs are withdrawn from the positive-sequence network. It is assumed that the impedances of the ammeters are negligible compared with the impedances of the branches of the equivalent mesh. These branches, *not* the

branches of the actual network, are indicated in Fig. 197. The angles of the admittances can be determined by wattmeters. The values of the branch admittances are

$$\left.\begin{aligned} Y_{ab} &= \frac{I_b}{E} \\ Y_{ac} &= \frac{I_c}{E} \\ Y_{aa} &= Y_{an} + Y_{ab} + Y_{ac} = \frac{I_a}{E} \end{aligned}\right\} \qquad (501)$$

It is important to note, for example, that $I_b$ is the current flowing out of terminal $b$ when potential is applied between terminal $a$ and neutral $n$ and both $b$ and $c$ are short-circuited to neutral. Also note that $Y_{ab}$ and $Y_{ac}$ are the admittances of the mesh branches of the *equivalent* network and are computed by applying the series, the parallel, and the star-mesh methods of reduction to the branches of the *actual* network.

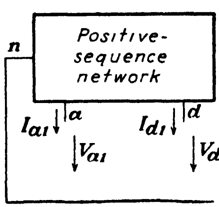


FIG. 198

In Fig. 198, the actual positive-sequence network, together with all the excitation emfs, is assumed to be within the rectangle. The neutral point of the network is $n$ and the points at which there are faults are $a$ and $d$. Owing to these faults there are positive-sequence components of potential and current at both $a$ and $d$, as indicated. The assumed positive directions of both current and fall in potential are from $a$ and $d$ *toward* the neutral $n$.

With *all* the excitation emfs acting, together with the positive-sequence potentials, $V_{a1}$ and $V_{d1}$, at the fault points, $a$ and $d$, the positive-sequence currents at $a$ and $d$ are $I_{a1}$ and $I_{d1}$.

With *all* the excitation emfs acting, but with the points $a$ and $d$ short-circuited *to the neutral* $n$, the currents at $a$ and $d$ are $I_{aE}$ and $I_{dE}$.

With *all* the excitation emfs *withdrawn* from the network and with an applied potential $V_{a1}$ between $a$ and $n$, but with $d$ short-circuited to $n$, the currents flowing *out* of the network at $a$ and $d$ are $-Y_{aa1}V_{a1}$ and $Y_{ad1}V_{a1}$ [see equations (501)].

With *all* the excitation emfs withdrawn from the network and with an applied potential $V_{d1}$ between $d$ and $n$, but with $a$ short-circuited to $n$, the currents flowing out of the network at $a$ and $d$

are $Y_{ad1}V_{d1}$ and $-Y_{dd1}V_{d1}$. These results can be tabulated as follows:

| Potentials acting | | | Currents flowing out of network | |
|---|---|---|---|---|
| Within the network | At $a$ | At $d$ | At $a$ | At $d$ |
| Excitation emfs | $V_{a1}$ | $V_{d1}$ | $I_{a1}$ | $I_{d1}$ |
| Excitation emfs | 0 | 0 | $I_{aE}$ | $I_{dE}$ |
| 0 | $V_{a1}$ | 0 | $-Y_{aa1}V_{a1}$ | $Y_{ad1}V_{a1}$ |
| 0 | 0 | $V_{d1}$ | $Y_{ad1}V_{d1}$ | $-Y_{dd1}V_{d1}$ |

Since the resulting current, when all the potentials are acting, equals the sum of the component currents produced by the individual action of each of the emfs,

$$\left.\begin{aligned} I_{a1} &= I_{aE} - Y_{aa1}V_{a1} + Y_{ad1}V_{d1} \\ I_{d1} &= I_{dE} + Y_{ad1}V_{a1} - Y_{dd1}V_{d1} \end{aligned}\right\} \tag{502}$$

The short-circuit currents $I_{aE}$ and $I_{dE}$ can be determined in terms of the circuit constants and the prefault potentials, $E_a$ and $E_d$, at the points $a$ and $d$. These are the line-to-neutral potentials at $a$ and $d$ prior to the faults and while the system is operating under balanced conditions. The method of applying the Superposition Principle is similar to that employed in deriving Thévénin's Theorem (Art. 159, page 487). The component potentials and currents can be tabulated as follows:

| Potentials acting | | | Currents flowing out of network | |
|---|---|---|---|---|
| Within the network | At $a$ | At $d$ | At $a$ | At $d$ |
| excitation emfs | $E_a$ | $E_d$ | 0 | 0 |
| excitation emfs | 0 | 0 | $I_{aE}$ | $I_{dE}$ |
| 0 | $E_a$ | 0 | $-Y_{aa1}E_a$ | $Y_{ad1}E_a$ |
| 0 | 0 | $E_d$ | $Y_{ad1}E_a$ | $-Y_{dd1}E_a$ |

Whence it follows that

$$\left.\begin{aligned} 0 &= I_{aE} - Y_{aa1}E_a + Y_{ad1}E_d \\ 0 &= I_{dE} + Y_{ad1}E_a - Y_{dd1}E_d \end{aligned}\right\} \qquad (503)$$

On combining equations (502) and (503),

$$I_{a1} = Y_{aa1}E_a - Y_{ad1}E_d - Y_{aa1}V_{a1} + Y_{ad1}V_{d1} \qquad (504)$$

$$I_{d1} = -Y_{ad1}E_a + Y_{dd1}E_d + Y_{ad1}V_{a1} - Y_{dd1}V_{d1} \qquad (505)$$

These are the general relations between the positive-sequence components of current and potential at the two fault points, the potentials at the fault points *before* the faults occur and the admittances of the mesh branches of the equivalent positive-sequence network.

Since neither the zero- nor negative-sequence network has excitation emfs acting within it, the relations for zero sequence corresponding to equations (504) and (504) are

$$I_{a0} = -Y_{aa0}V_{a0} + Y_{ad0}V_{d0} \qquad (506)$$

$$I_{d0} = Y_{ad0}V_{a0} - Y_{dd0}V_{d0} \qquad (507)$$

where the admittances are measured or computed for the zero-sequence equivalent network. The corresponding relations for negative sequence are

$$I_{a2} = -Y_{aa2}V_{a2} + Y_{ad2}V_{d2} \qquad (508)$$

$$I_{d2} = Y_{ad2}V_{a2} - Y_{dd2}V_{d2} \qquad (509)$$

where the admittances are measured or computed for the negative-sequence equivalent network.

The general equations (504) to (509), together with the particular relations or constraints imposed upon the sequence components of the potentials and currents at the faults by the character of the faults, are always sufficient to determine the values of the potentials and currents at the fault points in each of the three component networks. When the potentials or currents at the fault points of any one of the component networks are known, the currents throughout all the branches of that network can be determined.

**168. Zero-sequence Network.**—Since the zero-sequence network has no excitation emfs acting in it, the currents in the different branches can be determined by applying to the network at the points at fault the zero-sequence components of the

potentials at the faults in the actual network. The following is a convenient method for computing the zero-sequence component currents in the branches of the network. The actual zero-sequence network is reduced to the equivalent zero-sequence network by applying the star-mesh, the series, and the parallel transformations in a step-by-step process until the final equivalent circuit, Fig. 199, is reached. In this circuit, let the branch admittances be

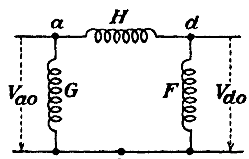


FIG. 199.

$$Y_{an} = G$$
$$Y_{dn} = F$$
$$Y_{ad} = H$$

The zero-sequence currents in the branches are

$$I_{an} = GV_{a0}$$
$$I_{dn} = FV_{d0}$$
$$I_{ad} = H(V_{a0} - V_{d0})$$

Assume that, in the step-by-step process of reduction, the final equivalent circuit of Fig. 199 was derived from the circuit of Fig. 200. In Fig. 200, the admittances of the parallel branches between $a$ and $n$ are $M$ and $N$, the admittances of the parallel branches between $d$ and $n$ are $P$ and $Q$, and the admittances of the parallel branches between $a$ and $d$ are $K$ and $L$.

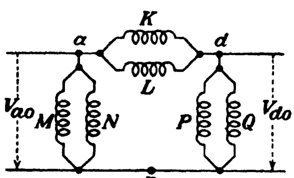


FIG. 200.

Note that

$$G = M + N$$
$$F = P + Q$$
$$H = K + L$$

The currents in the branches whose admittances are $M$, $P$, and $K$ are

$$\frac{M}{G}I_{an} = MV_{a0}$$
$$\frac{P}{F}I_{dn} = PV_{d0}$$
$$\frac{K}{H}I_{ad} = K(V_{a0} - V_{d0})$$

Assume that the branches whose admittances are $M$, $P$, and $K$, (Fig. 200), were derived from star branches $sa$, $sd$, and $sn$ (Fig. 201). In Fig. 201, the current $I_{as}$ in the star branch, $as$, is

$$I_{as} = I_{an} + I_{ad}$$

or

$$I_{as} = MV_{a0} + K(V_{a0} - V_{d0})$$

Similarly, the current in the star-branch, $ds$, is

$$\begin{aligned} I_{ds} &= I_{da} + I_{dn} \\ &= -K(V_{a0} - V_{d0}) + PV_{d0} \\ &= -KV_{a0} + (K + P)V_{d0} \end{aligned}$$

Also, the current in the star-branch, $ns$, is

$$\begin{aligned} I_{ns} &= I_{na} + I_{nd} \\ &= -MV_{s0} - PV_{d0} \end{aligned}$$

By continuing this step-by-step synthesizing process, the arrangement of the branches in the actual network will be reached and the zero-sequence current in each of the branches will be determined.

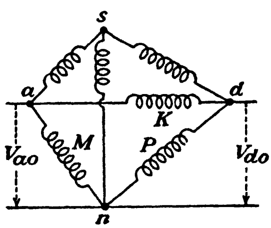


FIG. 201.

**169. Negative-sequence Network.**—Since the negative-sequence network has no excitation emfs acting in it, the negative-sequence currents in the different branches can be determined by a synthesizing process like that used in determining the zero-sequence currents.

**170. Positive-sequence Network.**—Since the positive-sequence network has excitation emfs acting within it, the currents in the different branches *cannot* be determined by the same process that is used in determining the zero-sequence and negative-sequence component currents. The positive-sequence currents are due to the action of the excitation emfs and the positive-sequence components, $V_{a1}$ and $V_{d1}$, of the line-to-neutral potentials at the points where the faults occur. It can be assumed that these component potentials are of the form

$$\begin{aligned} V_{a1} &= E_a + (V_{a1} - E_a) \\ V_{d1} &= E_d + (V_{d1} - E_d) \end{aligned}$$

where $E_a$ and $E_d$ are the line-to-neutral potentials at the fault points *before* the faults occur. In applying the principle of

superposition to the positive-sequence network, it can then be assumed the positive-sequence currents are due to two sets of emfs. The excitation emfs and the prefault emfs, $E_a$ and $E_d$, comprise the first set, and the *differences*, $(V_{a1} - E_a)$ and $(V_{d1} - E_d)$, between the positive-sequence components of the potentials at the faults and the prefault potentials at the same points, comprise the second set. The currents due to the action of the *first* set of emfs are the *balanced currents* flowing in the network *before* the faults occur. The balanced currents in the different branches of the network can be determined at the same time the prefault line-to-neutral potentials, $E_a$ and $E_d$, are determined. The currents due to the action of the *second* set of emfs are determined by applying the potentials $(V_{a1} - E_a)$ and $(V_{d1} - E_d)$ to the positive-sequence network at the fault points when *all* the *excitation emfs* are withdrawn from the network. Therefore, the second set of positive-sequence currents can be determined by the same synthesizing process that is used in determining the zero- and negative-sequence component currents. The resulting positive-sequence current in any branch of the network is the *sum* of the two positive-sequence components in that branch due to the action of the first set of emfs and the second set of emfs. The resulting positive-sequence current in any branch equals the sum of the current flowing in that branch *before* the faults occur and the positive-sequence current in that branch due to the action of the potentials $(V_{a1} - E_a)$ and $(V_{d1} - E_d)$ applied to the positive-sequence network at the points of fault when all of the excitation emfs are withdrawn. Consider the following illustrations.

I. *Line-to-line Faults on the Same Phase at a and at d.*—The constraints imposed by the faults are

$$V_{a1} = V_{a2} \qquad V_{d1} = V_{d2}$$
$$I_{a1} + I_{a2} = 0 \qquad I_{d1} + I_{d2} = 0$$

These equations apply to the line conductor *not* involved in the fault. If these relations between the positive- and negative-sequence components are substituted in equations (504), (505), (508), and (509),

$$(Y_{aa1} + Y_{aa2})V_{a1} - (Y_{ad1} + Y_{ad2})V_{d1} = Y_{aa1}E_a - Y_{ad1}E_d$$
$$-(Y_{ad1} + Y_{ad2})V_{a1} + (Y_{dd1} + Y_{dd2})V_{d1} = -Y_{ad1}E_a + Y_{dd1}E_d$$

The solution of these equations determines the positive-sequence components, $V_{a1}$ and $V_{d1}$, of the line-to-neutral potentials at the two faults.

II. *Line-to-neutral Fault on Phase a at the Point a and Line-to-neutral Fault on Phase e at the Point d.*—The constraints imposed by the faults are

$$V_{a0} + V_{a1} + V_{a2} = 0 \qquad (510)$$

$$I_{a0} = I_{a1} = I_{a2} \qquad (511)$$

$$V_{d0} + \alpha^2 V_{d1} + \alpha V_{d2} = 0 \qquad (512)$$

$$I_{d0} = \alpha^2 I_{d1} = \alpha I_{d2} \qquad (513)$$

If $I_{a1}$ and $I_{a2}$ are eliminated from equations (504) and (508) by means of equation (511),

$$Y_{aa1}V_{a1} - Y_{aa2}V_{a2} - Y_{ad1}V_{d1} + Y_{ad2}V_{d2} = Y_{aa1}E_a - Y_{ad1}E_d \qquad (514)$$

If $I_{d1}$ and $I_{d2}$ are eliminated from equations (505) and (509) by means of equation (513),

$$Y_{ad1}V_{a1} - \alpha^2 Y_{ad2}V_{a2} - Y_{dd1}V_{d1} + \alpha^2 Y_{dd2}V_{d2} = Y_{ad1}E_a - Y_{dd1}E_d \qquad (515)$$

If $V_{a0}$ and $V_{d0}$ are eliminated from equation (506) by means of equations (510) and (512), and if $I_{a0}$ and $I_{a2}$ are eliminated from equations (506) and (508) by means of equation (511),

$$Y_{aa0}V_{a1} + (Y_{aa0} + Y_{aa2})V_{a2} - \alpha^2 Y_{ad0}V_{d1} - (\alpha Y_{ad0} + Y_{ad2})V_{d2} = 0 \qquad (516)$$

By a similar process equations (507) and (509) are reduced to

$$Y_{ad0}V_{a1} + (Y_{ad0} + \alpha Y_{ad2})V_{a2} - \alpha^2 Y_{dd0}V_{d1} - \alpha(Y_{dd0} + Y_{dd2})V_{d2} = 0 \qquad (517)$$

Substitute the vector values of the admittances and of the rotating operators, $\alpha$ and $\alpha^2$, in equations (514), (515), (516), and (517). Solve the equations by a systematic method of elimination. For example, in order to eliminate $V_{a1}$, divide each equation by the coefficient of $V_{a1}$ and subtract the second equation from the first, the third equation from the second and the fourth equation from the third. Eliminate $V_{a2}$ in a similar manner from the three resulting equations and obtain two equations for $V_{d1}$ and $V_{d2}$. Eliminate $V_{d1}$ and so determine $V_{d2}$. Substitute this

value of $V_{d2}$ in either one of the last two equations and so determine $V_{d1}$. Substitute the values of $V_{d1}$ and $V_{d2}$ in any one of the group of three equations, obtained by eliminating $V_{a1}$, and so determine $V_{a2}$. Substitute the values of $V_{a2}$, $V_{d1}$, and $V_{d2}$ in one of the original equations and so determine $V_{a1}$ The entire process of solving the equations in this manner requires thirty-four operations of multiplication or division and twenty-one operations of addition or subtraction. If the equations are set down in the order (514), (516), (517), and (515), and the same process of elimination is carried out, the entire solution requires thirty-four operations of multiplication or division and nineteen operations of addition or subtraction. Furthermore the expressions on the right of the equality signs can be kept in their polar forms.

**171. Interconnection of Sequence Networks.**—It is desirable, if possible, to interconnect the sequence networks in such a way that they will represent the fault conditions, just as in the case of a single fault. It is necessary merely to make this interconnection so that the proper relations exist between the symmetrical components of the potential and the fault current at each fault. When the faults are not symmetrical about the same phase it is necessary to make the connections through a phase-shifting device in order to take account of the effect of the rotating operator $\alpha$, which then appears in the relations.[1] Sometimes it is necessary to connect the networks through one-to-one transformers to avoid short-circuiting parts of the networks. The methods of making the proper connections are shown in the following diagrams. As an aid in comparing the diagrams with the fault equations, the positive directions of the potentials and currents are indicated by arrows.

We shall first consider cases in which the faults are symmetrical about the same phase. Each network is represented by a rectangle, as in Fig. 187. The neutral conductor and the points on the line conductors at which the faults occur are $n$, $a$, $d$, with a subscript corresponding to the sequence of the network. The currents flowing from the network toward the fault are $I_a$ and $I_d$, while the fall in potential from line $a$ to the neutral is $V_a$ and from line $d$ to the neutral is $V_d$, as indicated by the

[1] KIMBARK, E. W., Experimental Analysis of Double Unbalances, *Am. Inst. Elec. Eng.*, **54**, 159, 1935.

arrows. This nomenclature, with subscripts indicating the sequence, is used in each of the component networks.

I. *Line-to-line Faults between b-c, and between e-f.*

$$I_{a1} + I_{a2} = 0 \qquad I_{d1} + I_{d2} = 0$$
$$V_{a1} = V_{a2} \qquad V_{d1} = V_{d2}$$

The zero-sequence network is not involved. It is easily seen that the potentials and currents in Fig. 202 satisfy the necessary conditions as expressed in the fault equations.

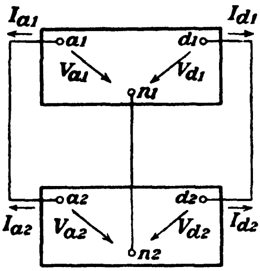



FIG. 202.

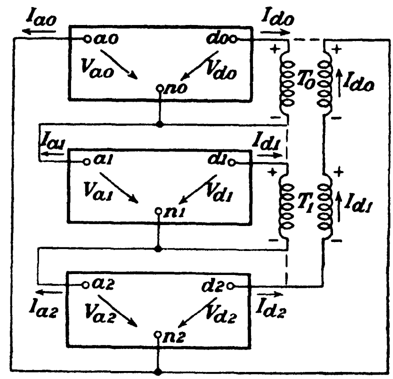



FIG. 203.

II. *Line-to-ground Faults on a and on d.*

$$I_{a0} = I_{a1} = I_{a2} \qquad I_{d0} = I_{d1} = I_{d2}$$
$$V_{a0} + V_{a1} + V_{a2} = 0 \qquad V_{d0} + V_{d1} + V_{d2} = 0$$

Were there but a *single* fault at *a*, the connections between the networks would be the same as in Fig. 189, page 501. If, in Fig. 203, the second fault at *d* were represented by a similar connection of the sequence networks, the points *a* and *d* in *each* of the networks would be short-circuited. This is shown by replacing the transformers $T_0$ and $T_1$ by the connections indicated by the dotted lines. These short circuits would introduce relations that do not actually exist, *viz.* that $V_{a0} = V_{d0}$, $V_{a1} = V_{d1}$ and $V_{a2} = V_{d2}$. It is for this reason that the transformers are necessary. Each of the transformers has a 1/1 ratio of transformation, and their corresponding terminals are marked with + signs. The positive directions of their secondary currents and potentials are correctly indicated by the arrows and

+ and − signs in Fig. 203. It is evident that the necessary relations between the symmetrical components of potentials at the faults and the fault currents are satisfied at both faults.

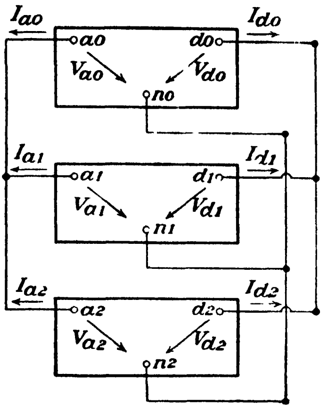


FIG. 204.

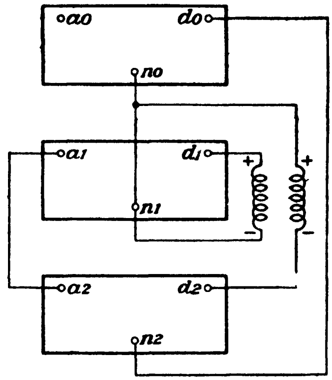


FIG. 205.

It is essential that the transformers should have relatively small copper and core losses, a small magnetizing current, and little magnetic leakage.

III. *Double Line-to-ground Faults on b and c and on e and f.*

$$I_{a0} + I_{a1} + I_{a2} = 0 \qquad I_{d0} + I_{d1} + I_{d2} = 0$$
$$V_{a0} = V_{a1} = V_{a2} \qquad V_{d0} = V_{d1} = V_{d2}$$

Observe that the interconnections of the sequence networks shown in Fig. 204 satisfy these equations.

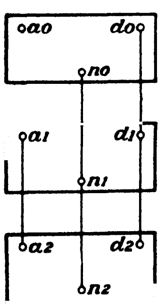


FIG. 206.

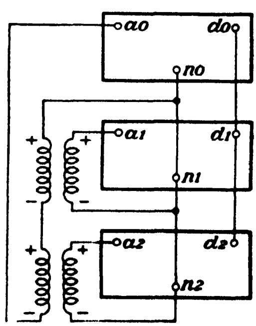


FIG. 207.

IV. Figure 205 shows the interconnections for a line-to-line fault between *b* and *c* and a line-to-ground fault on *d*.

V. Figure 206 shows the interconnections for a line-to-line fault between *b* and *c* and a double-line-to-ground fault on *e* and *f*.

VI. Figure 207 shows the interconnections for a line-to-ground fault on *a* and a double line-to-ground fault on *e* and *f*.

**172. Phase-shifting Device.**—When both faults are not symmetrical about the same phase, it has been suggested that the correct relations between the potentials and currents at one of the faults can be obtained by interconnecting the networks at this fault point through a phase-shifting device. In such a device, both the potentials and currents should be advanced or retarded in time-phase by 120 degrees without altering their magnitudes. Thus, if the rectangle in Fig. 208 incloses the phase shifting device,

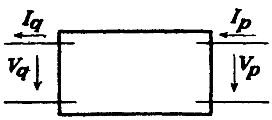


FIG. 208.

$$V_q = V_p\underline{/120^\circ} \qquad \text{or} \qquad V_q = V_p\underline{/240^\circ}$$
$$I_q = I_p\underline{/120^\circ} \qquad\qquad\quad I_q = I_p\underline{/240^\circ}$$

To be perfectly satisfactory, such a device should accomplish the result automatically. As an illustration of the use of such a

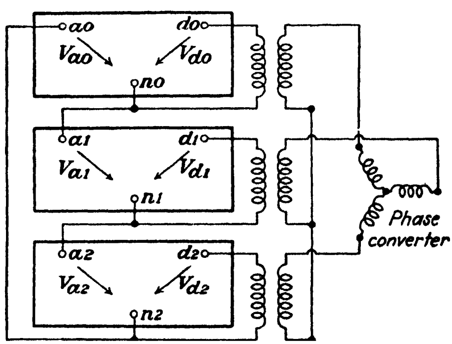


FIG. 209.

device, consider line-to-ground faults on *a* and on *e*. The fault equations are

$$I_{a0} = I_{a1} = I_{a2} \qquad I_{d0} = \alpha^2 I_{d1} = \alpha I_{d2}$$
$$V_{a0} + V_{a1} + V_{a2} = 0 \qquad V_{d0} + \alpha^2 V_{d1} + \alpha V_{d2} = 0$$

The interconnection of the component networks through phase-shifting devices in order to satisfy the fault conditions is shown in Fig. 209.

**173. Equivalent Positive-sequence Network.**—It has been shown (Art. 161, page 496) that, when there is a single shunt fault on the system, the positive-sequence component of the fault current can always be computed on the assumption that the positive-sequence network consists of a single emf in series with an impedance. It can also be shown in a similar manner

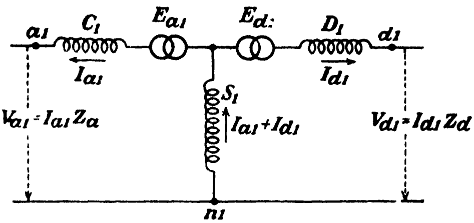



FIG. 210.

that when there are two simultaneous shunt faults occurring at different points, the *equivalent* positive-sequence network consists of two emfs and three impedances arranged as in Fig. 210. In the positive-sequence network of Fig. 211, the neutral conductor is $n1$ and the points at which the faults occur are $a1$ and $d1$. Before the impedances $Z_a$ and $Z_d$ are connected to the network, the line-to-neutral potentials at these points are $E_{a1}$ and $E_{d1}$. These are the line-to-neutral potentials which would be measured at the corresponding points in the actual network just before the faults occur. They are due to the action of the excitation emfs generated in the synchronous machines which are connected to the network.

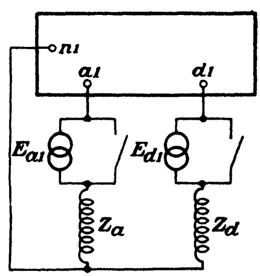



FIG. 211.

After the impedances $Z_a$ and $Z_d$ are connected to the network, the line-to-neutral potentials at these points become $V_{a1}$ and $V_{d1}$ and the currents in the impedances are $I_{a1}$ and $I_{d1}$. For the circuit condition shown in Fig. 211, it can be assumed that the currents in these impedances, even though the currents are both zero, are due to the action of the excitation emfs in the network and the emfs $E_{a1}$ and $E_{d1}$. By the principle of superposition each of these currents equals the sum of two components. The *first* of these components is due to the action of *all* the excitation emfs when $E_{a1}$ and $E_{d1}$ are withdrawn, and

the *second* is due to the action of $E_{a1}$ and $E_{d1}$ when all the excitation emfs are withdrawn. The *first* of these components are $I_{a1}$ and $I_{d1}$. When the network emfs are withdrawn, the network becomes passive and as far as the terminals $a_1$, $d_1$, $n_1$ are concerned can be reduced to an equivalent circuit of three branches connected either in $T$ or in $\pi$. Let us choose the equivalent $T$ form and designate the impedances of its branches as $C_1$, $D_1$, $S_1$, corresponding to the nomenclature used in the zero- and negative-sequence networks. These impedances can be computed or measured by the same methods that were used to determine the corresponding impedances in the zero- and negative-sequence networks. The currents flowing into this equivalent network (Fig. 212) at the points $a_1$ and $d_1$, due to the action of the emfs $E_{a1}$ and $E_{d1}$, can then be computed. Since the *second* components are opposite to the *first* and since the sums of the two components, both at $a_1$ and $d_1$, are zero, it is evident that the currents due to the action of the excitation emfs alone and flowing *out* of the network at $a_1$ and $d_1$ into $Z_a$ and $Z_d$ equal the currents that would flow *into* the network at these points, due to the action of $E_{a1}$ and $E_{d1}$ alone. It at once follows that, with respect to the points $a_1$ and $d_1$, the positive-sequence network, including all of its excitation emfs, is equivalent to the circuit shown in Fig. 210. It will be observed of this circuit that

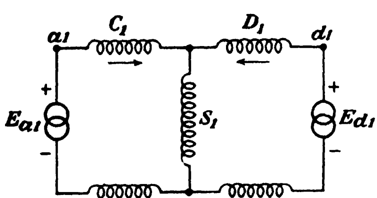


FIG. 212.

$$V_{a1} = E_{a1} - I_{a1}(C_1 + S_1) - I_{d1}S_1 \tag{518}$$
$$V_{d1} = E_{d1} - I_{d1}(D_1 + S_1) - I_{a1}S_1 \tag{519}$$

Compare these equations with equation (479), page 498. If these equations (518) and (519) are combined with the general positive-sequence equations (491) and (492), page 508, the positive-sequence components of the potentials and currents at the faults can be determined. Thus,

$$I_{a1} = \frac{(l + D_1 + S_1)E_{a1} - (m + S_1)E_{d1}}{(k + C_1 + S_1)(l + D_1 + S_1) - (m + S_1)(n + S_1)} \tag{520}$$
$$I_{d1} = \frac{(k + C_1 + S_1)E_{d1} - (n + S_1)E_{a1}}{(k + C_1 + S_1)(l + D_1 + S_1) - (m + S_1)(n + S_1)} \tag{521}$$

$$V_{a1} = \frac{[k(l + D_1 + S_1) - m(n + S_1)]E_{a1} + [m(C_1 + S_1) - kS_1]E_{d1}}{(k + C_1 + S_1)(l + D_1 + S_1) - (m + S_1)(n + S_1)} \tag{522}$$

$$V_{d1} = \frac{[l(k + C_1 + S_1) - n(m + S_1)]E_{d1} + [n(D_1 + S_1) - lS_1]E_{a1}}{(k + C_1 + S_1)(l + D_1 + S_1) - (m + S_1)(n + S_1)} \tag{523}$$

It is thus possible to compute the positive-sequence components of the potentials at each fault and the positive-sequence components of the currents flowing into each fault in terms of the impedances of the branches of the equivalent zero-, positive-, and negative-sequence networks and the line-to-neutral potentials at the fault points existing *before* the faults occur. These equations also enable us to determine two impedances, $V_{a1}/I_{a1}$ and $V_{d1}/I_{d1}$. When these impedances are connected to the positive-sequence network at the points where the faults occur, the currents and potentials throughout the network equal the positive-sequence components of the currents and potentials at the corresponding points in the actual network. The values of these impedances are readily computed. Rewrite the general positive-sequence equations (491) and (492) as

$$Z_a = \frac{V_{a1}}{I_{a1}} = k + m\frac{I_{d1}}{I_{a1}}$$

$$Z_d = \frac{V_{d1}}{I_{d1}} = l + n\frac{I_{a1}}{I_{d1}}$$

If the ratios of the positive-sequence components of the fault currents as determined by equations (520) and (521) are inserted in the equations

$$Z_a = k - m\frac{A(n + S_1) - (k + C_1 + S_1)}{A(l + D_1 + S_1) - (m + S_1)} \tag{524}$$

$$Z_d = l - n\frac{A(l + D_1 + S_1) - (m + S_1)}{A(n + S_1) - (k + C_1 + S_1)} \tag{525}$$

where $A = E_{a1}/E_{d1}$.

Since the zero- and negative-sequence networks include no excitation emfs, the total power *dissipated* in them must be positive. Thus the resistances of $Z_a$ and $Z_d$ cannot both be

negative. This may prove of considerable advantage when the solution is obtained by means of the network analyzer. Furthermore, it may be somewhat simpler to compute the positive-sequence components of potential and current when these individual impedances are connected to the network than it is by the other methods that have been suggested.

It is interesting to note that should the faults occur where the prefault line-to-neutral potentials are equal, both in magnitude and phase, the ratio of the positive-sequence components of the fault currents is

$$\frac{I_{a1}}{I_{d1}} = -\frac{l - m + D_1}{k - n + C_1}$$

**174. Negative Resistance.**—One of the difficulties encountered in setting up the equivalent fault circuit, connected to the positive-sequence network in place of the fault itself, is that the resistances of some of its branches may be negative. The resistances of the branches of the actual power network are always positive. When such branches are combined either in series or parallel, the resistance in the resulting impedance is always positive. However, by the $Y$-$\Delta$ or by the more general star-mesh transformation, a group of branches, all of which have positive resistances, are equivalent to another group of branches, some of which may have *negative* resistances. Furthermore, the process of deriving the impedances $k$, $l$, $m$, $n$ in equations (491) and (492) may result in some of these impedances having negative resistances. The difficulty in *computing* the fault currents and potentials is in no wise increased if some of the impedances involved have negative resistances, but this is not true when the solution is obtained from the network analyzer. It is then necessary to represent in miniature an impedance whose resistance is negative. There are several methods of accomplishing this result.

I. The impedance in question can be replaced by a device, such as a polyphase induction regulator, in which there is generated an emf whose magnitude and phase can be varied at will. It is then possible to adjust this emf so that there is the desired vector relation between the terminal potential of the device and the current through it. For example, if the fall in potential $V$ through the device leads the current $I$ as indicated

in Fig. 213, the device represents a negative resistance in series with an inductive reactance.

II. If the impedance has the form $-R + jX$, it can be replaced by an impedance $R - jX$ arranged as shown in Fig. 214. The transformer $T$ has two equal coils. $Z$ is a stabilizing impedance in series with an emf $E$. The magnitude and phase of this emf are adjusted so that the ammeter $A$ indicates zero. If the impedance of the ammeter is zero, the fall in potential in the direction of the current from $a$ to $b$ is $-I_{ab}(R - jX)$ or

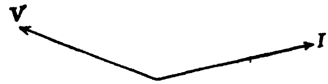


FIG. 213.

$$I_{ab}(-R + jX).$$

III. If in Fig. 192, page 511, the impedance $m$ has a negative resistance, the circuit can be modified as shown in Fig. 215,

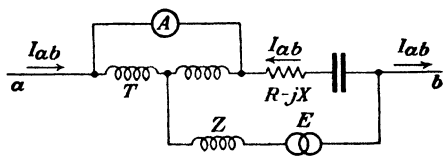


FIG. 214.

in which the impedance $-m$ has a positive resistance. The transformer $T$ has a 1:1 ratio and is used to invert both the potential and current as indicated. It will be noticed that the expressions for the potentials $V_{an}$ and $V_{dn}$ derived from Fig. 192 are identical with those derived from Fig. 215. It is evident that this method possesses no advantage if the values of $k$, $l$, $m$ are such that the resistance of either $k + m$ or $l + m$ is negative.

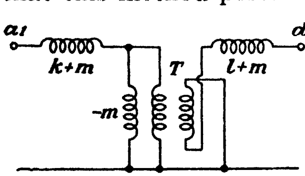


FIG. 215.

**175. Faults on Opposite Sides of a Y-Δ Bank of Transformers.**—There are two unique points of interest in regard to this arrangement of transformers.

I. Since no zero-sequence current can flow from the terminals of the delta, and since, therefore, zero-sequence current cannot be transmitted from one side of the transformers to the other, it is evident that the zero-sequence network consists of two independent parts, as in Fig. 216. If the impedances of these

independent parts are $A_0$ and $B_0$, the zero-sequence equations are

$$\left.\begin{aligned} V_{a0} &= -I_{a0}A_0 \\ V_{d0} &= -I_{d0}B_0 \end{aligned}\right\} \qquad (526)$$

These simpler relations replace equations (487) and (488), page 507, in the determination of the general fault impedances $k$, $l$, $m$, $n$.

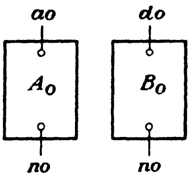


FIG. 216.

II. The relation between the positive- and negative-sequence components of both the line-to-neutral potentials and the line currents are not the same on the primary and secondary sides of the transformers.

It was shown in Art. 75, page 198, that it is best to designate the line terminals on the two sides of the transformers so that the line-to-neutral potentials are related as follows

$$\begin{aligned} E_{a1} &= -jkE_{d1} \\ E_{a2} &= jkE_{d2} \end{aligned}$$

where $E_a$ is the line-to-neutral potential on the $Y$ side, $E_d$ is the corresponding potential on the $\Delta$ side, and $k$ is the ratio of the line potentials. Since in the positive- and negative-sequence networks, all the line potentials are reduced to a common base, $k$ equals unity. Assume that at some point the positive-sequence network is divided into two independent parts by a single group of $Y$-$\Delta$-connected transformers, as in Fig. 217. Assume also that the faults at $a1$ and $d1$ are symmetrical with respect to phases of which it is true that $E_{p1} = -jE_{q1}$, where $E_{p1}$ and $E_{q1}$ are the line-to-neutral potentials at the terminals of the transformer group at no load. There is a similar relation between the line currents at $p$ and $q$, or $I_{p1} = -jI_{q1}$. Further, assume that under the given fault conditions the actual potential at $d1$ is $V_{d1}'$, and that the actual fault current is $I_{d1}'$. It is evident that, if the potentials $E_{q1}$, $V_{d1}'$ and all the excitation emfs in the right-hand portion of the network are retarded by 90 degrees in time-phase, the *potentials* and *currents* at $p$ and $q$ will be equal. An equivalent single positive-sequence

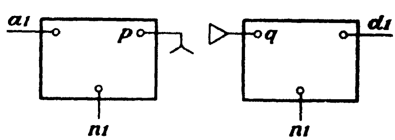


FIG. 217.

diagram can then be drawn as indicated in Fig. 218. Observe that the potential and current at the point $d_1$ are the actual potentials at the fault and the actual fault current each retarded by 90 degrees in time-phase.

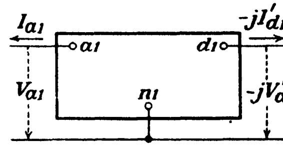



FIG. 218.

The negative-sequence network can be treated in a similar manner except that the potential at the fault and the fault current at $d_2$ are advanced by 90 degrees in time-phase. If $V_{d1}$, $I_{d1}$, $V_{d2}$, $I_{d2}$ are the potentials at the fault and the fault currents at $d$ in the positive- and negative-sequence *equivalent* networks,

$$V_{d1} = -jV_{d1}'$$
$$I_{d1} = -jI_{d1}'$$
$$V_{d2} = jV_{d2}'$$
$$I_{d2} = jI_{d2}'$$

The relations between $V_{d1}'$, $I_{d1}'$, $V_{d2}'$, $I_{d2}'$ are determined by the type of fault. Thus, for a line-to-line fault on $e$-$f$, when

$$V_{d1}' = V_{d2}' \qquad \text{and} \qquad I_{d1}' + I_{d2}' = 0$$
$$V_{d1} + V_{d2} = 0 \qquad \text{and} \qquad I_{d1} = I_{d2}$$

Again, for a line-to-ground fault on $d$, when

$$V_{d0} + V_{d1}' + V_{d2}' = 0 \qquad \text{and} \qquad I_{d0} = I_{d1}' = I_{d2}',$$

we have

$$V_{d0} + jV_{d1} - jV_{d2} = 0 \tag{527}$$
$$I_{d0} = jI_{d1} = -jI_{d2} \tag{528}$$

These values of the potentials and currents at $d$ in the *equivalent* positive- and negative-sequence networks should be used in determining the general impedance constants $k$, $l$, $m$, $n$ when the faults are on opposite sides of a $Y$-$\Delta$-transformer group. For example, if there are line-to-line faults on $b$-$c$ and $e$-$f$, the values of these constants are

$$k = C_2 + S_2 \qquad m = -S_2$$
$$l = D_2 + S_2 \qquad n = -S_2$$

Again, if there are line-to-ground faults on $a$ and on $d$, the constants are

$$k = A_0 + C_2 + S_2 \qquad m = -S_2$$
$$l = B_0 + D_2 + S_2 \qquad n = -S_2$$

Compare these values with those given in Art. 165, page 508.

If the fault at the point $d$ is symmetrical with respect to conductor $e$ rather than conductor $d$, the relations between the potentials and currents at $d$ should be further modified by the rotating operators $\alpha$ and $\alpha^2$ as indicated in Art. 165, page 508.

When one has determined the positive- and negative-sequence components of potential and current at any point in the equivalent networks on the $\Delta$ side of the $Y$-$\Delta$-transformer group, the actual values of these components are found by advancing the positive-sequence quantities 90 degrees and retarding the negative-sequence quantities 90 degrees in time-phase. The proper combination of these (actual) symmetrical components, together with the zero-sequence components, will give the resulting potential and current for each phase at this point in the network.

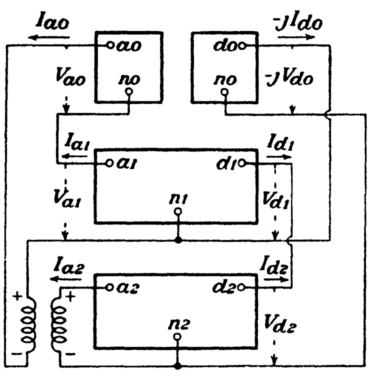



FIG. 219

It is interesting to note how the different sequence networks can be interconnected to represent the unbalanced fault conditions. If equations (527) and (528) are multiplied by $(-j)$, there results

$$-jV_{d0} + V_{d1} - V_{d2} = 0 \qquad (529)$$
$$-jI_{d0} = I_{d1} = -I_{d2} \qquad (530)$$

Since the zero-sequence network consists of two entirely independent parts, the zero-sequence current $I_{d0}$ will be retarded 90 degrees in time-phase if the zero-sequence potential $V_{d0}$ is retarded 90 degrees in time-phase. This agrees with equations (529) and (530). These equations show also that the connection between the positive- and negative-sequence equivalent networks should be made so that the potential and current at $d_2$ are reversed, as indicated in Fig. 219, in which it is assumed that there are line-to-ground faults on conductors $a$ and $d$. Compare

this with Fig. 203, page 522, where the faults are *not* separated by a $Y$-$\Delta$ group of transformers.

**176. Ungrounded Systems.**—Every power network is grounded through the transmission line capacitance to earth. Since, however, the impedance of this capacitance is relatively high, it may be desirable in many cases to assume that it is infinite. If the neutral points of the transformer groups and of the synchronous machines are not grounded, the system can then be treated as one having an isolated neutral. Under these conditions, a single line-to-ground fault will produce no other effect than to disturb the balance of the line-to-ground potentials. From this it follows that a line-to-line fault at $a$ and a simultaneous single line-to-ground fault at $d$ should be treated as a single line-to-line fault by the method described in Art. 161, page 496. When there are two simultaneous line-to-ground faults occurring at different points, the zero-sequence current at one point is equal and opposite to that at the other point. Therefore, it is evident that the zero-sequence network can be reduced to a single impedance connecting the two fault points. That is to say, the impedance $S_0$ of the equivalent zero-sequence network, Fig. 191, page 507, is infinite. Thus, the values of the general equivalent impedances $k$, $l$, $m$, $n$ can at once be determined by assuming that $S_0$ becomes infinite in the expressions for these impedances given in Art. 165, page 508. For example, if there are double line-to-ground faults on conductors $b$ and $c$ and on conductors $e$ and $f$, the limiting values of $k$, $l$, $m$, $n$ are

$$\left.\begin{aligned} k = l &= S_2 + \frac{C_2(C_0 + D_0 + D_2)}{C_0 + D_0 + C_2 + D_2} \\ m = n &= \frac{S_2(C_0 + D_0)}{C_0 + D_0 + C_2 + D_2} \end{aligned}\right\} \qquad (531)$$

There is some difficulty in this procedure when there are single line-to-ground faults at both $a$ and $d$, since in this case $k$, $l$, $m$, $n$ become infinite as $S_0$ becomes infinite. If the faults are on conductors $a$ and $d$, the difficulty is easily overcome by observing that the branches of the equivalent fault network, Fig. 192, page 511, $(k - m)$, $(l - m)$, and $m$ have the following values

$$\left.\begin{aligned} k - m &= C_0 + C_2 \\ l - m &= D_0 + D_2 \\ m &= S_0 + S_2 \end{aligned}\right\} \qquad (532)$$

As $S_0$ becomes infinite this network reduces to a single impedance $C_0 + C_2 + D_0 + D_2$ connecting the points at which the faults occur. If the faults are on conductors $a$ and $e$, it is best to determine the limiting values of the equivalent fault impedances $Z_a$ and $Z_b$ [equations (524) and (525), page 527] as $S_0$ becomes infinite. If the values of $k$, $l$, $m$, $n$, in equations (496), page 510, are substituted in these equations and it is assumed that $S_0$ is infinite, the limiting values of $Z_a$ and $Z_d$ become

$$\left.\begin{aligned} Z_a &= C_0 + C_2 + (1 - \alpha^2)S_2 \\ Z_d &= D_0 + D_2 + (1 - \alpha)S_2 \end{aligned}\right\} \qquad (533)$$

where $\alpha$ is the rotating operator for 120 degrees.

**177. Simultaneous Faults on Individual Lines.**—Consider a distribution system which has two individual lines radiating

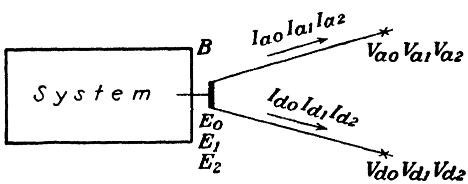



FIG. 220.

from a common bus point $B$, as in Fig. 220. Assume that these lines are unloaded and ungrounded and that there are simultaneous faults occurring on them at $a$ and at $d$. Owing to these faults the potentials and currents at $B$ will be unbalanced and the "system" will react as if an unsymmetrical load were connected to it at $B$. That is, the two faults at $a$ and $d$ can be replaced by a single fault at $B$. We will now determine the value of the impedance which should be connected to the positive-sequence network at $B$ to represent this single fault. Let $E_0$, $E_1$, $E_2$ be the symmetrical components of the resulting potentials at $B$. Also let $V_{a0}$, $V_{a1}$, $V_{a2}$ be the symmetrical components of the potentials at $a$ during the time of the fault and let $I_{a0}$, $I_{a1}$, $I_{a2}$ be the symmetrical components of the fault-currents at $a$. If the impedance of this line between $B$ and $a$ to positive and to negative sequence is $Z_a$ and to zero sequence is $Z_{a0}$,

$$\left.\begin{aligned} E_0 - I_{a0}Z_{a0} &= V_{a0} \\ E_1 - I_{a1}Z_a &= V_{a1} \\ E_2 - I_{a2}Z_a &= V_{a2} \end{aligned}\right\} \qquad (534)$$

Similar equations can be written for the other line, or

$$\left.\begin{aligned} E_0 - I_{d0}Z_{d0} &= V_{d0} \\ E_1 - I_{d1}Z_d &= V_{d1} \\ E_2 - I_{d2}Z_d &= V_{d2} \end{aligned}\right\} \qquad (535)$$

Furthermore, if the potential at $B$ before the faults occur is $E$ and the impedances of the "system" measured at $B$ to zero, positive, and negative sequence are $Z_{N0}$, $Z_{N1}$, $Z_{N2}$,

$$\left.\begin{aligned} 0 - (I_{a0} + I_{d0})Z_{N0} &= E_0 \\ E - (I_{a1} + I_{d1})Z_{N1} &= E_1 \\ 0 - (I_{a2} + I_{d2})Z_{N2} &= E_2 \end{aligned}\right\} \qquad (536)$$

In these nine equations there are fifteen unknowns. The remaining necessary relations are determined by the types of faults that occur at $a$ and at $d$. For example, if there is a line-to-line fault at $a$ and simultaneously a line-to-ground fault occurs at $d$, both of which are symmetrical with respect to the same phase, the fault relations are

$$\left.\begin{aligned} V_{a1} &= V_{a2} \\ I_{a0} &= 0 \\ I_{a1} + I_{a2} &= 0 \end{aligned}\right\} \qquad (537)$$

$$\left.\begin{aligned} V_{d0} + V_{d1} + V_{d2} &= 0 \\ I_{d0} = I_{d1} &= I_{d2} \end{aligned}\right\} \qquad (538)$$

If $V_{a2}$ and $I_{a2}$ are eliminated from equations (534),

$$E_1 - E_2 = 2I_{a1}Z_a \qquad (539)$$

Similarly, if $V_{d0}$, $V_{d1}$, $V_{d2}$, $I_{d0}$, $I_{d2}$ are eliminated from equations (535)

$$E_0 + E_1 + E_2 = I_{d1}(Z_{d0} + 2Z_d) \qquad (540)$$

Again, by means of the fault equations, the zero- and negative-sequence currents can be eliminated from equations (536)

$$\left.\begin{aligned} 0 - I_{d1}Z_{N0} &= E_0 \\ E - (I_{a1} + I_{d1})Z_{N1} &= E_1 \\ 0 - (-I_{a1} + I_{d1})Z_{N2} &= E_2 \end{aligned}\right\} \qquad (541)$$

Eliminate $E_0$, $E_1$, $E_2$ from equations (539), (540), (541),

$$\left.\begin{aligned} E &= I_{a1}(Z_{N1} + Z_{N2} + 2Z_a) + I_{d1}(Z_{N1} - Z_{N2}) \\ E &= I_{a1}(Z_{N1} - Z_{N2}) + I_{d1}(Z_{N0} + Z_{N1} + Z_{N2} + Z_{d0} + 2Z_d) \end{aligned}\right\} \qquad (542)$$

If these equations are treated as if they related to a coupled circuit, it is readily shown that

$$E = (I_{a1} + I_{d1})\left[Z_{N1} - Z_{N2} + \frac{2(Z_{N2} + Z_a)(Z_{N0} + 2Z_{N2} + Z_{d0} + 2Z_d)}{Z_{N0} + 4Z_{N2} + Z_{d0} + 2Z_a + 2Z_d}\right] \tag{543}$$

By applying Thévénin's theorem, it is seen that the impedance which should be connected to the positive-sequence network at $B$ in order to reproduce the conditions due to the faults at $a$ and $d$ is

$$-Z_{N2} + \frac{2(Z_{N2} + Z_a)(Z_{N0} + 2Z_{N2} + Z_{d0} + 2Z_d)}{Z_{N0} + 4Z_{N2} + Z_{d0} + 2Z_a + 2Z_d} \tag{544}$$

**178. Series Faults.**—In general, a series fault is produced when *unequal* impedances are inserted in series with the line conductors at some point in the circuit. These impedances constitute the faulted portion of the circuit. For example, if a line conductor breaks or a single-pole switch is opened, impedances having values infinity, zero, zero are inserted in the circuit. In the case of a single *shunt* fault, the impedances of the zero- and negative-sequence networks are measured between the neutral and the point at which the fault occurs. For the *series* fault, however, the impedances of these networks are measured *across the break in the circuit* that would be produced by the removal of the faulted portion of the circuit.

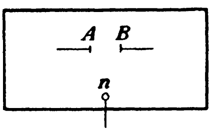


FIG. 221

Thus, if Fig. 221 represents the negative-sequence network, the faulted portion of which was between $A$ and $B$, the impedance of the network is measured between the points $A$-$B$. If $Z_{N0}$ and $Z_{N2}$ represent these impedances as measured in the zero- and negative-sequence networks,

$$V_{AB0} = -I_{AB0}Z_{N0} \tag{545}$$
$$V_{AB2} = -I_{AB2}Z_{N2} \tag{546}$$

where $V_{AB}$ is the impressed fall in potential from $A$ to $B$ and $I_{AB}$ is the current through this emf from $A$ to $B$. It is the current from $B$ to $A$ through the network. The corresponding relation that applies to the positive-sequence network is deduced by applying the Principle of Superposition in a manner similar to that used in the case of shunt faults. In Fig. 222, let $V$ be the

fall in potential appearing across a break $A$-$B$ in the positive-sequence network and let $Z$ be an impedance which will connect $A$ and $B$ when the switch is closed. By the principle of superposition, the current flowing from $A$ to $B$ with the switch open equals the sum of the currents which would be produced, first, by the action of the potential $V$ alone and, second, by the action of all the excitation emfs in the network alone. The first component is $-V/(Z_{N1} + Z)$ while the second component is an as yet undetermined current $I_{AB1}$. Since, with the switch open, the resultant current is zero

$$\frac{-V}{Z_{N1} + Z} + I_{AB1} = 0$$

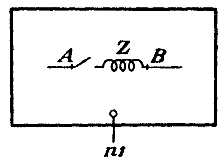


FIG. 222.

Note that $Z_{N1}$ is the impedance of the network measured between the terminals $A$-$B$ when all the excitation emfs are withdrawn. If the impedance $Z$ is zero, $I_{AB1}$ is the normal prefault current flowing in this branch of the network. We shall designate this value of the current as $I$. Thus,

$$V = Z_{N1}I$$

from which it follows that

$$I_{AB1} = \frac{Z_{N1}}{Z_{N1} + Z}I \tag{547}$$

This equation can be interpreted as indicating the effect produced when an impedance $Z$ is inserted in series with any branch of a network. If $Z$ is taken as the ratio $V_{AB1}/I_{AB1}$, equation (547) can be written as

$$V_{AB1} = Z_{N1}I - Z_{N1}I_{AB1} \tag{548}$$

This gives the relation that must exist between the positive-sequence components of the current through the fault and the fall in potential across it, in terms of the prefault current and the impedance of the positive-sequence network. Compare these three equations (545), (546), (548) with equations (476), (477), (479), page 497, used in determining the effect of a single shunt fault. The combination of equations (545), (546), (548) and those which depend upon the type of the fault is sufficient to determine the symmetrical components of the currents at the

fault. For exampl . if there is a break in one conductor, the fault equations relating to that conductor are

$$\left.\begin{aligned} V_{AB0} = V_{AB1} = V_{AB2} \\ I_{AB0} + I_{AB1} + I_{AB2} = 0 \end{aligned}\right\} \tag{549}$$

From the solution of equations (545), (546), (548), together with equations (549), the ratio $V_{AB1}/I_{AB1}$ is found to be

$$\frac{Z_{N0}Z_{N2}}{Z_{N0} + Z_{N2}}$$

Thus the effect of a break in one line conductor can be determined by assuming that there is connected across the break in the positive-sequence network an impedance equal to that of the parallel combination of the zero- and negative-sequence networks.

If two of the line conductors are broken, it can be shown that the zero- and negative-sequence networks should be connected in *series* across the break in the positive-sequence network in order to correctly represent the fault conditions.

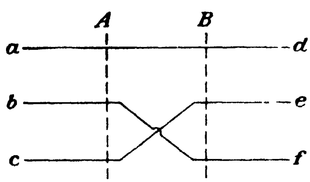



FIG. 223.

The solution of the general problem, in which three unequal impedances are inserted in series with the transmission conductors at some point in the system, can be obtained by means of the three equations [see equations (68), page 89] relating the symmetrical components of the currents in the impedances and the potentials across them, together with the three network equations (545), (546), and (548).

The symmetrical components of the currents on both sides of a series fault are the same. There is, however, an interesting modification of the series fault for which this is not true. This is illustrated in Fig. 223, which indicates that, whereas the corresponding phases are *a*,*b*,*c*, and *d*,*e*,*f*, phases *e* and *f* are wrongly connected to phases *c* and *b* instead of being rightly connected to phases *b* and *c*. That portion of the network between *A* and *B* constitutes the fault. Since $I_b = I_f$ and $I_c = I_e$, it follows that the positive-sequence component of $I_a$ equals the negative sequence component of $I_d$. Thus

$$I_{a1} = I_{d2}$$
$$I_{a2} = I_{d1}$$

Similarly, since $V_{bn} = V_{fn}$ and $V_{cn} = V_{en}$,

$$V_{a1} = V_{d2}$$
$$V_{a2} = V_{d1}$$

These relations between the symmetrical components of the potential and current on opposite sides of the fault can be realized by interconnecting the positive- and negative-sequence networks as indicated in Fig. 224. Both networks are set up with a break in the circuit between $A$ and $B$. The equivalent $Y$ form of the negative-sequence network, as shown by the dotted lines, can be substituted for the complete network.

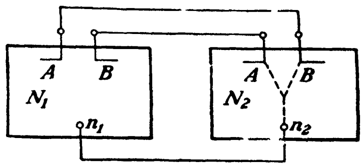



FIG. 224.

It will be observed that the positive-sequence component of the potential at the left of the fault equals the negative-sequence component of the potential at the right of the fault. Also, the positive-sequence component of the current flowing toward the fault at $A$ equals the negative-sequence component of the current flowing away from the fault at $B$.

**179. Simultaneous Shunt and Series Faults.**—Simultaneous shunt and series faults may occur at the same point as, for example, when a line conductor breaks (series fault) and one end of the broken conductor comes in contact with the ground

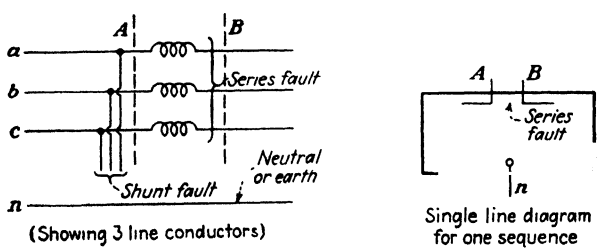



FIG. 225.

or with one of the other line conductors (shunt fault). The general relation of these faults to the network is shown diagrammatically in Fig. 225, in which the series fault occurs between the points $A$ and $B$ *in* the line conductors and the shunt fault occurs at the point $A$ *between* the line conductors, or between one or more of the line conductors and ground. If this diagram

represents either the zero- or the negative-sequence network, the entire network can be replaced by its equivalent $Y$ circuit, the terminals of which are $A$, $B$, and the neutral $n$. This is illustrated in Fig. 226. The potential $V_{An}$ is the fall in potential across the shunt fault from $A$ to $n$. Likewise, $V_{AB}$ is the fall in potential across the series fault from $A$ to $B$. The fault currents flowing *into the network* at $n$ and at $B$ are $I_{An}$ and $I_{AB}$, as indicated. If we designate the impedances of the branches of the negative-sequence equivalent circuit as $C_2$, $D_2$, $S_2$, the relations between the negative-sequence components of the fault potentials and the negative-sequence components of the fault currents are

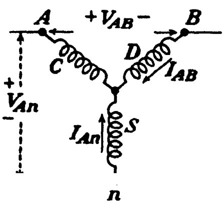


FIG. 226.

$$-V_{An2} = I_{An2}S_2 + (I_{An2} + I_{AB2})C_2 \tag{550}$$
$$-V_{AB2} = (I_{An2} + I_{AB2})C_2 + I_{AB2}D_2 \tag{551}$$

The corresponding relations between the zero-sequence components of the fault potentials and currents are

$$-V_{An0} = I_{An0}S_0 + (I_{An0} + I_{AB0})C_0 \tag{552}$$
$$-V_{AB0} = (I_{An0} + I_{AB0})C_0 + I_{AB0}D_0 \tag{553}$$

If desired these four equations (550) to (553) can be inverted as was done in Art. 164 page 506 [see equations (487) to (490)] so that the currents are expressed in terms of the potentials. It should be observed that the following method of analysis is similar to that used in the case of two simultaneous shunt faults. If all but the positive-sequence quantities are eliminated from these equations (550) to (553) and the equations [similar to equations (549)] which relate the symmetrical components of the fault potentials and currents at both faults, there will remain two general relations between the positive-sequence components of the potentials and currents at the faults as follows

$$\left.\begin{aligned} V_{An1} &= kI_{An1} + mI_{AB1} \\ V_{AB1} &= lI_{AB1} + nI_{An1} \end{aligned}\right\} \tag{554}$$

These are similar to the general equations developed in Art. 165 for two simultaneous shunt faults. We shall determine the values of the impedances $k$, $l$, $m$, $n$ in but two illustrative cases.

I. One end of a broken line conductor falls on the ground, causing a series fault of infinite impedance in one line conductor and a shunt fault of zero impedance between that line conductor and ground. These two faults require that

$$\left.\begin{aligned} I_{An0} = I_{An1} &= I_{An2} \\ V_{An0} + V_{An1} + V_{An2} &= 0 \\ I_{AB0} + I_{AB1} + I_{AB2} &= 0 \\ V_{AB0} = V_{AB1} &= V_{AB2} \end{aligned}\right\} \tag{555}$$

Substitution of these relations (555) in equations (550) to (553) shows that

$$k = S_0 + C_0 + S_2 + C_2 - \frac{(C_2 - C_0)^2}{C_0 + D_0 + C_2 + D_2}$$

$$l = \frac{(C_0 + D_0)(C_2 + D_2)}{C_0 + D_0 + C_2 + D_2}$$

$$m = n = -\frac{C_0(C_2 + D_2) + C_2(C_0 + D_0)}{C_0 + D_0 + C_2 + D_2}$$

II. One end of a broken line conductor falls on another line conductor, causing a series fault of infinite impedance in one line conductor and a shunt fault of zero impedance between two line conductors. In this case the two faults are not symmetrical with respect to the same phase. Let us assume that the conductor of phase *a* breaks and comes in contact with the conductor of phase *b*. The short circuit between conductors *a* and *b* is symmetrical with respect to phase *c*, for which the fault equations are

$$V_{cn1} = V_{cn2} \qquad I_{cn1} + I_{cn2} = 0$$

When these equations are written for phase *a* they become

$$\alpha V_{an1} = \alpha^2 V_{an2} \qquad \alpha I_{an1} + \alpha^2 I_{an2} = 0$$

Thus, it follows that these faults require that

$$\left.\begin{aligned} I_{An1} + \alpha I_{An2} &= 0 \\ V_{An1} &= \alpha V_{An2} \\ I_{AB0} + I_{AB1} + I_{AB2} &= 0 \\ V_{AB0} = V_{AB1} &= V_{AB2} \end{aligned}\right\} \tag{556}$$

Substitution of these relations [equations (556)] in equations (550) to (553) shows that

$$\begin{aligned} k &= S_2 + C_2 - \frac{C_2^2}{C_0 + D_0 + C_2 + D_2} \\ l &= \frac{(C_0 + D_0)(C_2 + D_2)}{C_0 + D_0 + C_2 + D_2} \\ m &= \frac{\alpha C_2(C_0 + D_0)}{C_0 + D_0 + C_2 + D_2} \\ n &= \frac{\alpha^2 C_2(C_0 + D_0)}{C_0 + D_0 + C_2 + D_2} \end{aligned} \qquad (557)$$

In these expressions (557) it should be observed that $C_2 + D_2$ is the impedance of the negative-sequence network between the points $A$-$B$ on opposite sides of the opening in the network where the series fault occurs. Likewise the impedance $S_2 + C_2$ is measured between $A$ and the neutral $n$ while the network is open between $A$ and $B$.

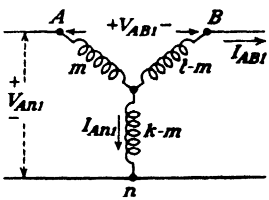



FIG. 227.

For faults of the *first* kind, such as case I, page 541, in which $m$ and $n$ are equal, it is possible to reproduce the effect of the fault by connect- the impedances $m$, $l - m$, $k - m$ between the points $A$, $B$, ing and $n$ in the positive-sequence network, as indicated in Fig. 227. The currents and potentials throughout the network will then have their proper positive-sequence values. An alternative method is to interconnect the zero-, positive-, and negative-sequence networks as indicated in Fig. 228. It will be

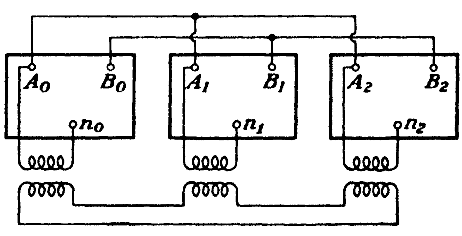



FIG. 228.

observed that with this interconnection the zero-, positive-, and negative-sequence components of the potential and current at each fault satisfy the conditions imposed upon them by the faults. Without the insulating transformers, each of the networks would be short-circuited between the terminal $A$ and the neutral. When $m$ and $n$ are not equal, as in case II,

the equivalent fault circuit can be modified as in Fig. 195, page 512. In this case, if the interconnection method (Fig. 229) is used, it is necessary to use a phase-shifting device in order to obtain the correct relations between the symmetrical components of the fault potentials and currents.

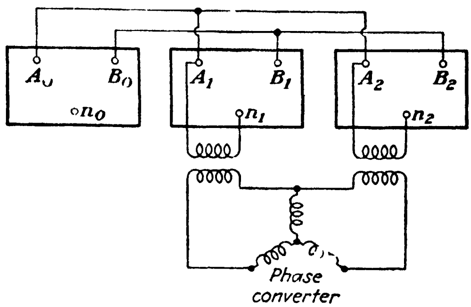



FIG. 229.

**180. Equivalent Positive-sequence Circuit.**—We shall first determine the relations that must exist between the positive-sequence components of the potentials and currents at the faults. This is accomplished by applying the principle of superposition. The circuit arrangement at the faults is indicated in Fig. 230. The fall in potential from $A$ to $B$ is $V_{AB1}$ and from $A$ to $n$ is $V_{An1}$. There are also the excitation emfs generated in the synchronous machines. The fault currents $I_{AB1}$ and $I_{An1}$ are due to the joint action of these excitation emfs and the potentials $V_{AB1}$ and $V_{An1}$. By the principle of superposition, each of these currents can be assumed to consist of three components. The first is that due to the action of the excitation emfs alone when both $V_{AB1}$ and $V_{An1}$ are zero. The second is due to the action of $V_{AB1}$ alone when the excitation emfs and $V_{An1}$ are all zero. The third is due to the action of $V_{An1}$ alone when the excitation emfs and $V_{Ab1}$ are zero. Before the faults occur we shall assume that the line-to-neutral potential at $A$ is $E$ and that the line current at the same point is $I$. The first component $I_{An1}'$ of $I_{An1}$ is $E/Z_N$, where $Z_N$ is the impedance of the positive-sequence network between $A$ and $n$ when all of the

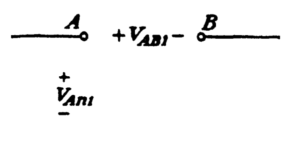



FIG. 230.

excitation emfs are withdrawn and the circuit is not open between $A$ and $B$. In order to determine the first component $I_{AB1}'$ of $I_{AB1}$, we shall resort to the following artifice. Assume that there are connected between $A$ and $n$ two emfs, each equal in magnitude to $E$ but in time opposition as indicated in Fig. 231. With both of these emfs acting, the line-to-neutral potential is zero. The current $I_{AB}$ can be assumed to consist of two components, one due to the action of the excitation emfs and $E'$ and the other due to the action of $E''$. The first of these components is $I$, the prefault current. When the excitation emfs are withdrawn and the circuit is open between $A$ and $B$, the positive-sequence network can be reduced to an equivalent $Y$ circuit whose terminals are $A$, $B$, $n$ as indicated in Fig. 226. The impedances of the branches of this equivalent circuit are $C_1$, $D_1$, and $S_1$. If the terminals $A$-$B$ are short-circuited and an emf $E''$ is applied between $A$ and $n$, the current $I_{AB}$ is

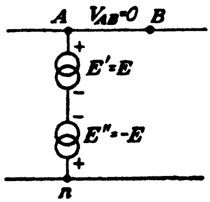


FIG. 231.

$$-\frac{E}{S_1 + \frac{C_1 D_1}{C_1 + D_1}} \frac{C_1}{C_1 + D_1}$$

That is,

$$I_{AB1}' = I - \frac{EC_1}{S_1C_1 + C_1D_1 + D_1S_1}$$

The second component, $I_{An1}''$, of $I_{An1}$ is due to the action of $V_{AB1}$ alone. It is

$$\frac{V_{Ab1}}{D_1 + \frac{S_1C_1}{S_1 + C_1}} \frac{C_1}{S_1 + C_1}$$

That is

$$I_{An1}'' = \frac{V_{AB1}C_1}{S_1C_1 + C_1D_1 + D_1S_1}$$

The second component, $I_{AB1}''$, of $I_{AB1}$ is likewise due to the action of $V_{AB1}$. It is

$$I_{AB1}'' = \frac{-V_{AB1}(S_1 + C_1)}{S_1C_1 + C_1D_1 + D_1S_1}$$

It is readily seen that the third components of these currents due to the action of $V_{An1}$ alone are

$$I_{An1}''' = \frac{-V_{An1}(C_1 + D_1)}{S_1C_1 + C_1D_1 + D_1S_1}$$

$$I_{AB1}''' = \frac{V_{An1}C_1}{S_1C_1 + C_1D_1 + D_1S_1}$$

Thus, it follows that the positive-sequence fault currents are

$$I_{An1} = \frac{E(C_1 + D_1)}{G} + \frac{V_{AB1}C_1}{G} - \frac{V_{An1}(C_1 + D_1)}{G} \qquad (558)$$

$$I_{AB1} = I - \frac{EC_1}{G} - \frac{V_{AB1}(S_1 + C_1)}{G} + \frac{V_{An1}C_1}{G} \qquad (559)$$

where $G = S_1C_1 + C_1D_1 + D_1S_1$. It should also be noted that

$$Z_N = S_1 + \frac{C_1D_1}{C_1 + D_1}$$

If these two equations (558), (559) are combined with the general equations (554), the values of the fault currents can be shown to be

$$I_{An1} = \frac{E(l + C_1 + D_1) + [C_1l - (C_1 + D_1)m]I}{(k + S_1 + C_1)(l + D_1 + C_1) - (m + C_1)(n + C_1)} \qquad (560)$$

$$I_{AB1} = \frac{E(n + C_1) + [C_1(n - D_1) - (k + S_1)(C_1 + D_1)]I}{(k + S_1 + C_1)(l_1 + D_1 + C_1) - (m + C_1)(n + C_1)} \qquad (561)$$

By applying the method used in Art. 173, page 525, we can determine two impedances which, if connected one between $A$ and $n$ and the other between $A$ and $B$, will correctly represent the effect of the fault on the positive-sequence network. These impedances are

$$Z_a = \frac{V_{An1}}{I_{An1}} = k + m\frac{I_{AB1}}{I_{An1}} \qquad (562)$$

$$Z_b = \frac{V_{AB1}}{I_{AB1}} = l + n\frac{I_{An1}}{I_{AB1}} \qquad (563)$$

where

$$\frac{I_{An1}}{I_{AB1}} = \frac{E(l + C_1 + D_1) + [C_1l - (C_1 + D_1)m]I}{E(n + C_1) + [C_1(n - D_1) - (k + S_1)(C_1 + D_1)]I}$$

If the solutions of equations (558) and (559) for the potentials in terms of the currents and circuit constants are made

$$V_{An1} = E + IC_1 - I_{An1}(S_1 + C_1) - I_{AB1}C_1 \qquad (564)$$
$$V_{AB1} = I(C_1 + D_1) - I_{An1}C_1 - I_{AB1}(C_1 + D_1) \qquad (565)$$

Inspection of these equations shows that they describe the conditions existing at the terminals of the $Y$ circuit of Fig. 232. The emfs in the branches of this circuit are $E$, $(IC_1)$, $(ID_1)$ and their assumed positive directions are indicated by the plus (+) and minus (−) signs. Since the emfs should be treated as vector quantities, their phase relations are determined by the phase relation of the initial line-to-neutral potential $E$ and the initial line current $I$ together with the impedances $C_1$ and $D_1$. Thus, it follows that, insofar as the terminals $A$, $B$, $n$ are concerned, this circuit is exactly equivalent to the positive-sequence network and may be substituted for it if desired.

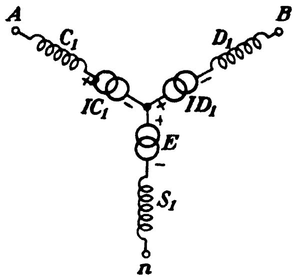



FIG 232

**Illustration of the Method of Computing Simultaneous Shunt Faults.**—The following problem illustrates the method of computing the currents when there are simultaneously shunt faults at different points on a distribution system. The arrangement of the distribution system is indicated in Fig. 233. This arrange-

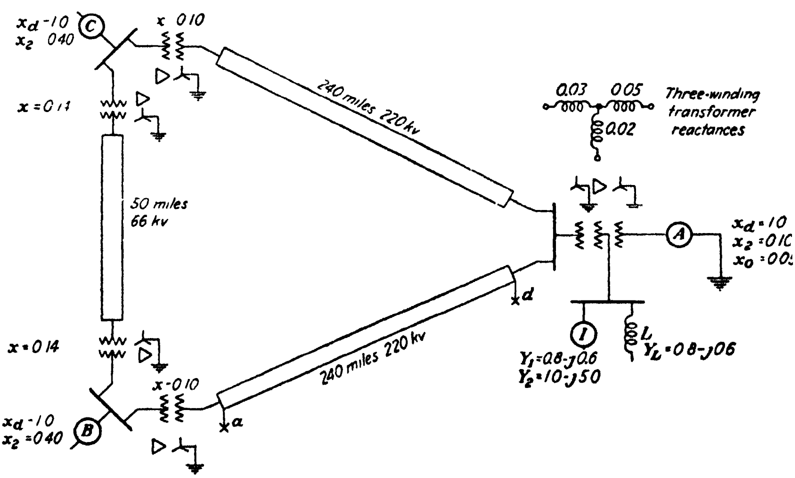



FIG. 233.

ment illustrates a simple combination of a variety of circuit elements.

As is customary in the solution of all problems, certain simplifying assumptions will be made. In this case, assumptions are chosen that lead to a direct solution, not unduly laborious nor encumbered by less important details, though none of the salient features of such a computation are ignored. It is important that the significance of each assumption should be considered carefully.

*Circuit Constants.*—The generators,[1] *A*, *B*, *C*, each have a rating of 125,000 kva. The part *I* of the load consists of 3-phase

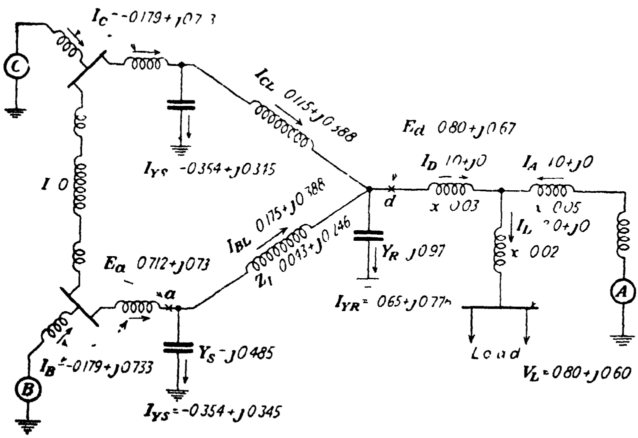



FIG. 234.

induction motors taking 125,000 kva. at 0.8 power factor. The admittance of this load to positive sequence is assumed constant. The part *L* of the load consists of a balanced arrangement of single-phase motor and resistance loads taking 125,000 kva. at 0.8 power factor. The admittance of this load is assumed constant. The load is divided among the three generators so that one-half the load current is supplied by generator *A* and the other half is supplied equally by generators *B* and *C*. This division of the load is indicated in Fig. 234. Generator *A* alone is grounded.

[1] It is more convenient to refer to these as "single generators" whereas they may each be combinations of two or more gnerators operating in parallel.

The transmission lines are described in Fig. 46, page 152, and Fig. 50, page 179.

In the following computation per unit values will be used.

*The unit of kva. is* 125,000/3 *kva.*

*The unit of potential is* $220/\sqrt{3}$ *kv.*

It follows that:

The unit of current is $125{,}000/220\sqrt{3} = 328$ amperes.

The unit of impedance is $220{,}000/328\sqrt{3} = 388$ ohms.

The unit of admittance is $\frac{1}{388} = 2.58 \times 10^{-3}$ mho.

The impedance to positive or negative sequence of the 66-kv. double-circuit line referred to the unit potential is

$$\left(\tfrac{220}{66}\right)^2(6.48 + j17.6) = 72 + j196 \text{ vector ohms}$$
$$= 0.186 + j0.504 \text{ per unit}$$

The impedance to zero sequence is not required in this problem.

If the 220-kv. line is represented by its equivalent $\pi$, the series impedance to positive or negative sequence is

$$96.7\underline{/80.2^\circ} = 0.043 + j0.246 \text{ per unit (ohm)}$$

The series impedance to zero sequence is

$$255\underline{/76.2^\circ} = 0.157 + j0.64 \text{ per unit (ohm)}$$

The shunt admittance to positive or negative sequence at *each* end of the line is

$$1.25 \times 10^{-3}\underline{/89.2^\circ} = 0 + j0.485 \text{ per unit (mho) approximately}$$

The shunt admittance to zero sequence at *each* end of the line is

$$19.9 \times 10^{-3}\underline{/89.3^\circ} = 0 + j0.385 \text{ per unit (mho) approximately}$$

The 66-kv. transformers each have an equivalent reactance to *each* sequence of 0.14 per unit. The two-winding transformers at the ends of the 220-kv. lines each have an equivalent reactance to *each* sequence of 0.10 per unit. The reactances assigned to the high-tension, low-tension, and load windings of the three-winding transformer at the generator *A* are, respectively, 0.03, 0.05, and 0.02 per unit. The resistances of the transformers are neglected.

The generators are all assumed to have "ideal" characteristics. Each generator has a synchronous reactance of 1.0 per unit.

Generator $A$ has a reactance to negative sequence of 0.10 per unit and a reactance to zero sequence of 0.05 per unit. Generators $B$ and $C$ each have a reactance to negative sequence of 0.40 per unit. The resistances of the generators are neglected.

The 3-phase induction motor load has an admittance to negative sequence of $1.0 - j5.0$ per unit.

*Fault Conditions.*—A line-to-ground fault occurs at each end of one of the 220-kv. lines as indicated in Fig. 233, the fault at $a$ being on phase $a$ and the fault at $d$ being on phase $e$ (see page 520). We shall determine the *steady state* fault currents on the assumption that the faults do not alter either the magnitude or phase of each excitation emf.

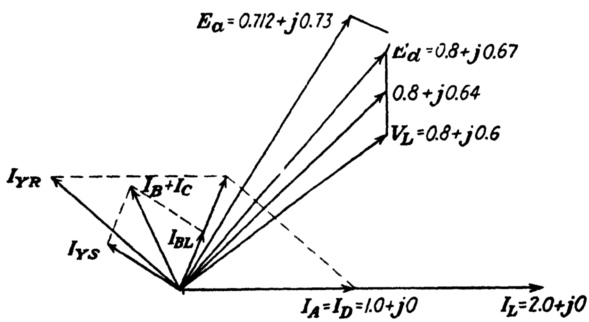



FIG. 235.

*Initial Conditions.*—We shall first determine the prefault line-to-ground potentials at the fault points $a$ and $d$. On assuming that the load current is $2.0 + j0$ per unit and that the line-to-neutral potential at the load is $0.8 + j0.6$ per unit, the equivalent circuit (Fig. 234) and the vector diagram (Fig. 235) are self-explanatory. The prefault potentials at the fault points $a$ and $d$ are

$$\left.\begin{aligned} E_a &= 0.712 + j0.730 \\ E_d &= 0.80 + j0.67 \end{aligned}\right\} \qquad (566)$$

The current delivered by generator $A$ is assumed to be $1.0 + j0$ per unit. Were it not for the $\Delta$-$Y$ transformations (see Art. 75, page 198), the currents delivered by generators $B$ and $C$ each would be $-0.179 + j0.733$ per unit. No current flows in the 66-kv. line.

***Reduction of Positive- and Negative-sequence Networks.***—When combining circuits in series, the circuit constants are expressed

as impedances, but when combining circuits in parallel and in the star-mesh transformation the circuit constants are expressed as admittances. The arrangement of the positive- and negative-sequence networks is indicated in Fig. 236, in which the circuit constants are expressed both as impedances and admittances.

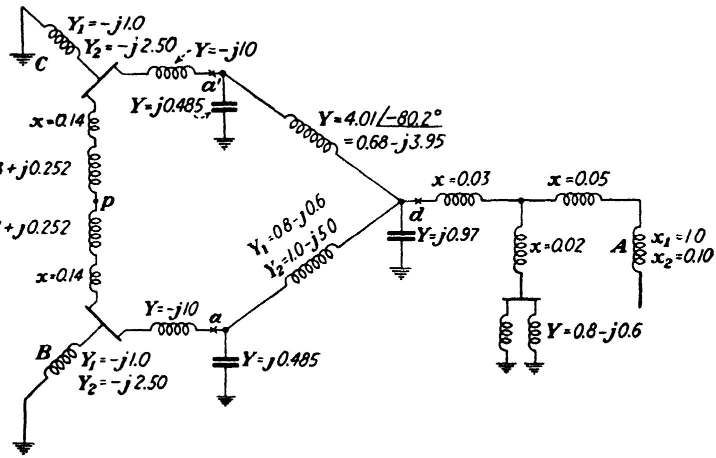



FIG. 236.

The impedances or admittances of the generators and induction motors both to positive and to negative sequence are indicated.

Consider the circuit to the right of the point $d$.

| | *Positive Sequence* | *Negative Sequence* |
|---|---|---|
| Load | $Y_1 = 1.60 - j1.20$ | $Y_2 = 1.80 - j5.60$ |
| | $Z_1 = 0.40 + j0.30$ | $Z_2 = 0.052 + j0.162$ |
| + Trans. | $0 + j0.02$ | $0 + j0.02$ |
| | $Z_1 = 0.40 + j0.32$ | $Z_2 = 0.052 + j0.182$ |
| | $Y_1 = 1.525 - j1.220$ | $Y_2 = 1.45 - j5.08$ |
| + (Gen A and trans.) | $Y_1 = 0 - j0.952$[1] | $Y_2 = 0 - j6.67$ |
| | $1.525 - j2.172$ | $1.45 - j11.75$ |
| | $Z_1 = 0.216 + j0.308$ | $Z_2 = 0.0103 + j0.0838$ |
| + Trans. | $0 + j0.03$ | $0 + j0.03$ |
| | $Z_1 = 0.216 + j0.338$ | $Z_2 = .0103 + j0.1138$ |
| | $Y_1 = 1.344 - j2.10$ | $Y_2 = 0.79 - j8.71$ |

[1] Generator A + transformer, $Z_1 = j1.0 + j0.05$, $Z_2 = j0.10 + j0.05$.

Consider the circuit at the left of the points $a$ and $a'$. Note that this circuit is symmetrical for both positive and negative sequence about the mid-point $p$ of the 66-kv. line as indicated in Fig. 237. If the admittances of the branches in this circuit are $A$, $B$, $C$, this circuit is shown to be equivalent to the circuit of Fig. 238 by means of the $Y$-$\Delta$ transformation. By applying the $Y$-$\Delta$ transformation a second time, the point $p$ is eliminated

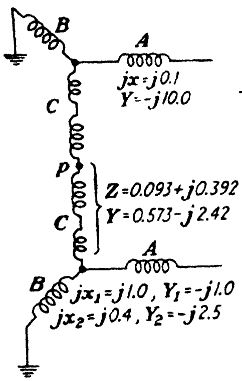



FIG. 237.

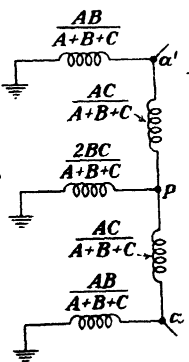



FIG. 238.

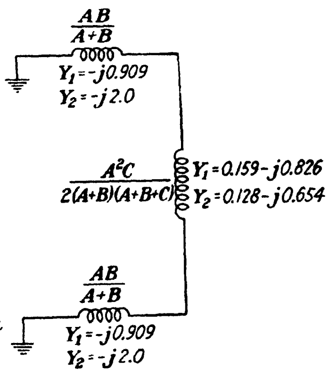



FIG. 239.

and the circuit of Fig. 238 reduces to the circuit of Fig. 239. Since

$$\begin{aligned} A &= -j10.0 \text{ for positive sequence} \\ &= -j10.0 \text{ for negative sequence} \\ B &= -j1.0 \text{ for positive sequence} \\ &= -j2.50 \text{ for negative sequence} \\ C^* &= 0.573 - j2.42 \text{ for positive sequence} \\ &= 0.573 - j2.42 \text{ for negative sequence} \end{aligned}$$

$$\begin{aligned} \frac{AB}{A+B} &= -j0.909 \text{ for positive sequence} \\ &= j2.0 \text{ for negative sequence} \end{aligned}$$

$$\begin{aligned} \frac{A^2C}{2(A+B)(A+B+C)} &= 0.159 - j0.826 \text{ for positive sequence} \\ &= 0.128 - j0.654 \text{ for negative sequence} \end{aligned}$$

Hence the complete positive- and negative-sequence circuits reduce to the combination of admittances in Fig. 240. On combining the parallel branches (Fig. 240), the positive-sequence

* $Z = j0.14 + (0.093 + j0.252)$ for positive and negative sequence.

network reduces to Fig. 241 and the negative-sequence network reduces to Fig. 242.

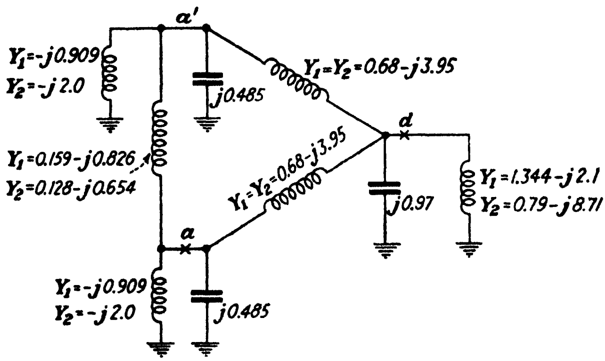



FIG. 240.

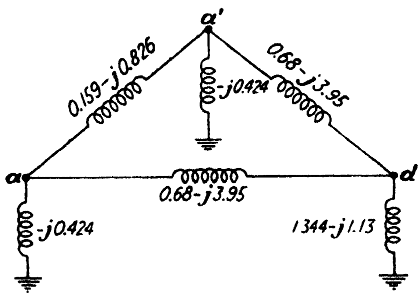



FIG. 241.

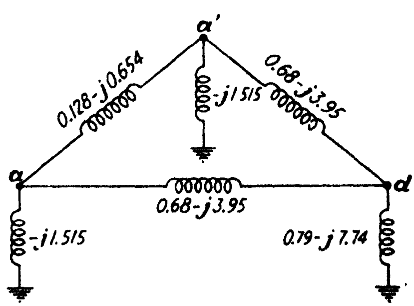



FIG. 242.

Eliminate the point $a'$ by means of the $Y$-$\Delta$ transformation. For example, the admittance, $Y_{ad}$, of the $\Delta$ branch between $a$ and $d$ is

$$Y_{ad} = \frac{(0.159 - j0.826)(0.68 - j3.95)}{(0.159 - j0.826) + (0.68 - j3.95) + (0 - j0.424)} \text{ for positive sequence}$$

$$= 0.128 - j0.629 \text{ for positive sequence}$$

Likewise

$Y_{ad} = 0.101 - j0.422$ for negative sequence
$Y_{an} = 0.02 - j0.0679$ for positive sequence
$Y_{dn} = 0.003 - j0.323$ for positive sequence
$Y_{an} = 0.01 - j0.164$ for negative sequence
$Y_{dn} = 0.04 - j0.985$ for negative sequence

When these $Y$-$\Delta$ transformations have been made, Fig. 241 becomes Fig. 243 and Fig. 242 becomes Fig. 244. On combining

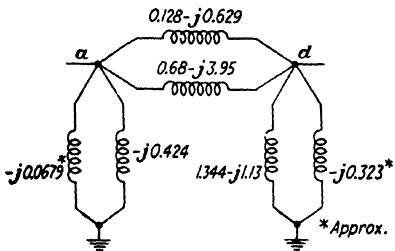



FIG. 243.

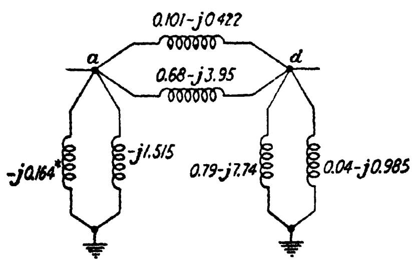



FIG. 244.

the parallel branches in Fig. 243 and in Fig. 244, the resulting positive- and negative-sequence equivalent circuits are as indicated in Fig. 245 and Fig. 246. By equations (501), page 514,

| *(Positive Sequence)* | *(Negative Sequence)* |
|---|---|
| $Y_{aa1} = 0.81 - j5.07$ | $Y_{aa2} = 0.78 - j6.05$ |
| $Y_{dd1} = 2.16 - j6.03$ | $Y_{dd2} = 1.61 - j13.10$ |
| $Y_{ad1} = 0.81 - j4.58$ | $Y_{ad2} = 0.78 - j4.37$ |

*Reduction of Zero-sequence Network.*—The arrangement of the zero-sequence network is indicated in Fig. 247. This network

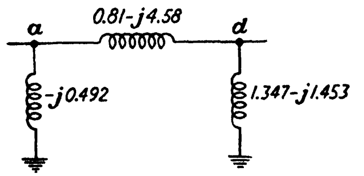



FIG. 245.

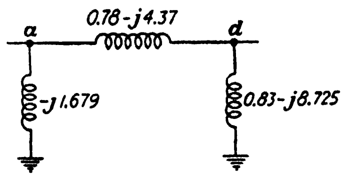



FIG. 246.

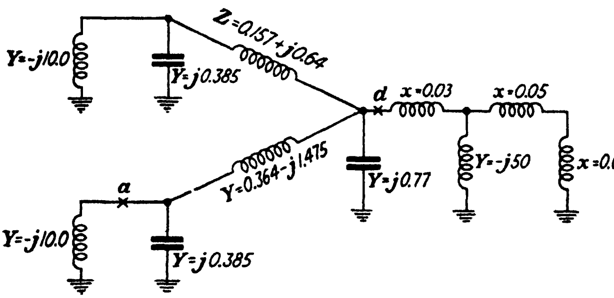



FIG. 247.

is readily reduced to the equivalent circuit of Fig. 248, from which

$$Y_{aa0} = 0.36 - j11.09$$
$$Y_{dd0} = 0.63 - j23.43$$
$$Y_{ad0} = 0.364 - j1.475$$

*Symmetrical Components of the Fault Potentials.*—The equations relating the symmetrical components of the fault currents and the potentials at the faults are

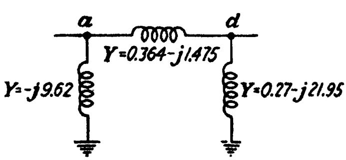



FIG. 248.

$$V_{a0} + V_{a1} + V_{a2} = 0 \quad (567)$$
$$I_{a0} = I_{a1} = I_{a2} \quad (568)$$
$$V_{d0} + \alpha^2 V_{d1} + \alpha V_{d2} = 0 \quad (569)$$
$$I_{d0} = \alpha^2 I_{d1} = \alpha I_{d2} \quad (570)$$

The equations which relate the positive- and negative-sequence components of the potentials at the faults are equations (514), (515), (516), and (517), page 520. If each equation is divided by the coefficient of $V_{a1}$ and the numerical values of the admittances are substituted,

$$\left.\begin{aligned}
V_{a1} - (1.19 - j0.037)V_{a2} - (0.906 + j0.014)V_{d1} \\
+ (0.866 - j0.015)V_{d2} = -0.003 + j0.11 \\
V_{a1} + (0.476 + j0.828)V_{a2} - (1.36 + j0.232)V_{d1} \\
- (1.55 + j2.38)V_{d2} = -0.218 - j0.367 \\
V_{a1} + (1.55 + j0.052)V_{a2} + (0.042 + j0.13)V_{d1} \\
- (0.304 - j0.16)V_{d2} = 0 \\
V_{a1} - (0.294 - j2.62)V_{a2} + (10.39 + j11.4)V_{d1} \\
+ 8.12 - j22.65)V_{d2} = 0
\end{aligned}\right\} \quad (571)$$

Subtract the second equation from the first, the third equation from the second, and the fourth equation from the third, and divide each of the three resulting equations by the coefficient of $V_{a2}$.

$$\left.\begin{aligned}
V_{a2} - (0.272 + j0.002)V_{d1} - (1.74 + j0.613)V_{d2} = \\
-0.217 - j0.184 \\
V_{a2} + (0.697 + j0.843)V_{d1} - (0.224 - j1.91)V_{d2} = \\
-0.029 + j0.322 \\
V_{a2} + (0.982 - j4.74)V_{d1} - (7.32 - j1.99)V_{d2} = 0
\end{aligned}\right\} \quad (572)$$

Subtract the second equation from the first and the third equation from the second. Eliminate $V_{d1}$ from the two resulting equations and obtain

$$V_{d2} = 0.121 - j0.018$$

On substituting this value of $V_{d2}$ in the second of the last two equations

$$V_{d1} = 0.091 + j0.155$$

On substituting the values of $V_{d1}$ and $V_{d2}$ in the first equation of (572)

$$V_{a2} = 0.030 - j0.099$$

On substituting the values of $V_{d1}$, $V_{d2}$, and $V_{a2}$ in the third equation of (571)

$$V_{a1} = 0.005 + j0.147$$

The values of $V_{a0}$ and $V_{d0}$ are now determined from equations (567) and (569)

$$V_{a0} = -0.035 - j0.048$$
$$V_{d0} = -0.044 + j0.042$$

By equations (508) and (509), page 516, the negative-sequence fault currents are

$$I_{a2} = 0.593 - j0.288$$
$$I_{d2} = -0.367 + j1.42$$

By equations (568) and (570), page 554, the positive- and zero-sequence fault currents are

$$I_{a1} = 0.59 - j0.29$$
$$I_{a0} = 0.59 - j0.29$$
$$I_{d1} = 1.41 - j0.39$$
$$I_{d0} = -1.04 - j1.03$$

As a check, the zero-sequence fault currents can be computed by means of equations (506) and (507), and the positive-sequence

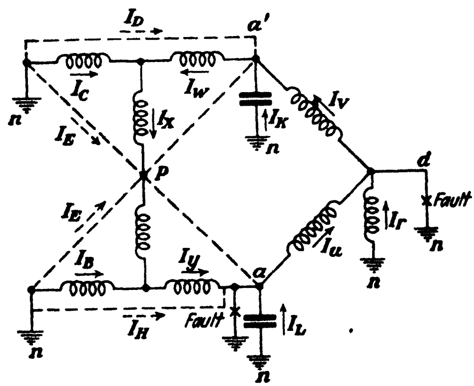



FIG. 249.

fault currents can be computed by means of equations (504) and (505), page 516.

The current in the faulted phase at $a$ is

$$I_{a0} + I_{a1} + I_{a2} = 1.95\underline{/-25.4^\circ}$$

The current in the faulted phase at $d$ is

$$I_{d0} + a^2 I_{d1} + a I_{d2} = 4.40/-135.5^\circ$$

*Network Currents.*—Because of the symmetry of this network, we shall determine the symmetrical components of the currents supplied by the generators by a procedure somewhat different from that described in Art. 168, page 516. The arrangement of

the branches of the positive- and negative-sequence networks, after the branches at the right of $d$ have been replaced by a single branch, is indicated in Fig. 249. The point $p$ is the mid-point of the 66-kv. line. The dotted lines indicate the $\Delta$ branches which, in the process of reducing the circuit, replace the existing $Y$ branches. Since these two $Y$ circuits are alike, the two $\Delta$ circuits are also alike and the currents $I_E$, in the $\Delta$ branches

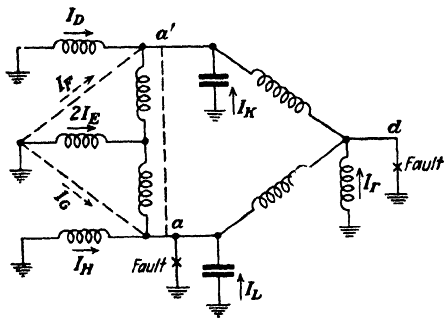



FIG. 250.

between $p$ and $n$ are equal, as indicated in Fig. 249. Note that the generator currents are

$$I_B = I_H + I_E$$
$$I_C = I_D + I_E$$

When the replacements indicated in Fig. 249 are made, the result is Fig. 250. Again the dotted lines indicate the $\Delta$ branches which replace the existing $Y$ branches. Note that

$$2I_E = I_F + I_G$$

Therefore

$$I_B = I_H + \tfrac{1}{2}(I_F + I_G)$$

or

$$I_B = I_H + I_G - \tfrac{1}{2}(I_G - I_F)$$

Also

$$I_C = I_D + \tfrac{1}{2}(I_F + I_G)$$

or

$$I_C = I_D + I_F + \tfrac{1}{2}(I_G - I_F)$$

On replacing the $Y$ branches by their equivalent $\Delta$ branches as indicated in Fig. 250, we have Fig. 251 or Fig. 252. Note that

$$I_B = I_N - \tfrac{1}{2}(I_G - I_F)$$
$$I_C = I_M + \tfrac{1}{2}(I_G - I_F)$$

Also

$$I_G = \frac{Y_G}{Y_N} I_N$$

$$I_F = \frac{Y_F}{Y_M} I_M$$

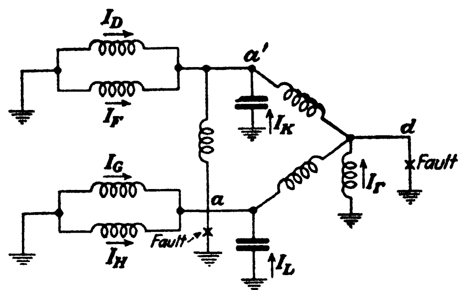



FIG. 251.

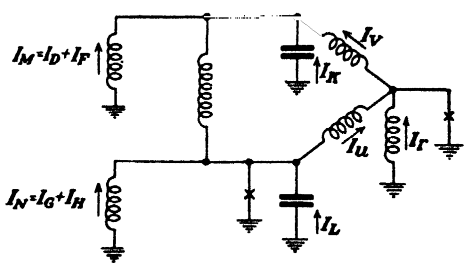



FIG. 252.

On referring to Fig. 239, it will be noted that

$$Y_M = Y_N = \frac{AB}{A + B}$$

On referring to Figs. 238 and 250, it will be noted that

$$Y_F = Y_G = \frac{ABC}{(A + B)(A + B + C)}$$

Thus,

$$\frac{Y_F}{Y_M} = \frac{Y_G}{Y_N} = \frac{C}{A + B + C}$$

Hence

$$I_B = I_N - \frac{C}{2(A + B + C)}(I_N - I_M) \qquad (573)$$

$$I_C = I_M + \frac{C}{2(A + B + C)}(I_N - I_M) \qquad (574)$$

Note that $I_N$ is determined by $Y_N$ and the line-to-neutral potential at $a$. In order to determine the current $I_M$ the network must be reduced further. If the parallel branches between $a'$

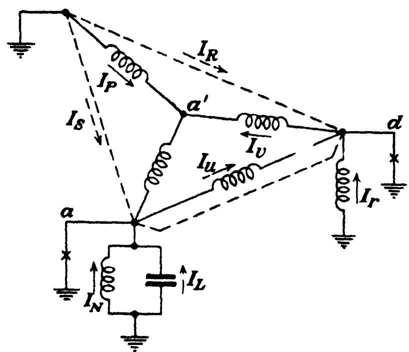



FIG. 253.

and $n$ are combined, we have Fig. 253 in which $I_P = I_M + I_K$. From which

$$I_M = \frac{Y_M}{Y_P} I_P$$

The values of $Y_M$ for positive and negative sequence are (see Fig. 240)

$$Y_{M1} = 0 - j0.909$$
$$Y_{M2} = 0 - j2.00$$

The values of $Y_P$ for positive and negative sequence are (see Figs. 241 and 242)

$$Y_{P1} = 0 - j0.424$$
$$Y_{P2} = 0 - j1.515$$

On eliminating the point $a'$ by a $Y$-$\Delta$ transformation, as indicated by the dotted lines in Fig. 253, we have Fig. 254. Note that

$$I_P = I_R + I_S$$

The currents $I_R$ and $I_S$ are determined by the admittances $Y_R$ and $Y_S$ and the line-to-neutral potentials at $d$ and $a$ (from

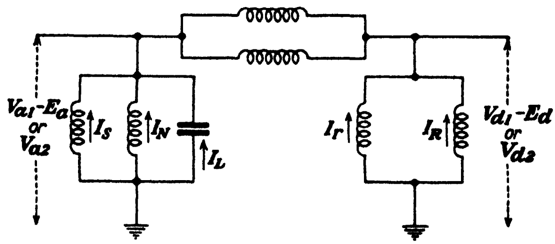



FIG. 254.

Figs. 243 and 244).

$$Y_{R1} = -j0.323 \text{ for positive sequence}$$
$$Y_{S1} = -j0.068 \text{ for positive sequence}$$
$$V_{a1} - E_a = -0.707 - j0.583$$
$$V_{d1} - E_d = -0.709 - j0.515$$
$$I_{S1} = -(V_{a1} - E_a)Y_{S1} = 0.040 - j0.048$$
$$I_{R1} = -(V_{d1} - E_d)Y_{R1} = 0.186 - j0.229$$
$$I_{P1} = 0.226 - j0.277$$
$$Y_{R2} = -j0.986 \text{ for negative sequence}$$
$$Y_{S2} = -j0.164 \text{ for negative sequence}$$
$$V_{a2} = 0.03 - j0.099$$
$$V_{d2} = 0.121 - j0.018$$
$$-V_{a2}Y_{S2} = 0.016 + j0.005$$
$$-Y_{d2}Y_{R2} = 0.018 + j0.119$$
$$I_{P2} = 0.034 + j0.124$$
$$I_{M1} = \frac{-j0.909}{-j0.424}I_{P1} = 0.485 - j0.594$$
$$I_{M2} = \frac{-j2.00}{-j1.515}I_{P2} = 0.045 + j0.164$$

The values of $Y_N$ for positive and negative sequence are indicated in Fig. 239 and are $Y_{N1} = -j0.909$, $Y_{N2} = -j2.0$.

$$I_{N1} = -(V_{a1} - E_a)Y_{N1} = 0.53 - j0.642$$
$$I_{N2} = -V_{a2}Y_{N2} = 0.198 + j0.060$$

Refer to Fig. 237, page 551.

$$\frac{C}{2(A + B + C)} = 0.0926/\underline{10.9°} \text{ for positive sequence}$$
$$= 0.0833/\underline{11.1°} \text{ for negative sequence}$$

By equation (573), page 559,

$$\text{(Fault current) } I_{B1} = 0.525 - j0.638$$
$$I_{B2} = 0.184 + j0.066$$

Since the load current delivered by generator $B$ is

$$-0.179 + j0.733,$$

the symmetrical components of the current delivered by generator $B$ during the fault are

$$I_{B1} = 0.346 + j0.095$$
$$I_{B2} = 0.184 + j0.066$$

Thus the effect of the fault is to *reduce* the current supplied by generator $B$.

By equation (574), page 559,

$$\text{(Fault current) } I_{C1} = 0.49 - j0.598$$
$$I_{C2} = 0.059 + j0.158$$

Since the load current delivered by generator $C$ is

$$-0.179 + j0.733,$$

the symmetrical components of the current delivered by generator $C$ during the fault are

$$I_{C1} = 0.311 + j0.135$$
$$I_{C2} = 0.059 + j0.158$$

Thus the effect of the fault is to reduce the current delivered by generator $C$ also.

The symmetrical components of the current delivered by generator $A$ during the fault are

$$I_{A0} = -0.15 - j0.16$$
$$I_{A1} = 1.43 - j0.65$$
$$I_{A2} = 0.11 + j0.59$$

The labor involved in the foregoing computations would be considerably reduced if all the resistances in the network were neglected. Fault computations are often made on the assumption that there are connected to the network no loads except those due to synchronous machines. It is then customary to neglect the resistances of the network and the connected synchronous machines. Furthermore, if all the synchronous machines are operating at no load, the excitation emfs of the machines are in time-phase, though the emfs need not be equal. Under these conditions, if the faults are symmetrical about the same phase, so that the rotating operators $\alpha$ and $\alpha^2$ are not used, the computation involves real numbers only, a great simplification.

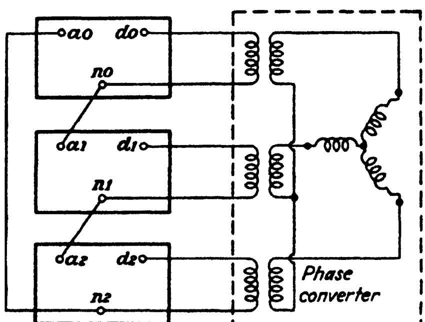



FIG. 255.

This network was also represented on the "network analyzer" at the Massachusetts Institute of Technology by E. W. Kimbark. The three sequence networks were connected at the point *d*, as indicated in Fig. 255, through a phase-shifting device designed by Mr. Kimbark. For convenience the power factor of the load was taken as 0.75 instead of 0.8 as used in the computation.

A comparison of the computed and measured fault currents is given in Table I. The agreement between the computed and measured values of the sequence currents and potentials is good.

Mr. Kimbark also obtained a solution on the "network analyzer" for the symmetrical components of the *transient* fault currents and potentials. In this solution the synchronous reactance, $x_d$, of each generator was replaced by its transient reactance, $x_d'$, and the excitation emfs were replaced by the emfs behind the transient reactances. The latter emfs were

determined by reproducing the initial balanced load conditions. The transient reactance of generator *A* was assumed to be 0.23 per unit. The transient reactances of generators *B* and *C* were assumed to be 0.35 per unit. The excitation emf of genera-

TABLE I.—COMPARISON OF CALCULATED AND MEASURED FAULT CURRENTS AND POTENTIALS

| Quantity | Calculated value | Measured value |
|---|---|---|
| **Potentials at fault points, polar form (per unit)** | | |
| $V_{a1}$ | 0.147/88.0° | 0.150/87.1° |
| $V_{a2}$ | 0.104/286.7° | 0.105/285.1° |
| $V_{a0}$ | 0.059/233.9° | 0.0585/231.6° |
| $V_{d1}$ | 0.180/59.7° | 0.185/56.8° |
| $V_{d2}$ | 0.123/351.4° | 0.122/349.4° |
| $V_{d0}$ | 0.061/136.3° | 0.0605/132.8° |
| **Currents at fault points, polar form (per unit)** | | |
| $I_{a1} = I_{a2} = I_{a0}$ | 0.65/334.6° | 0.66/335.6° |
| $I_{d1}$ | 1.465/344.5° | 1.464/347.5° |
| $I_{d2}$ | 1.465/104.5° | 1.476/106.4° |
| $I_{d0}$ | 1.465/224.5° | 1.476/227.0° |
| **Generator currents, rectangular form (per unit)** | | |
| Gen. *A*: $I_1$ | $+1.43 - j0.65$ | $+1.48 - j0.63$ |
| $I_2$ | $+0.11 + j0.59$ | $+0.14 + j0.60$ |
| $I_0$ | $-0.15 - j0.16$ | $-0.15 - j0.16$ |
| Gen. *B*: $I_1$ | $+0.346 + j0.095$ | $+0.354 + j0.127$ |
| $I_2$ | $+0.184 + j0.066$ | $+0.188 + j0.070$ |
| Gen. *C*: $I_1$ | $+0.311 + j0.135$ | $+0.281 + j0.170$ |
| $I_2$ | $+0.059 + j0.158$ | $+0.052 + j0.158$ |

tor *A* was $0.71 + j1.76$ whereas the emf behind the transient reactance was $0.73 + j1.00$. The excitation emfs of generators *B* and *C* were each $-0.15 + j0.56$ whereas the emfs behind their transient reactances were each $0.32 + j0.70$. It was assumed that the magnitude and phase of each of the emfs behind the

transient reactances were maintained constant before and during the fault. The per unit values of the transient currents and potentials are given in Table II.

TABLE II.—CURRENTS AND POTENTIALS (PER UNIT) UNDER FAULT CONDITIONS

| Branch | Positive | Negative | Zero |
|---|---|---|---|
| | | Sequence currents | |
| Gen. *A* | +2.36 − *j*1.66 | +0.51 + *j*1.06 | −0.38 − *j*0.22 |
| Gen. *B* | +0.84 − *j*0.42 | +0.40 − *j*0.06 | 0 |
| Gen. *C* | +0.62 − *j*0.09 | +0.20 + *j*0.25 | 0 |
| Fault *a* | +0.98 − *j*1.10 | +0.98 − *j*1.10 | +0.98 − *j*1.10 |
| Fault *d* | +2.52 − *j*1.56 | +0.10 + *j*2.98 | −2.56 − *j*1.48 |
| | | Sequence potentials | |
| Fault *a* | +0.14 + *j*0.29 | −0.03 − *j*0.21 | −0.11 − *j*0.08 |
| Fault *d* | +0.25 + *j*0.25 | +0.22 − *j*0.09 | −0.07 + *j*0.10 |

# CLASSIFIED INDEX

# CLASSIFIED INDEX

## HISTORICAL OUTLINE



## GENERAL PRINCIPLES

### RESOLUTION OF N-PHASE QUANTITIES

## RESOLUTION OF 3-PHASE QUANTITIES

### MEASUREMENT OF SEQUENCE COMPONENTS

### Potential



### Current



### Circuit Constants



### Power



### Products IZ and EY



## RESOLUTION OF 4-PHASE QUANTITIES

## RESOLUTION OF 2-PHASE QUANTITIES



## TRANSMISSION LINES

### RESOLUTION OF LINE CONSTANTS

### Line Impedance, Neglecting Effect of Earth



### Line Impedance, Considering Effect of Earth



* 0 = zero, 1 = positive, 2 = negative.

## Mutual Impedance between Parallel Circuits



## Capacitance



* 0 = zero, 1 = positive, 2 = negative.

## Long Lines



# TRANSFORMERS

## THREE-PHASE COMBINATIONS OF SINGLE-PHASE TRANSFORMERS

### Two-winding Transformers



### Three-winding Transformers



* 0 = zero, 1 = positive, 2 = negative.

# INDUCTION MACHINES

## TERMINAL CHARACTERISTICS OF MOTORS AND PHASE CONVERTERS

### Motors

## Phase Converters



# INDUCTION MACHINES

## INTERNAL REACTIONS

## Rotating Magnetic Field



## Stator and Rotor Frequencies



## Losses

## Torque



## Vector Diagrams and Equivalent Circuits



## Unsymmetrical Rotor Circuit



## Zero Sequence



## Cage Winding



## Saturation

## INDUCTION MACHINES

### UNSYMMETRICAL STATOR AND ROTOR WINDINGS



## SYNCHRONOUS MACHINES

### Ideal Synchronous Machine

## Single-phase Field Winding



## Solid Cylindrical Rotor



## Salient Poles

## Negative-sequence Equivalent Circuit



## Measurement of Sequence Constants



## Saturation



## POWER NETWORKS



## Relations among Sequence Components of Potential and Current

## Series Faults



## Simultaneous Shunt and Series Faults

# CHART FOR DE  MININ  YMMETRICAL COMPONENTS

The chart on the back cover is designed to give the complex values of the positive-, the negative-, and the zero-sequence components of the line-to-neutral potential. This chart is derived from Fig. 14, page 54. The use of the chart is best illustrated by a numerical example. Let the numerical values of the line potentials be $V_{ab} = 176$, $V_{bc} = 220$, $V_{ca} = 154$. The ratios of the smaller potentials to the *greatest* potential are

$$\frac{V_{ab}}{V_{bc}} = 0.8 \qquad \text{and} \qquad \frac{V_{ca}}{V_{bc}} = 0.7$$

For these ratios of 0.8 and 0.7 the positive-sequence component of the line-to-neutral potential $V_{an}$ is

$$\begin{aligned} V_{an1} &= (0.822 + j0.0425)220/\sqrt{3} \\ &= 104.4 + j5.4 \end{aligned}$$

The negative-sequence component of the line-to-neutral potential $V_{an}$ is

$$\begin{aligned} V_{an2} &= (-0.178 + j0.0425)220/\sqrt{3} \\ &= -22.6 + j5.4 \end{aligned}$$

It should be noted that in deriving these complex expressions it is assumed that the complex expression for the greatest potential $V_{bc}$ is $(0 - j220)$. If $V_{ca}$ is greater than $V_{ab}$, the sign before the imaginary part of each of the components is negative instead of positive.

In order to determine the zero-sequence component of the line-to-neutral potential $V_{an}$, measure the potentials $V_{bn}$ and $V_{cn}$. Let us assume that $V_{bn} = 143$ volts and $V_{cn} = 88$ volts. The procedure is as follows: Locate the vertex $a$ of the line-potential triangle with reference to the coordinate axes having $O$ as the origin. In our example the vector drawn from $O$ to $a$ is $0.322 + j0.0425$. Dividing this by 3 locates the center of gravity of the line-potential triangle at $0.107 + j0.0142$. Since $V_{bn} = 0.65V_{bc}$ and $V_{cn} = 0.4V_{bc}$, the neutral point $n$ lies at the terminus of $0.089 + j0.075$. The difference between the vectors drawn to the center of gravity and to the neutral point is the zero-sequence component. In this example it is $(0.107 + j0.0142) - (0.089 + j0.075)$. On account of the scale to which the coordinate axes are drawn, the magnitude of the zero-sequence component is obtained by multiplying this difference by $220\sqrt{3}$. Thus $V_{an0}$ is $(0.018 - j0.061)220\sqrt{3} = 6.9 - j23.2$. Had the values of $V_{bn}$ and $V_{cn}$ been interchanged, the vector to the neutral point would have been $0.089 - j0.075$ and $V_{an0}$ would have been $(0.018 + j0.089)220\sqrt{3} = 6.9 + j33.9$.